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HYPER-KÄHLER METRICS AND HARMONIC SUPERSPACE

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1. Introduction

Supersymmetry severely restricts a form of matter self-couplings. The scalar fields of any supersymmetric matter theory[x)] in four dimensions are described by nonlinear σ-models, Kählerian in the N=1 case[1], hyper-Kählerian in the rigid N=2 case[2] and quaternionic in the local one[3]. These remarkable geometric properties are to be revealed most transparently within manifestly supersymmetric formulations based on unconstrained off-shell superfields. Indeed, any admissible superfield self-interactions should necessarily lead to the above-mentioned σ-models.

There is an exhaustive description of the Kähler geometry of N=1 matter in superspace[1,4]. The bosonic manifold metric was shown to be related in a simple way to the superfield Lagrangian. These results were successfully used in phenomenological applications in N=1 supersymmetric GUT's[5]. Now, attempts of utilization of N=2 supersymmetry get starting (see, e.g.,[6]). Until the last year N=2 matter Lagrangians have been constructed either at the component level[2,3,7] or in terms of N=1 superfields[8] with at most one manifest supersymmetry. In the latter case an off-shell formulation was achieved for some hyper-Kählerian σ-models, and some new hyper-Kählerian metrics were found.

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x) We mean the supermultiplets with the propagating spins 0, 1/2.

However, N=0 and N=1 formulations give no recipes how to write down general N=2 supersymmetric Lagrangians so as to automatically get hyper-Kählerian metrics for scalar fields. Therefore it is highly desirable to have a complete N=2 superspace desctiption both having in mind future phenomenological applications and purely mathematical reasons. Indeed, manifest N=2 supersymmetry opens a way to explicitly construct hyper-Kählerian metrics (even for the simplest, 4-dimensional manifolds metrics are not known in a number of important cases including famous K_3-manifold).

In/9/ we have developed a manifestly N=2 supersymmetric off-shell description of self-interacting N=2 matter[x)] (q and ω -hypermultiplets) in harmonic superspace in terms of unconstrained analytic N=2 superfields. Thus, listing all the possible hypermultiplet self-couplings we may, in principle, list all possible hyper-Kählerian metrics and find their explicit form.

In the present paper we do the first steps in this direction and compute the metric for the simplest U(2) invariant quartic self-interaction of a q-hypermultiplet. The problem of finding the metric amounts to eliminating an infinite number of auxiliary fields. In the case under consideration we obtain the known hyper-Kählerian Taub-NUT metric. Details of computation are given in Sect. 2,3. To make a closer contact with the hyper-Kähler geometry we pass in Sect. 4 to another equivalent representation of self-interacting q-hypermultiplet which reveals unexpected analogies with the τ-description of N=2 Yang-Mills theory/9,10/. For the U(2)-example we recover in this way the on-shell constrained N=2 superfield formulation of supersymmetric hyper-Kählerian σ-models given by Sierra and Townsend/11/ within which the hyper-Kähler properties are manifest (Sect. 5). Section 6 contains a discussion of the most general self-coupling of hypermultiplets based on the dimensionality and analyticity considerations. We conjecture that action is an analytic superspace integral of arbitrary analytic Lagrange density. Finally, the appendix treats a general harmonic conservation law which may be of use in future calculations of bosonic metrics for more complicated hypermultiplet self-couplings.

2. In the present section we compute the bosonic metric associated with the U(2)-invariant self-coupling of a single q-hypermultiplet. The harmonic superspace action and the corresponding equations of motion are/9/:

$$S = \int d\zeta^{(-4)} du \left[\overset{*}{q}{}^+ D^{++} q^+ + \frac{\lambda}{2} (q^+)^2 (\overset{*}{q}{}^+)^2 \right] \tag{2.1}$$

x) As well as N=2 Yang-Mills and supergravity theories.

$$D^{++} q^+ + \lambda (q^+ \overset{*}{q}{}^+) q^+ = 0 \quad , \quad D^{++} \overset{*}{q}{}^+ - \lambda (q^+ \overset{*}{q}{}^+) \overset{*}{q}{}^+ = 0 \tag{2.2}$$

Here[x)] q^+ is a complex unconstrained N=2 superfield defined on the analytic N=2 superspace $\{\zeta_A, u^\pm_i\} = \{x^{\alpha\dot\alpha}_A, \theta^+_\alpha, \bar\theta^+_{\dot\alpha}, u^\pm_i\}$, $q^+ = q^+(\zeta_A, u)$, $*$ is the analyticity preserving conjugation $\left((\overset{*}{q}{}^+)^* = -q^+ \right)$ and D^{++} is the harmonic derivative in the analytic basis:

$$D^{++} = \partial^{++} - 2i\theta^+ \sigma^a \bar\theta^+ \partial_a \quad , \quad \partial^{++} \equiv u^{+i} \frac{\partial}{\partial u^{-i}} \, . \tag{2.3}$$

Besides the standard SU(2) invariance realized in N=2 superspace/9/, this model has U(1) invariance

$$q^{+'} = e^{i\alpha} q^+ \quad , \quad \overset{*}{q}{}^{+'} = e^{-i\alpha} \overset{*}{q}{}^+ \tag{2.4}$$

leading to the conserved Noether current j^{++}

$$D^{++} j^{++} = 0 \quad , \quad j^{++} = i\, q^+ \overset{*}{q}{}^+ \, . \tag{2.5}$$

This U(1) invariance will substantially simplify the computation of metric.

Since we are interested in the pure bosonic part of the action, we may omit the fermions in θ^+, $\bar\theta^+$ expansion of q^+

$$\begin{aligned} q^+(\zeta_A, u) = {} & F^+(x_A, u) + i\theta^+ \sigma^a \bar\theta^+ A^-_a(x_A, u) + \theta^+\theta^+ M^-(x_A, u) + \\ & + \bar\theta^+\bar\theta^+ N^-(x_A, u) + \theta^+\theta^+ \bar\theta^+\bar\theta^+ P^{(-3)}(x_A, u) \, . \end{aligned} \tag{2.6}$$

Substituting this into (2.2), one gets the equations of motion in (x_A, u) space:

$$\partial^{++} F^+ + \lambda (F^+ \overset{*}{F}{}^+) F^+ = 0 \tag{2.7a}$$

$$\partial^{++} A^-_a - 2\partial_a F^+ + \lambda F^+ \overset{*}{F}{}^+ A^-_a + \lambda (F^+)^2 \overset{*}{A}{}^-_a = 0 \tag{2.7b}$$

$$\partial^{++} M^- + \lambda (F^+)^2 \overset{*}{N}{}^- + 2\lambda F^+ \overset{*}{F}{}^+ M^- = 0 \tag{2.7c}$$

$$\partial^{++} N^- + \lambda (F^+)^2 \overset{*}{M}{}^- + 2\lambda F^+ \overset{*}{F}{}^+ N^- = 0 \tag{2.7d}$$

$$\begin{aligned} & \partial^{++} P^{(-3)} + \partial^a A^-_a + \lambda (F^+)^2 \overset{*}{P}{}^{(-3)} + 2\lambda F^+ \overset{*}{F}{}^+ P^{(-3)} - \frac{\lambda}{2} A^{-a} A^-_a \overset{*}{F}{}^+ - \\ & - \lambda A^{-a} \overset{*}{A}{}^-_a F^+ + 2\lambda \overset{*}{F}{}^+ M^- N^- + 2\lambda F^+ (M^- \overset{*}{M}{}^- + N^- \overset{*}{N}{}^-) = 0 \, . \end{aligned} \tag{2.7e}$$

x) For the notation and details concerning harmonic superspace, see refs./9,10/.

All these equations except (2.7e) are kinematical and serve to eliminate an infinite tail of auxiliary fields appearing in harmonic expansion with respect to $u^{\pm}_i$. The last equation contains dynamics and hence will not be used in what follows.

Now we integrate in (2.1) over $\theta^+, \bar\theta^+$ using eqs. (2.6),(2.7a-d). Contribution proportional to M^-, N^- and $\rho^{(-3)}$ drop out, and the bosonic action reduces to

$$S_B = \frac{1}{2}\int d^4x\, du \left(\overset{*}{A}{}^-_a \partial^a F^+ - A^-_a \partial^a \overset{*}{F}{}^+ \right). \qquad (2.8)$$

where $F^+(x,u)$ and $A^-_a(x,u)$ obey eqs (2.7a,b). The latter are easily solved due to U(1) invariance (2.4). Indeed the conservation law (2.5) implies $\partial^{++}(F^+\overset{*}{F}{}^+)=0$. Whence

$$F^+(x,u)\overset{*}{F}{}^+(x,u) = c^{(ij)}(x)\, u^+_i u^+_j$$

$$\left(F^+\overset{*}{F}{}^+\right)^* = -F^+\overset{*}{F}{}^+ \quad\Rightarrow\quad \overline{c^{(ij)}} = -\varepsilon_{il}\,\varepsilon_{jn}\, c^{(ln)} . \qquad (2.9)$$

This suggest the following change of variables

$$F^+(x,u) = f^+(x,u)\, e^{\lambda\varphi} , \qquad \varphi(x,u) = -c^{(ij)}(x)\, u^+_i u^-_j = -\overset{*}{\varphi}(x,u) \qquad (2.10)$$

which reduces (2.7a) to the linear equation

$$\partial^{++} f^+(x,u) = 0 \quad\Rightarrow\quad f^+(x,u) = f^i(x)\, u^+_i . \qquad (2.11)$$

Taking into account that

$$F^+\overset{*}{F}{}^+ = f^+\overset{*}{f}{}^+ \quad\Rightarrow\quad c^{(ij)}(x) = -f^{(i}(x)\,\bar f^{j)}(x) , \qquad (2.12)$$

where $\bar f^i = \varepsilon^{ij}\bar f_j ,\ \bar f_j \equiv \overline{(f^j)}$, we obtain the general solution of (2.7a) in the form

$$F^+(x,u) = f^i(x)\, u^+_i\, e^{\lambda\varphi} = f^i(x)\, u^+_i \exp\left(\lambda f^{(j}(x)\,\bar f^{k)}(x)\, u^+_j u^-_k\right). \qquad (2.13)$$

Thus, all the components in $u^{\pm}_i$ -expansion of $F^+(x,u)$ are expressed in terms of $f^i(x)$ which is the physical bosonic field.

The remaining equation (2.7b) is simplified by substitution $A^-_a(x,u) = B^-_a(x,u)\, e^{\lambda\varphi}$. Equation (2.7b) implies that harmonic expansion of B^-_a contains only linear $(\sim u^-)$ and trilinear $(\sim u^-u^-u^+)$ terms. Finally,

$$A^-_a = e^{\lambda\varphi}\Big\{ 2\lambda f^i u^+_i\, \partial_a\left(f^{(k}\bar f^{j)} u^-_k u^-_j\right) + 2\,\partial_a f^i\, u^-_i + $$

$$+ \frac{\lambda f^i u^-_i}{1+\lambda f\bar f}\left(f^j\partial_a \bar f_j - \bar f_j \partial_a f^j\right)\Big\} . \qquad (2.14)$$

Let us emphasize once more that this simple form of the solution is due to U(2) invariance of the action (2.1). More general self-couplings lead to much more complicated equations (see Sect. 4).

To find the action in terms of $f^i(x)$, we integrate (2.8) over $u^{\pm}_i$ using (2.13), (2.14), the reduction identities /9/

$$u^+_i u^+_{(j_1}\cdots u^+_{j_n} u^-_{k_1}\cdots u^-_{k_m)} = u^+_{(i} u^+_{j_1}\cdots u^-_{k_m)} + \frac{m}{m+n+1}\,\varepsilon_{i(k_1} u^+_{j_1}\cdots u^+_{j_n} u^-_{k_2}\cdots u^-_{k_m)} ,$$

$$u^-_i u^+_{(j_1}\cdots u^+_{j_n} u^-_{k_1}\cdots u^-_{k_m)} = u^-_{(i} u^+_{j_1}\cdots u^-_{k_m)} - \frac{n}{m+n+1}\,\varepsilon_{i(j_1} u^+_{j_2}\cdots u^-_{k_m)}$$

and the $u^{\pm}_i$ integration rules /9/

$$\int du\, (u^+)^{(m}(u^-)^{n)}(u^+)_{(k}(u^-)_{l)} = \begin{cases} \dfrac{(-1)^n\, m!\, n!}{(m+n+1)!}\,\delta^{(i_1}_{(j_1}\cdots\delta^{i_{m+n})}_{j_{k+l})} & \text{if } m=l,\ n=k \\ 0 & \text{otherwise} \end{cases}$$

$$(u^+)^{(m}(u^-)^{n)} \equiv u^{+(i_1}\cdots u^{+i_m} u^{-j_1}\cdots u^{-j_n)} .$$

As a result, we arrive at the following bosonic action

$$S_B = -\frac{1}{2}\int d^4x \left(g_{ij}\,\partial_a f^i \partial^a f^j + \bar g^{ij}\,\partial_a \bar f_i \partial^a \bar f_j + 2 h^i{}_j\, \partial_a f^j \partial^a \bar f_i \right), \qquad (2.15)$$

where

$$g_{ij} = \frac{\lambda(2+\lambda f\bar f)}{2(1+\lambda f\bar f)}\,\bar f_i \bar f_j , \qquad \bar g^{ij} = \frac{\lambda(2+\lambda f\bar f)}{2(1+\lambda f\bar f)}\, f^i f^j ,$$

$$h^i{}_j = \delta^i_j(1+\lambda f\bar f) - \frac{\lambda(2+\lambda f\bar f)}{2(1+\lambda f\bar f)}\, f^i \bar f_j \ ; \quad f\bar f = f^i \bar f_i . \qquad (2.16)$$

It is remarkable that the extremely simple monomial N=2 superfield interaction (2.1) entails a complicated nonpolynomial Lagrangian for the physical bosons. Note the manifest U(2)-invariance of (2.15), (2.16), which reflects the U(2)-invariance of the original action.

3. It is not so easy to see that the metric (2.16) is hyper-Kählerian, especially, because it is not manifestly Kählerian in coordinates $f^i, \bar f_i$. By simple (though lengthy) calculations one can verify that it is Ricci-flat. However, it is the necessary condition, not the sufficient one. One should also pick up three linearly independent covariantly constant complex structures and this is less trivial. In Sect. 5 we shall visualize these geometric properties of metric (2.6) by passing to the new, τ -representation of eqs. (2.2). Here we prefer to proceed in a different way. Namely, we demonstrate that (2.16) is reduced by a change of variables to the well-known Taub-NUT metric, which belongs to the class of four-dimensional Euclidean gravitational instantons and is known to be hyper-Kähler.

To this end, let us first introduce "spherical" coordinates in the R^4-space $\{f^i, \bar{f}_i$:

$$f^1 = \rho \cos\frac{\theta}{2} \cdot \exp\frac{i}{2}(\psi+\varphi)$$

$$f^2 = \rho \sin\frac{\theta}{2} \exp\frac{i}{2}(\psi-\varphi) \quad , \quad f\bar{f} = \rho^2 . \tag{3.1}$$

Then

$$ds^2 = g_{ij}\, df^i df^j + \bar{g}^{ij}\, d\bar{f}_i d\bar{f}_j + 2h^i{}_j\, df^j d\bar{f}_i =$$

$$= 2(1+\lambda\rho^2)d\rho^2 + \frac{1}{2}\rho^2(1+\lambda\rho^2)(d\theta^2 + \sin^2\theta\, d\varphi^2) +$$

$$+ \frac{\rho^2}{2(1+\lambda\rho^2)}\left(d\psi + \cos\theta\, d\varphi\right)^2 . \tag{3.2}$$

We assume that (3.2) has no singularities in ρ, so $\lambda > 0$. Then one makes a change of variables

$$\rho^2 = 2(r-m)m \quad , \quad r \geq m = \frac{1}{2\sqrt{\lambda}} \tag{3.3}$$

recasting ds^2 in the form

$$ds^2 = 2\left\{\frac{1}{4}\frac{r+m}{r-m}\, dr^2 + \frac{1}{4}(r^2-m^2)(d\theta^2 + \sin^2\theta\, d\varphi^2) + \right.$$

$$\left. + m^2 \frac{r-m}{r+m}\,(d\psi + \cos\theta\, d\varphi)^2 \right\} \tag{3.4}$$

which, up to a numerical coefficient, is the standard Taub-NUT metric (see, e.g.,/12/).

4. We have seen above that the hyper-Kähler geometry in the N=2 analytic superspace description arises only upon eliminating an infinite tower of auxiliary fields, i.e., with partially putting the theory on-shell x). One may inquire how to expose the hyper-Kähler properties directly in terms of N=2 superfields. Clearly, it should essentially involve the use of superfield equations of motion. Here we derive another on-shell superfield representation of self-interacting

x) This has to be compared with the N=1 case where the Kähler properties are manifest already at the level of off-shell N=1 superfield action. Elimination of auxiliary fields there does not influence the form of the bosonic Lagrangian.

q-hypermultiplets in which the geometry is expected to reveal itself more transparently and which bears an interesting analogy with the τ-description of N=2 gauge theory/9,10/.

For simplicity we restrict our study to a single q^+ self-interacting in a manifestly SU(2)-invariant manner (the general case will be treated elsewhere). The action is

$$S_q = \int dZ^{(-4)} du \left(\overset{*}{\bar{q}}{}^+ D^{++} q^+ + \mathcal{L}^{(+4)}_{int} \right) \tag{4.1}$$

$$\mathcal{L}^{(+4)}_{int} = \frac{\lambda_1}{2}(q^+)^2(\overset{*}{\bar{q}}{}^+)^2 + \lambda_2 (q^+)^3 \overset{*}{\bar{q}}{}^+ - \bar{\lambda}_2 (\overset{*}{\bar{q}}{}^+)^3 q^+ + \lambda_3 (q^+)^4 + \bar{\lambda}_3 (\overset{*}{\bar{q}}{}^+)^4 . \tag{4.2}$$

Note that the kinetic term in (4.1) is invariant under some extra $\widetilde{SU}(2)$ group (containing the U(1)-subgroup (2.4)) which is an analogue of the known Pauli-Gürsey group. With respect to this group q^+ and $\overset{*}{\bar{q}}{}^+$ form an isodoublet. If

$$q^{+a} \equiv (q^+, -\overset{*}{\bar{q}}{}^+) \quad , \quad \overline{(q^{+a})}^{\,*} = \overset{*}{\bar{q}}{}^+_a = -\varepsilon_{ab}\, q^{+b} \tag{4.3}$$

then the kinetic term can be written in the form

$$\frac{1}{2}\left(\overset{*}{\bar{q}}{}^+ D^{++} q^+ - q^+ D^{++} \overset{*}{\bar{q}}{}^+\right) = \frac{1}{2}\, q^{+a} D^{++} q^+_a . \tag{4.4}$$

Though self-couplings in (4.1) break this SU(2) symmetry the $\widetilde{SU}(2)$-notation is useful in that it allows one to write the equations of motion in a compact 2x2 matrix form x)

$$(D^{++} + iV^{++})\, q^+ = \left[D^{++}\delta_a^b + i(V^{++})_a{}^b\right] q^+_b = 0 \tag{4.5}$$

$$V^{++} = \overset{*}{\overline{V^{++}}} = \frac{1}{i}\begin{pmatrix} -\lambda_1 q^+\overset{*}{\bar{q}}{}^+ - \lambda_2 (q^+)^2 + \bar{\lambda}_2(\overset{*}{\bar{q}}{}^+)^2 & -2\lambda_2 q^+\overset{*}{\bar{q}}{}^+ - 4\lambda_3 (q^+)^2 \\ -2\bar{\lambda}_2 q^+\overset{*}{\bar{q}}{}^+ + 4\bar{\lambda}_3(\overset{*}{\bar{q}}{}^+)^2 & \lambda_1 q^+\overset{*}{\bar{q}}{}^+ - \bar{\lambda}_2(\overset{*}{\bar{q}}{}^+)^2 + \lambda_2 (q^+)^2 \end{pmatrix} . \tag{4.5a}$$

Let us also remind the analyticity conditions

$$D^+_\alpha q^{+a} = 0 \quad , \quad \bar{D}^+_{\dot\alpha}\, q^{+a} = 0. \tag{4.6}$$

x) Besides this, extra $\widetilde{SU}(2)$ group effectively reduces also the number of independent coupling constants in (4.2) from 5 to 2.

The quantity V^{++}, being a real analytic superfield in the adjoint representation of $\widetilde{SU}(2)$, can be regarded as a composite N=2 Yang-Mills prepotential[9]. Correspondingly, equation (4.5) is similar to the equation for the "bridge" between λ- and τ-representations of N=2 gauge theory ([9], eq (IV.16b)). This suggests the following substitution for q^+:

$$q^+ = e^{iv}\,\tilde{q}^+ , \tag{4.7}$$

$$\left(D^{++}+iV^{++}\right)e^{iv}=0 \quad , \text{or} \quad V^{++}=-ie^{iv}D^{++}e^{-iv}. \tag{4.8}$$

In terms of $\tilde{q}^+$ eqs. (4.5), (4.6) reduce to

$$D^{++}\tilde{q}^+=0 \Rightarrow \tilde{q}^{+a}=\tilde{q}^{ia}(z)u^+_i , \tag{4.9}$$

$$\mathcal{D}^+_{\alpha(\dot\alpha)}\tilde{q}^+ \equiv \left(D^+_{\alpha(\dot\alpha)}+iA^+_{\alpha(\dot\alpha)}\right)\tilde{q}^+=0 , \tag{4.10}$$

$$A^+_{\alpha(\dot\alpha)} \equiv -ie^{-iv}D^+_{\alpha(\dot\alpha)}e^{iv}=A^i_{\alpha(\dot\alpha)}(z)u^+_i . \tag{4.11}$$

Thus, we arrive at the on-shell description of the self-interacting hypermultiplet in terms of the ordinary N=2 superfield $\tilde{q}^i(z)$ constrained by the "covariantized" analyticity conditions (4.10). It directly generalizes the standard N=2 superfield formulation of a free hypermultiplet[9] and is related to the original analytic superspace description given by eqs. (4.5), (4.6) like the τ-representation of N=2 Yang-Mills is related to the λ one[9]. In the q^+-language, analyticity is purely kinematic while the dynamics is concentrated in eq. (4.5) which can be interpreted as the condition of "covariant" $u^\pm_i$-independence of q^+. On the contrary, in the $\tilde{q}^+$-language, the notion of $u^\pm_i$-independence is kinematic. The theory is put on-shell by the constraints (4.10) stating that $\tilde{q}^+$ is "covariantly" analytic.

Let us emphasize that Eq.(4.8) in different descriptions comes out as a definition of different objects. In the λ-description it defines the bridge e^{iv} while in the τ-description it defines the "prepotential" V^{++}. The expression of e^{iv} in forms of V^{++} can be obtained iteratively, by a general recipe given by us for the N=2 gauge theory[10]. This solution is nonlocal in harmonics and is independent of a specific form of V^{++}.

Note that again in a close analogy with the N=2 Yang-Mills theory[13,9] we may define the τ-representation of q^+-hypermultiplet in more

abstract terms, namely, by adding to eq. (4.10) the constraints

$$\{\mathcal{D}^+_\alpha,\mathcal{D}^+_\beta\}=\{\bar{\mathcal{D}}^+_{\dot\alpha},\bar{\mathcal{D}}^+_{\dot\beta}\}=\{\mathcal{D}^+_\alpha,\bar{\mathcal{D}}^+_{\dot\beta}\}=0 , \tag{4.12}$$
$$[D^{++},\mathcal{D}^+_\alpha]=[D^{++},\bar{\mathcal{D}}^+_{\dot\alpha}]=0 .$$

Equations (4.10), with any $A^+_{\alpha(\dot\alpha)}$ composed of $\tilde{q}^+$ and satisfying (4.12), are reduced after the redefinitions (4.7), (4.8) to the manifestly analytic equations (4.5), (4.6). However, for (4.5) to be derivable from an action, V^{++} and, respectively, e^{iv} and $A^+_{\alpha(\dot\alpha)}$ have to obey certain integrability, conditions whose implications are not clear to us at the moment.

These considerations can be easily extended to the case of n hypermultiplets. Superfield q^{+a} (4.3) then acquires additional indices and so do V^{++} and $A^+_{\alpha(\dot\alpha)}$ which become 2nx2n matrices.

In the next Section the usefulness of the τ-representation will be illustrated by the $(q^+)^2(\overset{*}{q}{}^+)^2$-example.

5. In the τ-description of q^+ proposed above the basic geometric object is the composite spinor connection $A^+_{\alpha(\dot\alpha)}(\tilde{q})$ restricted by the constraints (4.12), (4.10). On the other hand, Sierra and Townsend[11] have given a different on-shell superfield formulation of q^+-hypermultiplet, also in terms of the ordinary constrained N=2 superfields. For one hypermultiplet their constraint is as follows[11]:

$$E^{(ia}_{jb}(\tilde{q})\,D^{k)}_{\alpha(\dot\alpha)}\tilde{q}^{jb}(z)=0 \qquad \overline{\tilde{q}_{jb}}=\varepsilon_{je}\varepsilon_{bc}q^{ec} \tag{5.1}$$
$$(i,j=1,2;\ a,b=1,2)$$

where the real superfields q^{jb} are assumed to parametrize a four-dimensional real Riemann space, $E^{ia}_{jb}(\tilde{q})$ is the corresponding inverse vielbein with the world indices jb and the tangent space indices ia. In terms of E^{ia}_{jb} the hyper-Kählerian geometry of self-interacting hypermultiplet manifests itself most clearly[11]. So it would be desirable to put our constraints (4.9)-(4.11) in the form (5.1). For the time being, we do not know, whether it is always possible (the σ-models associated with the constraint (5.1) seem to require the SU(2) automorphism group to be unbroken while eqs. (4.9)-(4.11) do not imply such a restriction). Our aim here is to explicitly demonstrate that for the U(2)-case treated above this equivalence really takes place.

The relevant V^{++} is diagonal

$$V^{++}=\frac{1}{i}\begin{pmatrix}\lambda q^+\overset{*}{q}{}^+ & 0\\ 0 & -\lambda q^+\overset{*}{q}{}^+\end{pmatrix} \tag{5.2}$$

and there arises an analogy with the Abelian N=2 gauge theory. Such a simplification allows us to obtain the bridge and spinor connections in a closed form

$$v = \frac{i\lambda}{2}\left(\tilde q^{+}\overset{*}{\tilde q}{}^{-} + \tilde q^{-}\overset{*}{\tilde q}{}^{+}\right), \quad \tilde q^{\pm} = \tilde q^{i}(z)u^{\pm}_{i}, \quad \overset{*}{\tilde q}{}^{\pm} = -\bar{\tilde q}{}^{i}(z)u^{\pm}_{i} \tag{5.3}$$

$$A^{+}_{\alpha(\dot\alpha)} = \begin{pmatrix} D^{+}_{\alpha(\dot\alpha)}v & 0 \\ 0 & -D^{+}_{\alpha(\dot\alpha)}v \end{pmatrix}. \tag{5.4}$$

Note that

$$\tilde q^{i}(z)\big|_{\theta=0} = f^{i}(x), \quad iv\big|_{\theta=0} = \lambda\varphi(x,u),$$

where f^{i} and φ are the same as in eqs (2.9), (2.10).

By means of some easy algebra the constraints (4.10) with $A^{+}_{\alpha(\dot\alpha)}$ (5.4) can be reduced to

$$E^{+a}_{kb}D^{+}_{\alpha(\dot\alpha)}\tilde q^{kb}(z) = 0, \quad E^{+a}_{kb} = E^{ia}_{kb}(z)u^{+}_{i} \tag{5.5}$$

$$E^{ia}_{kb} = \begin{pmatrix} E^{i1}_{k1} & E^{i2}_{k1} \\ E^{i1}_{k2} & E^{i2}_{k2} \end{pmatrix} = \begin{pmatrix} \delta^{i}_{k}\left(1+\frac{\lambda}{2}\tilde q\bar{\tilde q}\right) - \frac{\lambda}{2}\tilde q_{k}\bar{\tilde q}{}^{i} & \frac{\lambda}{2}\bar{\tilde q}_{k}\bar{\tilde q}{}^{i} \\ -\frac{\lambda}{2}\tilde q_{k}\tilde q^{i} & \delta^{i}_{k}\left(1+\frac{\lambda}{2}\tilde q\bar{\tilde q}\right) + \frac{\lambda}{2}\bar{\tilde q}_{k}\tilde q^{i} \end{pmatrix}. \tag{5.6}$$

Taking off the zweibeins u^{+}_{i} from the r.h.s. of (5.5) we may cast the latter equation just into the form (5.1). To achieve a complete agreement with ref./11/, one should also take into account a freedom of rescaling (5.5) by a scalar function of $\tilde q$. It turns out that the hyper-Kähler properties become manifest in terms of the vielbeins

$$\tilde E^{ia}_{kb} = (\det E)^{-\frac{1}{6}}\cdot E^{ia}_{kb} = \frac{1}{(1+\lambda\tilde q\bar{\tilde q})^{1/2}}E^{ia}_{kb}. \tag{5.7}$$

One may explicitly check that the two-forms:

$$\Omega^{ij} = \tilde E^{(ia}_{kb}\tilde E^{j)c}_{ed}\,\varepsilon_{ac}\,d\tilde q^{kb}\,d\tilde q^{ed} \tag{5.8}$$

are closed, constitute a SU(2)-triplet, and are covariantly constant

with respect to the connection constructed by the metric

$$g_{ib,ka} = \tilde E^{jc}_{ib}\tilde E^{ed}_{ka}\,\varepsilon_{je}\,\varepsilon_{cd}. \tag{5.9}$$

These properties are just characteristic of a hyper-Kähler manifolds. The purely bosonic metric defined as the θ-independent part of (5.9) exactly coincides with (2.16)

$$g_{ik}(f) = \begin{pmatrix} \frac{\lambda(2+\lambda\rho^{2})}{2(1+\lambda\rho^{2})}\bar f_{i}\bar f_{k} & \varepsilon_{ik}(1+\lambda\rho^{2}) + \frac{\lambda(2+\lambda\rho^{2})}{2(1+\lambda\rho^{2})}\bar f_{i}f_{k} \\ -\varepsilon_{ik}(1+\lambda\rho^{2}) + \frac{\lambda(2+\lambda\rho^{2})}{2(1+\lambda\rho^{2})}f_{i}\bar f_{k} & \frac{\lambda(2+\lambda\rho^{2})}{2(1+\lambda\rho^{2})}f_{i}f_{k} \end{pmatrix}. \tag{5.10}$$

Thus, there exists a possibility to expose the structure of metrics associated with the q-self-couplings also in the τ-representation by passing to the constraints of the form (5.1). One may derive a general formula relating the vielbein E^{ia}_{jb} to the bridge e^{iv}. However, to restore e^{iv} by V^{++} is in general not easier than to compute the metric in the λ-representation. Perhaps, it would be more fruitful to deal at once with the N=2 Yang-Mills-like constraints (4.10), without transforming them to the form (5.1) or (and it would be most desirable) to learn how to reveal the geometric structures directly in the λ-representation which provides the natural framework for handling hypermultiplets.

In any case, there remains an actual and interesting task of computing the metrics for other self-couplings of q and ω-hypermultiplets by applying the straight forward method of Sect. 2. In particular, it is an intriguing question which self-coupling corresponds to the more familiar hyper-Kähler metric, that of Equchi and Hansen/14/. It appeared in the early investigations on supersymmetric hyper-Kähler σ-models and, like the Taub-NUT metric, exhibits U(2)-invariance.

6. Finally, we discuss the most general self-interactions of hypermultiplets. The dimensionality and analyticity arguments seem to completely determine their form. Indeed let us start with the case N=0. The standard N=0 σ-model action is

$$S = \frac{1}{\kappa^{2}}\int d^{4}x\, g_{ij}(f)\,\partial_{a}f^{i}\partial^{a}f^{j}, \tag{6.1}$$

where $\varkappa$ is a coupling constant (dimension mass^{-1}), f^i are dimensionless and are considered as coordinates of some manifold. To pass from f^i to physical scalar field one has to rescale it as $f^i = \varkappa f^i_{phys}$. The metric $g_{ij}(f)$ is dimensionless and does not explicitly depend on $\varkappa$.

Correspondingly in the N=1 case matter is described by dimensionless chiral superfields ϕ^i ($\phi^i(x,\theta) = f^i + \theta\psi^i + \dots$) which again play the rôle of coordinates of some (Kähler) manifold. The most general N=1 σ-model action is

$$S' = \frac{1}{\varkappa^2}\int d^4x\, d^4\theta\, K(\phi, \bar{\phi}). \tag{6.2}$$

The dimensional parameter $\varkappa$ enters again via the factor $\varkappa^{-2}$ and the structure of Lagrange density is controlled by dimension of measure $d^4x\, d^4\theta$.

As we know N=2 matter is represented by N=2 analytic superfields $q^+(\zeta_A, u)$ and $\omega(\zeta_A, u)$.

Their θ-expansion again begins with geometric fields $f^i(x)$ so q^+ and ω are dimensionless as well. Under the natural assumption that the most general N=2 σ-model is formulated via q^+ or ω superfields the only possible superspace action which results (after elimination of auxiliary fields) in (6.1) is

$$S = \frac{1}{\varkappa^2}\int d\zeta^{(-4)} du\, \mathcal{L}^{(+4)}(q^+, \omega, u^\pm, D^{++}q^+, D^{++}\omega, \dots), \tag{6.3}$$

where $\mathcal{L}^{(+4)}$ is dimensionless function of q^+, ω, harmonics $u^\pm$ and analyticity preserving derivatives $D^{++}q^+, D^{++}\omega$, etc. Note that $\mathcal{L}^{(+4)}$ cannot contain harmonic nonlocalities[x)], because for analyticity such terms would inevitably include spinor derivatives $(D^+)^4$. The latter is forbidden by the above dimensionality arguments.

Thus, we conjecture that any hyper-Kählerian σ-model is supersymmetrized to some $\mathcal{L}^{(+4)}$ (6.3). This provides a technique to explicitly compute hyper-Kählerian metrics by choosing a Lagrangian and eliminating auxiliary bosonic fields[xx)].

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x) E.g., like those occurring in the N=2 Yang-Mills action[/10/].

xx) Recently Rosly and Schwarz[/15/] have suggested a geometric action for hyper-Kähler supersymmetric σ-models in the analytic N=2 superspace starting with the Sierra-Townsend approach[/11/] where hyper-Kähler metrics are assumed to be given in advance.

Concluding the paper we wish to emphasize importance of establishing a classification of the hyper-Kähler metrics according to N=2 superfield Lagrange densities. The simplest case considered above is an example.

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Appendix

We derive here a general conservation law for self-interacting q-hypermultiplets which may be useful in practical calculations.

We start with the most general q-hypermultiplet action containing no more than one harmonic derivative.

$$S = S_{free} + S_{int} = \int d\zeta^{(-4)} du \left[\overset{*}{q}{}^{+a} D^{++} q^{+a} + \mathcal{L}^{(+4)}_{int}(q^+, \overset{*}{q}{}^+, u^\pm, D^{++}q^+) \right]. \tag{A.1}$$

The relevant equations of motion are

$$D^{++}q^{+a} + \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial \overset{*}{q}{}^{+a}} - D^{++}\frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}\overset{*}{q}{}^{+a})} = 0,$$

$$D^{++}\overset{*}{q}{}^{+a} - \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial q^{+a}} + D^{++}\frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}q^{+a})} = 0. \tag{A.2}$$

Let us compute

$$D^{++}\mathcal{L}^{(+4)}_{int} = \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial q^{+a}} D^{++}q^{+a} + \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial \overset{*}{q}{}^{+a}} D^{++}\overset{*}{q}{}^{+a} + \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}q^{+a})}(D^{++})^2 q^{+a} +$$
$$+ \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}\overset{*}{q}{}^{+a})}(D^{++})^2 \overset{*}{q}{}^{+a} + \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial u^{-i}} \cdot u^{+i} =$$

$$= D^{++}\left\{ \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}\overset{*}{q}{}^{+a})} \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial q^{+a}} - \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}q^{+a})} \cdot \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial \overset{*}{q}{}^{+a}} + \right. \tag{A.3}$$
$$\left. + \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}q^{+a})} \cdot D^{++}\frac{\partial \mathcal{L}^{(+4)}}{\partial (D^{++}\overset{*}{q}{}^{+a})} - \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}\overset{*}{q}{}^{+a})} \cdot D^{++}\frac{\partial \mathcal{L}^{(+4)}}{\partial (D^{++}q^{+a})} \right\} + \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial u^{-i}} u^{+i}.$$

Using once more eqs (A.2) we observe that the quantity

$$T^{(+4)} = \mathcal{L}^{(+4)}_{int} - \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}q^{+a})} D^{++}q^{+a} - \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial (D^{++}\overset{*}{q}^{+a})} D^{++}\overset{*}{q}^{+a} \tag{A.4}$$

obeys the conversation-like identity

$$D^{++} T^{(+4)} = \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial u^{-i}} u^{+i} \tag{A.5}$$

which becomes exact if $\mathcal{L}^{(+4)}_{int}$ contains no explicit u^- -dependence.

$$u^{+i} \frac{\partial \mathcal{L}^{(+4)}_{int}}{\partial u^{-i}} = 0 \Rightarrow D^{++} T^{(+4)} = 0 . \tag{A.6}$$

Eq. (A.6) implies that in coordinates of the central basis

$$T^{(+4)} = T^{(ijke)}(z)\, u^+_i u^+_j u^+_k u^+_e . \tag{A.7}$$

In the case when $\mathcal{L}^{(+4)}_{int}$ does not contain derivatives, $T^{(+4)}$ coincides with $\mathcal{L}^{(+4)}_{int}$. This conservation law is especially simple for U(2)-invariant coupling (2.1):

$$D^{++}\left[(q^+\overset{*}{q}^+)\right]^2 = 0 \Rightarrow D^{++}(q^+\overset{*}{q}^+) = 0 . \tag{A.8}$$

An interesting point about the conservation law (A.6) is that it can be related by the standard Noether procedure to the invariance of action (A.1) with respect to the following transformations

$$\delta u^-_i = c^{--} u^+_i , \quad \delta u^+_i = 0$$
$$\delta x^m_A = -2i c^{--} \theta^+ \sigma^m \bar\theta^+ , \quad \delta\theta^+ = \delta\bar\theta^+ = 0 \tag{A.9}$$

$$\delta^* q^+ = -c^{--} D^{++} q^+ \tag{A.10}$$

provided c^{--} is a double U(1) charged constant independent of u $(D^{++}c^{--}=0 ,\ c^{--}\neq 0)$. Such a constant looks rather unusually. However, one may recall the familiar isospin transformations. Here, e.g., in the transformation of proton via neutron $\delta p^{(+)} = i\alpha^{(+)} n^{(0)}$ parameter $\alpha^{(+)}$ also has an electric charge +1. We prefer to postpone a discussion of the exact meaning of transformations (A.9), (A.10) to the future.

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Гальперин А. и др. E2-85-514

Гиперкэлеровы метрики и гармоническое суперпространство

После исключения бесконечного набора вспомогательных полей наиболее общее самодействие N=2 мультиплетов материи, описываемых суперполями без связей, ведет к наиболее общей N=2 гиперкэлеровой модели. Это дает новую возможность классификации гиперкэлеровых метрик по соответствующим N=2 аналитическим самодействиям, и эффективный способ явного вычисления этих метрик. В качестве простого примера анализируется U(2)-инвариантное самодействие одного q-гипермультиплета. Показано, что оно ведет к известной метрике Тауб-НУТ. Чтобы геометрическая картина была видна прямо на N=2 суперполях, мы вводим новое представление для q-гипермультиплетов на массовой оболочке, похожее на τ-описание N=2 калибровочных теорий. Для U(2)-примера это описание соответствует формулировке Сьерра и Таунсенда.

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Hyper-Kähler Metrics and Harmonic Superspace

The most general unconstrained superfield action for self-interacting N=2 matter hypermultiplets in analytic N=2 superspace is argued to produce a most general N=2 hyper-Kähler σ-model after eliminating an infinite set of auxiliary fields. This suggests a new possibility of classifying hyper-Kähler metrics according to the N=2 analytic superfield self-interactions and provides an effective tool to compute these metrics explicity. As a simplest example the U(2)-invariant quartic self-coupling of a single q-hypermultiplet is analyzed and is shown to yield the familiar Taub NUT metric. To see the geometric pattern directly in terms of N=2 superfields we introduce a new on-shell representation of q-hypermultiplets in N=2 harmonic superspace similar to the τ-description of N=2 gauge theories. For U(2)-example this formulation is checked to coincide with that by Sierra and Townsend.

The investigation has been performed at the Laboratory of Theoretical Physic, JINR

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