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EXCEPTIONAL QUANTUM MECHANICS

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1. In quantum mechanics observables (operators) are usually presented by some matrices or differential operators with the property of associativity. One can construct nonassociative octonionic quantum mechanics with octonionic observables [22]. Other generalizations are possible, where higher hypercomplex number serve as observables [23].

One more quantum mechanics which uses octonions is the famous exceptional quantum mechanics by Jordan, von Neumann and Wigner [1] (see also [2,4,18,19]). In the present paper we give a short and elementary exposition of the exceptional quantum mechanics including the corresponding equation of motion.

In conclusion, we discuss possibilities of matrix representation (representations in terms of usual matrices with real entries) in the exceptional quantum mechanics. We deal in the spirit of the Dirac representation theory in the same manner as matrix (and also analytic) representations were introduced for quaternions, octonions and higher hypercomplex numbers in refs. [22,23].

2. Observables (algebra $\mathfrak{M}_3^{8+}$). In exceptional quantum mechanics a general form of any observable is the Hermitian 3x3 matrix

$$\mathfrak{J} = \begin{pmatrix} \xi & c & \bar{b} \\ \bar{c} & \eta & a \\ b & \bar{a} & \zeta \end{pmatrix}, \tag{1}$$

where ξ, η, ζ are real numbers, and a, b, c are octonions ($a = a_0 e_0 + a_1 e_1 + \dots + a_7 e_7$, etc.). The bar denotes octonionic conjugate: $e_0 \to \bar{e}_0 = e_0$, $e_j \to \bar{e}_j = -e_j$, $j = 1, 2, \dots, 7$ being the involution operation ($\bar{\bar{a}} = a$). There are 27 independent matrices of this form, 26 of them being traceless. It is convenient to introduce basis matrices as follows. Let us take direct products of the Gell-Mann 3x3 matrices λ_a and octonionic units e_i (in all 72 matrices)

$$\lambda_a e_i \qquad (a = 0, 1, \dots, 8;\ i = 0, 1, \dots 7) \tag{2}$$

This set can be divided into two sets (we eliminate the imaginary unit from $\lambda_g = i\varkappa_g$, $g = 2, 5, 7$):

1) $\mathfrak{M}_3^{8-}$, the set of 45 antihermitian matrices,

$$\varkappa_{0i} = \lambda_0 e_i, \quad \varkappa_{ai} = \lambda_a e_i, \qquad a = 1, 3, 4, 6, 8;\ i = 1, \dots, 7$$
$$\varkappa_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} e_0, \quad \varkappa_5 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} e_0, \quad \varkappa_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} e_0 . \tag{3}$$

2) $\mathfrak{M}_3^{8+}$, the set of 27 Hermitian matrices,

$$\mu_0=\lambda_0 e_0, \quad \mu_\alpha=\lambda_\alpha e_0, \qquad (\alpha=1,3,4,6,8)$$
$$\mu_{2i}=\varkappa_2 e_i, \quad \mu_{5i}=\varkappa_5 e_i, \quad \mu_{7i}=\varkappa_7 e_i, \quad (i=1,\dots,7). \tag{4}$$

These Hermitian matrices form the desired basis of observables which is convenient, since one can use the well-known Gell-Mann multiplication table for the algebra of matrices λ /21/ . Now we can write matrix (1) of the general form as follows:

$$J = j_0\mu_0 + j_\alpha\mu_\alpha + j_{fi}\mu_{fi} \quad (\alpha=1,3,4,6,8;\ f=2,5,7;\ i=1,\dots,7). \tag{5}$$

where j_0, j_α , j_{fi} are real coefficients.
With the Jordan product

$$A\cdot B = \frac{1}{2}(AB+BA) \tag{6}$$

matrices (1) or (5) form a nonassociative but commutative algebra (algebra $\mathfrak{M}_3^{8+}$). One way to show this fact is to perform a direct calculation of $A\cdot B$ with matrices of the form (1) and to see that $A\cdot B$ results in a matrix of the form (1) (this is the original proof by Albert /2/). Another way of proof is to refer to the multiplication table of the basis matrices of $\mathfrak{M}_3^{8+}$:

$$\mu_\alpha\cdot\mu_\beta = \frac{2}{3}\delta_{\alpha\beta}\mu_0 + d_{\alpha\beta c}\mu_c \qquad (\alpha,\beta,c=1,3,4,6,8),$$
$$\mu_\alpha\cdot\mu_{fi} = d_{\alpha fg}\mu_{gi} \qquad (f,g=2,5,7), \tag{7}$$
$$\mu_{fi}\cdot\mu_{gj} = \delta_{ij}\left(\frac{2}{3}\delta_{fg}\mu_0 + d_{fgc}\mu_c\right) + f_{fgh}\varepsilon_{ijk}\mu_{hk} \quad (f,g,h=2,5,7),$$

which is easily calculated using the Gell-Mann multiplication table for the matrices λ ($d_{\dots}$ and $f_{\dots}$ are Gell-Mann structure constants) /21/, and the multiplication table for octonions

$$e_0 e_0 = e_0, \qquad e_i e_j = -\delta_{ij}e_0 + \varepsilon_{ijk}e_k , \tag{8}$$

where ε_{ijk} can be chosen, for example, to be (see ref. /23/)

$$\varepsilon_{123}=1, \varepsilon_{147}=1, \varepsilon_{156}=-1, \varepsilon_{246}=1, \varepsilon_{257}=1, \varepsilon_{345}=-1, \varepsilon_{367}=1 \tag{9a}$$

or

$$\varepsilon_{123}=1, \varepsilon_{145}=1, \varepsilon_{167}=-1, \varepsilon_{246}=1, \varepsilon_{257}=1, \varepsilon_{347}=1, \varepsilon_{356}=-1. \tag{9b}$$

Note that the matrices of the $\mathfrak{M}_3^{8+}$ algebra satisfy the equations

$$(A\cdot B)\cdot A^2 = A\cdot(B\cdot A^2), \qquad (\!(A,B,A^2)\!)=0, \tag{10}$$
$$J^3 - J^2\,\mathrm{Tr}\,J + J\,Q(J) - I\det J = 0, \tag{11}$$

where $\mathrm{Tr}\,J=\xi+\eta+\zeta$, $Q(J)=\xi\eta+\eta\zeta+\zeta\xi-|a|^2-|b|^2-|c|^2=\frac{1}{2}((\mathrm{Tr}\,J)^2-\mathrm{Tr}\,J^2)$,

$$\det J = \xi\eta\zeta - \xi|a|^2 - \eta|b|^2 - \zeta|c|^2 + 2\mathrm{Re}(abc) = \tag{12}$$
$$=\frac{1}{6}\left[(\mathrm{Tr}\,J)^3 - 3(\mathrm{Tr}\,J)(\mathrm{Tr}\,J^2) + 2\mathrm{Tr}\,J^3\right] = \frac{1}{3}\mathrm{Tr}((J\times J)\cdot J), \tag{13}$$

$J\times J = J(J-\mathrm{Tr}\,J) - \frac{1}{2}\mathrm{Tr}(J(J-\mathrm{Tr}\,J))$ is the Freudental product /8/ :

$$A\times B = A\cdot B - \frac{1}{2}A\,\mathrm{Tr}\,B - \frac{1}{2}B\,\mathrm{Tr}\,A + \frac{1}{2}(\mathrm{Tr}\,A)(\mathrm{Tr}\,B) - \frac{1}{2}\mathrm{Tr}(A\cdot B). \tag{14}$$

For the definition of the Jordan associator $(\!(X,Y,Z)\!)$ and a proof of property (10) see Appendix A. Equations (10) and (11) serve as a source of many useful formulas and are usually taken as a basis when the algebra $\mathfrak{M}_3^{8+}$ is defined axiomatically.

3. Automorphism group of algebra $\mathfrak{M}_3^{8+}$. The algebra $\mathfrak{M}_3^{8+}$ has the automorphism group x) isomorphic to the exceptional group F_4. Infinitesimal transformations of F_4 can be written in the following two forms /5,7,9,13,17-19/ (see Appendices A and B):

$$\delta J = (\!(A,J,B)\!) \qquad (J,A,B\in\mathfrak{M}_3^{8+}) \tag{15.a}$$
$$= [C,J]+[[ab]J]-3(a,b,J) \quad (C\in\mathfrak{M}_3^{8-},\ \mathrm{Tr}\,C=0, \tag{15.b}$$
$a=a_i e_i$, $b=b_i e_i$ are imaginary octonions)
$$\equiv [\omega_f\varkappa_f + \omega_{\alpha i}\varkappa_{\alpha i}, J] + [[ab]J] - 3(a,b,J) \qquad f=2,5,7,\ \alpha=1,3,4,6,8. \tag{15.b'}$$

The finite transformation is represented as follows:

$$J' = e^{\mathrm{ad}}J \qquad \mathrm{ad}\,J \equiv (\!(A,J,B)\!). \tag{16}$$

For the definition and relevant properties of the Jordan associator $(\!(X,Y,Z)\!)$ see Appendix A. In eqs. (15.b) and (15.b′) (a,b,J) is the usual associator, $(a,b,J)=(ab)J-a(bJ)$, and the square brackets denote the usual commutators. In eq. (15.b′) it is emphasized that the matrices $\varkappa$ of $\mathfrak{M}_3^{8-}$ serve as generators of the F_4 transformations. Only the traceless matrices ($\varkappa_f$ and $\varkappa_{\alpha i}$) generate the transformations in the usual manner (the first term of (15.b′)

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x) Automorphism is such a transformation that $(A\cdot B)' = A'\cdot B'$, where the prime denotes transformed quantities, $\delta(A\cdot B)=\delta A\cdot B + A\cdot\delta B$.

$$[\varkappa_f,\mu_\alpha]=2f_{f\alpha\beta}\mu_\beta\,,\;[\varkappa_f,\mu_{gi}]=2f_{fgh}\mu_{hi}\,,$$
$$[\varkappa_{\alpha i},\mu_\beta]=2f_{\alpha\beta f}\mu_{fi}\,,\;[\varkappa_{\alpha i},\mu_{fj}]=-2\delta_{ij}f_{\alpha f\beta}\mu_\beta+2\varepsilon_{ijk}d_{\alpha fg}\mu_{gk}. \tag{17}$$

i.e., as the transformations of all the nonexceptional groups. The second and third terms in eq. (15.b′) form an infinitesimal transformation of the subgroup G_2 of the group F_4. One can say that this transformation is generated by the missing matrices $\varkappa_{oi}=\lambda_o e_i$, however, its unusual form is due to the nonassociativity of the octonions.

4. Eigenstates and eigenvalues. In the algebra $\mathfrak{M}_3^{8+}$ there exist the triples of the pairwise orthogonal idempotents E_i [6]

$$E_1\cdot E_1=E_1\,,\quad E_2\cdot E_2=E_2\,,\quad E_3\cdot E_3=E_3\,,\quad \mathrm{Tr}\,E_i=1,$$
$$E_1\cdot E_2=0,\quad E_2\cdot E_3=0,\quad E_3\cdot E_1=0. \tag{18}$$

with the completeness property

$$E_1+E_2+E_3=\mathbb{1}\equiv\mu_o\,. \tag{19}$$

The simplest example is

$$E_1^o=\begin{pmatrix}e_o&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad E_2^o=\begin{pmatrix}0&0&0\\0&e_o&0\\0&0&0\end{pmatrix},\qquad E_3^o=\begin{pmatrix}0&0&0\\0&0&0\\0&0&e_o\end{pmatrix}. \tag{20}$$

Any other triple of idempotents E_1, E_2, E_3 differs from E_1^o, E_2^o, E_3^o by some automorphism transformation (which we denote by the prime)

$$E_1'=E_1^o\,,\quad E_2'=E_2^o\,,\quad E_3'=E_3^o \tag{21}$$

since any matrix of $\mathfrak{M}_3^{8+}$ can be diagonalized by a transformation of F_4 [5,18] (see Appendix D).

For any observable J there exists a decomposition into the corresponding triple of pairwise orthogonal idempotents [6]

$$J=\lambda_1E_1^{(J)}+\lambda_2E_2^{(J)}+\lambda_3E_3^{(J)}\,, \tag{22}$$

the idempotents $E_i^{(J)}$ being the eigenstates of J and λ_i the eigenvalues

$$J\cdot E_i^{(J)}=\lambda_iE_i^{(J)}\,. \tag{23}$$

If E is an irreducible idempotent, $E\cdot E=E$, $\mathrm{Tr}E=1$, then

$$E\times E=0\,,\qquad \det E=0 \tag{24}$$

(det is defined by eq. (13)) and vice versa. Any irreducible idempotent E can be represented as follows (P.Jordan[3], see also[18])

$$E=\begin{pmatrix}a\\b\\c\end{pmatrix}(\bar a\;\bar b\;\bar c)=\begin{pmatrix}a\bar a&a\bar b&a\bar c\\b\bar a&b\bar b&b\bar c\\c\bar a&c\bar b&c\bar c\end{pmatrix},\quad \bar aa+\bar bb+\bar cc=1, \tag{25}$$

where one of the octonions a, b, c is a pure real number.

5. Trace of matrix of $\mathfrak{M}_3^{8+}$ is defined as the sum of its diagonal elements (they are real). In particular, for the basis matrices we have

$$\mathrm{Tr}\,\mu_o=3\,,\quad \mathrm{Tr}\,(\mu_\alpha\cdot\mu_\beta)=2\delta_{\alpha\beta}\,,\quad \mathrm{Tr}(\mu_{fi}\cdot\mu_{gj})=2\delta_{fg}\delta_{ij} \tag{26}$$

and traces of μ_α, μ_{fi} and $\mu_\alpha\cdot\mu_{fi}$ are zero. The trace of a product of matrices of $\mathfrak{M}_3^{8+}$ has the properties [5]

$$\mathrm{Tr}\,(X\cdot Y)=0\qquad \forall Y\Rightarrow X=0\ ^{x)} \tag{27}$$
$$\mathrm{Tr}\,(X\cdot Y)=\mathrm{Tr}(Y\cdot X) \tag{28}$$
$$\mathrm{Tr}\,(X\cdot(Y\cdot Z))=\mathrm{Tr}((X\cdot Y)\cdot Z)\equiv\mathrm{Tr}(X\cdot Y\cdot Z)\ \text{ is symmetric in } X,Y,Z, \tag{29}$$
$$\mathrm{Tr}\,(X,Y,Z)=0. \tag{30}$$

The last two statements are equivalent. Equation (30) means that the trace is invariant under the transformations F_4

$$\mathrm{Tr}\,(\delta Y)=0,\qquad \mathrm{Tr}\,Y'=\mathrm{Tr}\,Y. \tag{31}$$

This is valid for any number of factors

$$\mathrm{Tr}(Y_1'\cdot Y_2')=\mathrm{Tr}(Y_1\cdot Y_2),\quad \mathrm{Tr}(Y_1'\cdot Y_2'\cdot Y_3')=\mathrm{Tr}(Y_1\cdot Y_2\cdot Y_3),\dots \tag{32}$$

There is the inverse theorem: if some transformation leaves both the quadratic and cubic forms (i.e., traces $\mathrm{Tr}(X\cdot Y)$ and $\mathrm{Tr}(X\cdot(Y\cdot Z))$) invariant, then it is automorphism and belongs to F_4 [5].

6. Transition probabilities. Let us specify initial and final states by some density "matrices" ρ_i and ρ_f which belongs to $\mathfrak{M}_3^{8+}$. As ρ_i and ρ_f we take idempotents which are eigenstates of some observables (pure states) in the sense of eqs. (23). Define

$$w_{fi}=\mathrm{Tr}\,(\rho_f\cdot\rho_i) \tag{33}$$

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x) To prove this insert into $\mathrm{Tr}\,(X\cdot Y)$

$$X=x_o\mu_o+x_\alpha\mu_\alpha+x_{fi}\mu_{fi}\,,\qquad Y=y_o\mu_o+y_\alpha\mu_\alpha+y_{fi}\mu_{fi}\,.$$

Then $\mathrm{Tr}\,(X\cdot Y)=3\,x_oy_o+2\,(x_\alpha y_\alpha+x_{fi}y_{fi})$

whence immediately there follows eq. (27).

as the probability to find the state f in the state i . Let us verify that this interpretation is consistent. Summation over three pairwise orthogonal final idempotents gives

$$\sum_f \mathrm{Tr}(\rho_f \cdot \rho_i) = \mathrm{Tr}(\mathbb{1}\cdot \rho_i) = \mathrm{Tr}(\rho_i) = 1, \tag{34}$$

where we have used eq. (19) [x)] and the normalization condition of ρ_i . Thus, the sum of the probabilities equals unity. Let us demonstrate also the positivity of the probabilities

$$0 \leq \mathrm{Tr}(\rho_f \cdot \rho_i) \leq 1 . \tag{35}$$

If we reduce triple of ρ_f to the triple E_f° (20) (using a suitable F_4 transformation) we obtain

$$0 \leq \mathrm{Tr}(E_f^\circ \rho_i') \leq 1 \qquad (\mathrm{Tr}(\rho_f \cdot \rho_i) = \mathrm{Tr}(E_f^\circ \cdot \rho_i')), \tag{36}$$

i.e.,

$$0 \leq (\rho_i')_{11} \leq 1 , \quad 0 \leq (\rho_i')_{22} \leq 1 , \quad 0 \leq (\rho_i')_{33} \leq 1 . \tag{37}$$

The positivity of these diagonal elements is clear, since the diagonal elements of any idempotent are positive

$$(E)_{aa} = (E \cdot E)_{aa} > 0 , \quad ((E^2)_{aa} = E_{ab} E_{ba} = E_{ab} \bar{E}_{ab}) \tag{38}$$

with no summation over a . Thus, property (35) is proved.

7. Equations of motion. Usually an evolution law in quantum mechanics is a transformation of a one-parameter subgroup of an automorphism group of algebra of observables. In accord with this rule we assume equations of motion for the density "matrix" $\rho(t)$ in the Schrödinger picture and for any observable $J(t)$, that does not depend explicitly on time, in the Heisenberg picture to be

$$\frac{d}{dt}\rho(t) = -(\!(A, \rho(t), B)\!) , \qquad \frac{d}{dt} J = 0 , \tag{39}$$

$$\frac{d}{dt} J(t) = (\!(A, J(t), B)\!) , \qquad \frac{d}{dt}\rho = 0 , \tag{40}$$

where $A, B \in \mathfrak{M}_3^{8+}$. Formal solutions of these equations can be written

x) It is possible, using a suitable transformation of the automorphism group, to reduce ρ_f to E_f° (eq. (20)): $\rho_f' = E_f^\circ$, to obtain

$$\sum_{f=1,2,3} \mathrm{Tr}(\rho_f \cdot \rho_i) = \sum_{f=1,2,3} \mathrm{Tr}(E_f^\circ \cdot \rho_i') = \mathrm{Tr}\,\rho_i' = \mathrm{Tr}\,\rho_i = 1 .$$

$$\rho(t) = e^{-t\,ad}\rho(0), \tag{41}$$

$$J(t) = e^{t\,ad} J(0), \tag{42}$$

where

$$ad\,\rho(0) \equiv (\!(A, \rho(0), B)\!) = -\tfrac{1}{4}[[AB]\rho(0)] + \tfrac{1}{4}\sum \pm (A, \rho(0), B) \tag{43}$$

(the sum in the last term is defined more accurately in Appendix A). Two operators A and B play here the same role that is usually played by Hamiltonian. For the law of evolution defined above the usual property

$$\mathrm{Tr}(J_1(0)\cdot \ldots \cdot J_n(0) \cdot \rho(t)) = \mathrm{Tr}(J_1(t) \cdot \ldots \cdot J_n(t) \cdot \rho(0)) \tag{44}$$

is valid with any arrangement of brackets that fix an order of multiplication.

The condition of conservation in time is given by

$$(\!(A, J(t), B)\!) = 0 \tag{45}$$

The role of equations (39) and (40) are the same as that of equations Neumann (Liouville) for the density matrix in the Schrödinger picture and Heisenberg-Born-Jordan-Dirac equations for observables, which do not depend explicitly on time, in the Heisenberg picture.

8. Matrix representations for elements of $\mathfrak{M}_3^{8+}$. One type of representatives of any operator (observable, density matrix)

$$F = a_o \mu_o + a_a \mu_a + a_{fi}\mu_{fi} \quad (a = 1,3,4,6,8 ; f = 2,5,7) \tag{46}$$

is set of traces arranged in 27-component column

$$\mathrm{Tr}(\mu \cdot F) = \begin{pmatrix} \mathrm{Tr}\,F \\ \mathrm{Tr}(\mu_a \cdot F) \\ \mathrm{Tr}(\mu_{fi} \cdot F) \end{pmatrix} \tag{47}$$

$\mathrm{Tr}(\mu \cdot F)$ is a short notation of this column. The reconstruction theorem is

$$F = \tfrac{1}{3}\mu_o \mathrm{Tr}\,F + \tfrac{1}{2}\mu_a \mathrm{Tr}(\mu_a \cdot F) + \tfrac{1}{2}\mu_{fi}\mathrm{Tr}(\mu_{fi} \cdot F). \tag{48}$$

where the number coefficients are clear from eq. (26).

Other types of representatives, the left and right representatives, are defined as follows. At first we find the representatives of basis elements (4) of algebra $\mathfrak{M}_3^{8+}$ [x)]

x) One can obtain symmetric matrices taking $\mu_o, \sqrt{\frac{3}{2}}\mu_a, \sqrt{\frac{3}{2}}\mu_{fi}$ instead of μ_o, μ_a, μ_{fi} .

$$\begin{bmatrix} \mathrm{Tr}(\mu_b\cdot F) \\ \mathrm{Tr}(\mu_a\cdot\mu_b\cdot F) \\ \mathrm{Tr}(\mu_{fi}\cdot\mu_b\cdot F) \end{bmatrix} = \left[\begin{array}{c|c|c} 0 & \delta_{bc} & 0 \\ \hline \frac{2}{3}\delta_{ab} & d_{abc} & 0 \\ \hline 0 & 0 & \delta_{ik} d_{fbg} \end{array}\right] \begin{bmatrix} \mathrm{Tr}\,F \\ \mathrm{Tr}(\mu_c\cdot F) \\ \mathrm{Tr}(\mu_{gk}\cdot F) \end{bmatrix} = \mu_b^{\ell} \begin{bmatrix} \mathrm{Tr}\,F \\ \mathrm{Tr}(\mu_a\cdot F) \\ \mathrm{Tr}(\mu_{fi}\cdot F) \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Tr}(F\cdot\mu_b) \\ \mathrm{Tr}(\mu_a\cdot F\cdot\mu_b) \\ \mathrm{Tr}(\mu_{fi}\cdot F\cdot\mu_b) \end{bmatrix} = \mu_b^{r} \begin{bmatrix} \mathrm{Tr}\,F \\ \mathrm{Tr}(\mu_a\cdot F) \\ \mathrm{Tr}(\mu_{fi}\cdot F) \end{bmatrix} \tag{49}$$

$$\begin{bmatrix} \mathrm{Tr}(\mu_{gj}\cdot F) \\ \mathrm{Tr}(\mu_a\cdot\mu_{gj}\cdot F) \\ \mathrm{Tr}(\mu_{fi}\cdot\mu_{gj}\cdot F) \end{bmatrix} = \left[\begin{array}{c|c|c} 0 & 0 & \delta_{jk}\delta_{gh} \\ \hline 0 & 0 & \delta_{jk} d_{agh} \\ \hline \frac{2}{3}\delta_{ij}\delta_{fg} & \delta_{ij} d_{fgc} & \varepsilon_{ijk} f_{fgh} \end{array}\right] \begin{bmatrix} \mathrm{Tr}\,F \\ \mathrm{Tr}(\mu_c\cdot F) \\ \mathrm{Tr}(\mu_{hk}\cdot F) \end{bmatrix}$$

$$= \mu_{gj}^{\ell} \begin{bmatrix} \mathrm{Tr}\,F \\ \mathrm{Tr}(\mu_a\cdot F) \\ \mathrm{Tr}(\mu_{fi}\cdot F) \end{bmatrix} = \begin{bmatrix} \mathrm{Tr}(F\cdot\mu_{gj}) \\ \mathrm{Tr}(\mu_a\cdot F\cdot\mu_{gj}) \\ \mathrm{Tr}(\mu_{fi}\cdot F\cdot\mu_{gj}) \end{bmatrix} = \mu_{gj}^{r} \begin{bmatrix} \mathrm{Tr}\,F \\ \mathrm{Tr}(\mu_a\cdot F) \\ \mathrm{Tr}(\mu_{fi}\cdot F) \end{bmatrix} . \tag{50}$$

where F is an arbitrary element of $\mathfrak{M}_3^{8+}$. Due to commutativity of algebra $\mathfrak{M}_3^{8+}$ the right representatives are equal to the left ones[x)]

$$\mu_0^{r} = \mu_0^{\ell} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad \mu_a^{r} = \mu_a^{\ell}, \quad \mu_{fi}^{r} = \mu_{fi}^{\ell} . \tag{51}$$

These matrix representatives are 27x27 matrices (μ_0^{ℓ} is the unit 27x27 matrix). Dimensions of blocks are the following

$$\left[\begin{array}{c|c|c} 1\times 1 & 1\times 5 & 1\times 21 \\ \hline 5\times 1 & 5\times 5 & 5\times 21 \\ \hline 21\times 1 & 21\times 5 & 21\times 21 \end{array}\right] \tag{52}$$

Traces of the representatives are

$$\mathrm{Tr}\,\mu_0^{\ell} = 27, \ \mathrm{Tr}\,\mu_a^{\ell} = \mathrm{Tr}\,\mu_{fi}^{\ell} = 0, \ \mathrm{Tr}(\mu_a^{\ell}\mu_b^{\ell}) = 6\,\delta_{ab}, \ \mathrm{Tr}(\mu_{fi}^{\ell}\mu_{gj}^{\ell}) = 6\,\delta_{fg}\delta_{ij}. \tag{53}$$

It is clear that any 3x3 matrix (over octonions) of $\mathfrak{M}_3^{8+}$ can be converted into a usual real 27x27 matrix

$$\mathrm{Tr}(\mu\cdot X\cdot F) = \mathrm{Tr}(\mu\cdot F\cdot X) = X^{\ell}\,\mathrm{Tr}(\mu\cdot F), \tag{54}$$

x) Note that our notation of the left and right representatives (say, x^{ℓ}, x^{r}, e_1^{ℓ}, μ_{fi}^{ℓ}) arose[/22/] following the spirit the Dirac representation theory. It seems to be more indicative than notation used in algebra (L_x , R_x , L_{e_1} , $L_{\mu_{fi}}$) and called the left and right multiplications (see, e.g., ref./9/).

with

$$X = x_0\mu_0 + x_a\mu_a + x_{fi}\mu_{fi} \tag{55}$$

$$X^{\ell} = X^{r} = x_0\mu_0 + x_a\mu_a^{\ell} + x_{fi}\mu_{fi}^{\ell} . \tag{56}$$

If we know X^{ℓ} we can easily reconstruct X as follows:

$$X = \frac{1}{27}\mu_0\,\mathrm{Tr}(\mu_0^{\ell}X^{\ell}) + \frac{1}{6}\mu_a\,\mathrm{Tr}(\mu_a^{\ell}X^{\ell}) + \frac{1}{6}\mu_{fi}\,\mathrm{Tr}(\mu_{fi}^{\ell}X^{\ell}) . \tag{57}$$

Some of more complicated products can be easily expressed in terms of the above representatives

$$\mathrm{Tr}(\mu\cdot(X\cdot(Y\cdot F))) = X^{\ell}Y^{\ell}\,\mathrm{Tr}(\mu\cdot F), \tag{58}$$

$$\mathrm{Tr}(\mu\cdot(\!(X,F,Y)\!)) = -[X^{\ell},Y^{\ell}]\,\mathrm{Tr}(\mu\cdot F), \tag{59}$$

$$\mathrm{Tr}(\mu\cdot(\!(X,Y,Z)\!)\cdot F) = -[[X^{\ell}Z^{\ell}]Y^{\ell}]\,\mathrm{Tr}(\mu\cdot F) . \tag{60}$$

i.e.,

$$(\!(X,Y,Z)\!)^{\ell} = -[[X^{\ell}Z^{\ell}]Y^{\ell}] . \tag{60.a}$$

When calculating eq. (60) the property

$$(\!(X,Y,Z)\!)\cdot F = (\!(X,Y\cdot F,Z)\!) - Y\cdot(\!(X,F,Z)\!) \tag{61}$$

is used.

However we cannot find a representative $(X\cdot Y)^{\ell}$ for $X\cdot Y$

$$T(\mu\cdot(X\cdot Y)\cdot F) = (X\cdot Y)^{\ell}\,\mathrm{Tr}(\mu\cdot F) \tag{62}$$

in terms of X^{ℓ} and Y^{ℓ} only. Let us use now the decomposition

$$(\!(\mu_A,\mu_B,\mu_C)\!) = \mathcal{A}_{ABCD}\,\mu_D \qquad A,B,C,D = 0,a,fi \tag{63}$$

where $\mathcal{A}_{ABCD}$ are real coefficients. They vanish, if any subscript of A, B, C or D equals zero. Then we have

$$\mathrm{Tr}(\mu_A\cdot(\!(X,Y,F)\!)) = \mathrm{Tr}(\mu_A\cdot(X\cdot Y)\cdot F - \mu_A\cdot X\cdot(Y\cdot F)) =$$

$$= -\mathrm{Tr}((\!(\mu_A,X,Y)\!)\cdot F) = -X_B Y_C\,\mathcal{A}_{ABCD}\,\mathrm{Tr}(\mu_D\cdot F), \tag{64}$$

$$(X\cdot Y)^{\ell}_{AD}\,\mathrm{Tr}(\mu_D\cdot F) = \mathrm{Tr}(\mu_A\cdot(X\cdot Y)\cdot F) =$$

$$= \mathrm{Tr}(\mu_A\cdot((\!(X,Y,F)\!) + X\cdot(Y\cdot F))) = ((X^{\ell}Y^{\ell})_{AD} - X_B Y_C\,\mathcal{A}_{ABCD})\,\mathrm{Tr}(\mu_D\cdot F) \tag{65}$$

and we find the desired representative of the product $X\cdot Y$:

$$(X\cdot Y)^{\ell}_{AD} = (X^{\ell}Y^{\ell})_{AD} - X_B Y_C\,\mathcal{A}_{ABCD} \tag{65.a}$$

It includes in addition to matrices X^1 and Y^1 (constructed of μ_B^{ℓ}) the two-index 27x27 matrices

$$\mathcal{A}_{BC} = \| \mathcal{A}_{ABCD} \| \tag{66}$$

with zeros in 0-row and 0-column. Thus, the desired matrix representation was obtained.

Another way to construct a matrix representation is to start with 72 matrices (2) (see Appendix F).

All the relations can be translated into terms of the representatives obtained. For examples, for the column-representatives

$$\tilde{\rho}(t) = \mathrm{Tr}\left(\mu \cdot \rho(t)\right), \qquad \tilde{J}(t) = \mathrm{Tr}\left(\mu \cdot J(t)\right) \tag{67}$$

the equations of motion (39) and (40) and their solutions take the form

$$\frac{d}{dt}\tilde{\rho}(t) = [A^{\ell}B^{\ell}]\,\tilde{\rho}(t), \qquad \tilde{\rho}(t) = e^{t[A^{\ell}B^{\ell}]}\,\tilde{\rho}(0), \tag{68}$$

$$\frac{d}{dt}\tilde{J}(t) = -[A^{\ell}B^{\ell}]\,\tilde{J}(t), \qquad \tilde{J}(t) = e^{-t[A^{\ell}B^{\ell}]}\,\tilde{J}(0), \tag{69}$$

and for the matrix representatives the form x)

$$\frac{d}{dt}\rho^{\ell}(t) = [[A^{\ell}B^{\ell}]\rho^{\ell}(t)] \equiv \left(\mathrm{ad}\,[A^{\ell}B^{\ell}]\right)\rho^{\ell}(t), \tag{70}$$

$$\frac{d}{dt}J^{\ell}(t) = -[[A^{\ell}B^{\ell}]J^{\ell}(t)] \equiv -\left(\mathrm{ad}\,[A^{\ell}B^{\ell}]\right)J^{\ell}(t), \tag{71}$$

$$\rho^{\ell}(t) = e^{t\,\mathrm{ad}\,[A^{\ell}B^{\ell}]}\,\rho^{\ell}(0) = e^{t[A^{\ell}B^{\ell}]}\,\rho^{\ell}(0)\,e^{-t[A^{\ell}B^{\ell}]}, \tag{72}$$

$$J^{\ell}(t) = e^{-t\,\mathrm{ad}\,[A^{\ell}B^{\ell}]}\,J^{\ell}(0) = e^{-t[A^{\ell}B^{\ell}]}\,J^{\ell}(0)\,e^{t[A^{\ell}B^{\ell}]}. \tag{73}$$

A role of the object $\Psi(t) = e^{t[A^{\ell}B^{\ell}]}\,\Psi(0)$ is problematic.

<u>Conservation of the probabilities</u>. From eqs. (39),(68) and (70) there follow the conservation laws

$$\mathrm{Tr}\,\dot{\rho}(t) = 0, \qquad \mathrm{Tr}\,\rho(t) = \mathrm{Tr}\,\rho(0), \tag{74}$$

$$\dot{\tilde{\rho}}_0(t) = 0 \quad \text{(zero component)}, \qquad \tilde{\rho}_0(t) = \tilde{\rho}_0(0), \tag{75}$$

$$\mathrm{Tr}\,\dot{\rho}^{\ell}(t) = 0, \qquad \mathrm{Tr}\,\rho^{\ell}(t) = \mathrm{Tr}\,\rho^{\ell}(0). \tag{76}$$

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x) Note that in the octonionic quantum mechanics (see refs.[22,23]) the equation of motion in the Heisenberg picture

$$\frac{d}{dt}J(t) = [[\alpha\beta]J(t)] - 3(\alpha, \beta, J(t))$$

converted into the matrix representation takes the similar form

$$\frac{d}{dt}\overset{\ell}{J}{}^{r}(t) = [[\alpha^{\ell}\beta^{\ell}] + [\alpha^{r}\beta^{r}] + [\alpha^{\ell}\beta^{r}],\ \overset{\ell}{J}{}^{r}(t)]$$

with the formal solution

$$\overset{\ell}{J}{}^{r}(t) = e^{t\,\mathrm{ad}\,([\alpha^{\ell}\beta^{\ell}] + [\alpha^{r}\beta^{r}] + [\alpha^{\ell}\beta^{r}])}\,\overset{\ell}{J}{}^{r}(0) = e^{t([\alpha^{\ell}\beta^{\ell}] + \dots)}\,\overset{\ell}{J}{}^{r}(0)\,e^{-t([\alpha^{\ell}\beta^{\ell}] + \dots)}.$$

<u>Appendix A. Properties of the Jordan associator</u>. The Jordan associator is defined by

$$(\!(X,Y,Z)\!) \overset{def}{=} (X \cdot Y) \cdot Z - X \cdot (Y \cdot Z) = \tag{A.1}$$

$$= -\tfrac{1}{4}[[XZ]Y] + \tfrac{1}{4}\Big((X,Y,Z) - (Z,X,Y) - (Y,Z,X) - (Z,Y,X) + (Y,X,Z) + (X,Z,Y)\Big) \tag{A.2}$$

($(X,Y,Z) = (XY)Z - X(YZ)$ are usual associators). It is subjected to the following identities.

a) Identities that follow only from definition (A.1) (and are true for any quantities)

$$(\!(X,Y,Z)\!) + (\!(Z,Y,X)\!) = 0 \tag{A.3}$$

$$(\!(X,Y,Z)\!) + (\!(Z,X,Y)\!) + (\!(Y,Z,X)\!) = 0 \tag{A.4}$$

$$(\!(X,Y,Z\cdot W)\!) + (\!(W,Y,X\cdot Z)\!) - (\!(X,Z,Y\cdot W)\!) - (\!(W,Z,X\cdot Y)\!) = (\!(Y,X,Z)\!)\cdot W + X\cdot(\!(Y,W,Z)\!). \tag{A.5}$$

Let us note that (A.4) is equivalent to the identity

$$[[XY]Z] + [[ZX]Y] + [[YZ]X] =$$

$$= (X,Y,Z) + (Z,X,Y) + (Y,Z,X) - (Z,Y,X) - (Y,X,Z) - (X,Z,Y) \tag{A.6}$$

(here the signs correspond to parities of permutations).

b) Identities that are valid only for the Jordan algebra ($X,Y,Z,W \in \mathcal{M}_3^{8+}$)

$$(\!(X,Y,X^2)\!) = 0, \tag{A.7}$$

$$2(\!(X,Y,Z\cdot X)\!) + (\!(Z,Y,X^2)\!) = 0, \tag{A.8}$$

$$(\!(X,Y,Z\cdot W)\!) + (\!(W,Y,X\cdot Z)\!) + (\!(Z,Y,W\cdot X)\!) = 0, \tag{A.9}$$

$$(\!(Y,W\cdot X,Z)\!) = (\!(Y,W,Z)\!)\cdot X + W\cdot(\!(Y,X,Z)\!). \tag{A.10}$$

We can easily prove identity (A.7) by referring to the theorem, that any matrix of $\mathcal{M}_3^{8+}$ can be diagonalized by suitable F_4 transformation. When X is diagonalized, it is real and we have associative situation. If (A.7) is written in the form (A.2), all the terms turn to zero x). Original proof of eq. (A.7) by Albert [2] is more direct. Identities (A.8) and (A.9) follow from eq. (A.7), using the process of polarization ($X \to X + \lambda Z$). Identity (A.10) follows from eq. (A.5) if we use there eq. (A.9) and then eq. (A.4). It means, that $(\!(X,Y,Z)\!)$ is infinitesimal automorphism transformation of Y.

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x) Instead of this argument we can directly perform multiplication in transformed $(\!(X,Y,X^2)\!)$ since X is now a real diagonal matrix.

Due to the property (A.10) the operation

$$D_{B,A}\mathcal{J} = (A,\mathcal{J},B) \qquad (A.11)$$

is named in algebra the derivation. One can be convinced that $D_{B,A}$ form a Lie algebra

$$[D_{D,C}, D_{B,A}]\mathcal{J} = D_{D,C}(A,\mathcal{J},B) - (A, D_{D,C}\mathcal{J}, B) =$$

$$= (D_{D,C}A,\mathcal{J},B) + \cancel{(A,D_{D,C}\mathcal{J},B)} + (A,\mathcal{J},D_{D,C}B) - \cancel{(A,D_{D,C}\mathcal{J},B)} =$$

$$= (D_{B,A'} + D_{B',A})\mathcal{J} \qquad (A' = D_{D,C}A\ ,\ B' = D_{D,C}B), \qquad (A.12)$$

$$[D_{D,C}, D_{B,A}] = D_{B,A'} + D_{B',A}\ . \qquad (A.13)$$

It is the F_4 algebra (52 parameters).

Appendix B. Infinitesimal automorphisms of the algebra $\mathfrak{M}_3^{8+}$.

According to Appendix A one form of the automorphism (F_4) transformations of $\mathfrak{M}_3^{8+}$ is

$$1)\ \delta\mathcal{J} = (A,\mathcal{J},B)\ . \qquad (B.1)$$

Other (particular) forms of these automorphism (F_4) transformations are /5/

$$2)\ \delta\mathcal{J} = [C,\mathcal{J}], \qquad C\in\mathfrak{M}_3^{8-}, \qquad \mathrm{Tr}\, C = 0, \qquad (B.2)$$

$$3)\ \delta\mathcal{J} = [[CD] - \tfrac{1}{3}\mathrm{Tr}[CD], \mathcal{J}], \quad C,D\in\mathfrak{M}_3^{8-}, \quad \mathrm{Tr}\,C = \mathrm{Tr}\,D = 0, \qquad (B.3)$$

$$4)\ \delta\mathcal{J} = [[\mathcal{J}C]D] + [[D\mathcal{J}]C], \quad C,D\in\mathfrak{M}_3^{8-}, \quad \mathrm{Tr}\,C = \mathrm{Tr}\,D = 0, \qquad (B.4)$$

$$5)\ \delta\mathcal{J} = [[CD] - \tfrac{1}{3}\mathrm{Tr}[CD], \mathcal{J}] + [[\mathcal{J}C]D] + [[D\mathcal{J}]C] = \qquad (B.5a)$$

$$C,D\in\mathfrak{M}_3^{8-}, \quad \mathrm{Tr}\,C = \mathrm{Tr}\,D = 0$$

$$= \sum_{\text{antisymm.}}(C,D,\mathcal{J}) - \tfrac{1}{3}[\mathrm{Tr}[CD],\mathcal{J}] = \qquad (B.5b)$$

$$= \tfrac{2}{3} c_{ai} d_{bj}\delta_{ab}\left(-[[e_i e_j]\mathcal{J}] + 3(e_i,e_j,\mathcal{J})\right) = \qquad (B.5c)$$

(after substitution $C = c_f\varkappa_f + c_{ai}\varkappa_{ai}$, $D = d_g\varkappa_g + d_{bj}\varkappa_{bj}$)

$$= -[[ab]\mathcal{J}] + 3(a,b,\mathcal{J})\ . \qquad (B.5d)$$

Make some comments:

to 2) Check the automorphism property of transformation (B.2)

$$[C, \mathcal{J}_1\cdot\mathcal{J}_2] = [C\mathcal{J}_1]\cdot\mathcal{J}_2 + \mathcal{J}_1\cdot[C\mathcal{J}_2] \qquad (B.6)$$

In fact (see Appendix E, eq. (E.4))

$$[C, \mathcal{J}_1\cdot\mathcal{J}_2] = [C\mathcal{J}_1]\cdot\mathcal{J}_2 + \mathcal{J}_1\cdot[C\mathcal{J}_2] -$$

$$-\tfrac{1}{2}\left((C,\mathcal{J}_1,\mathcal{J}_2) + (C,\mathcal{J}_2,\mathcal{J}_1) - (\mathcal{J}_1,C,\mathcal{J}_2) - (\mathcal{J}_2,C,\mathcal{J}_1) + (\mathcal{J}_1,\mathcal{J}_2,C) + (\mathcal{J}_2,\mathcal{J}_1,C)\right) \qquad (B.7)$$

Insert $C = c_f\varkappa_f + c_{ai}\varkappa_{ai}$, $\mathcal{J} = j_0\mu_0 + j_a\mu_a + j_{fi}\mu_{fi}$.

In the most of the cases these associators equal zero separately due to associativity. The worst situation $C\sim\varkappa_{ai}$, $\mathcal{J}_1\sim\mu_{fj}$, $\mathcal{J}_2\sim\mu_{gk}$:

$$(\lambda_a e_i, \varkappa_f e_j, \varkappa_g e_k) + (\lambda_a e_i, \varkappa_g e_k, \varkappa_f e_j) - (\varkappa_f e_j, \lambda_a e_i, \varkappa_g e_k) -$$

$$-(\varkappa_g e_k, \lambda_a e_i, \varkappa_f e_j) + (\varkappa_f e_j, \varkappa_g e_k, \lambda_a e_i) + (\varkappa_g e_k, \varkappa_f e_j, \lambda_a e_i) =$$

$$= (\lambda_a\varkappa_f\varkappa_g - \lambda_a\varkappa_g\varkappa_f + \varkappa_f\lambda_a\varkappa_g - \varkappa_g\lambda_a\varkappa_f + \varkappa_f\varkappa_g\lambda_a - \varkappa_g\varkappa_f\lambda_a)\cdot$$

$$\cdot(e_i,e_j,e_k) = 0 \qquad (B.8)$$

due to the relation (specific to the Gell-Mann matrices)

$$2(\lambda_a\varkappa_f\varkappa_g - \lambda_a\varkappa_g\varkappa_f + \varkappa_f\lambda_a\varkappa_g - \varkappa_g\lambda_a\varkappa_f + \varkappa_f\varkappa_g\lambda_a - \varkappa_g\varkappa_f\lambda_a) =$$

$$= [\varkappa_f\{\varkappa_g\lambda_a\}] - [\varkappa_g\{\varkappa_f\lambda_a\}] + \{\lambda_a[\varkappa_f\varkappa_g]\} = 0\ . \qquad (B.9)$$

to 3) $[CD]\in\mathfrak{M}_3^{8-}$, but in general $\mathrm{Tr}\,[CD] \neq 0$ and we extract the trace. Then according to eq. (B.2) we have the automorphism property

$$[[CD] - \tfrac{1}{3}\mathrm{Tr}[CD], \mathcal{J}_1\cdot\mathcal{J}_2] = [[CD] - \tfrac{1}{3}\mathrm{Tr}[CD], \mathcal{J}_1]\cdot\mathcal{J}_2 +$$

$$+ \mathcal{J}_1\cdot[[CD] - \tfrac{1}{3}\mathrm{Tr}[CD], \mathcal{J}_2]\ . \qquad (B.10)$$

to 4) The commutators $[[\mathcal{J}C]D]$ and $[[D\mathcal{J}]C]$ are not automorphisms separately

$$[[\mathcal{J}_1\cdot\mathcal{J}_2, C]D] = [[\mathcal{J}_1 C]\cdot\mathcal{J}_2 + \mathcal{J}_1\cdot[\mathcal{J}_2 C], D] =$$

$$= [[\mathcal{J}_1 C]D]\cdot\mathcal{J}_2 + [\mathcal{J}_1 C]\cdot[\mathcal{J}_2 D] + [\mathcal{J}_1 D]\cdot[\mathcal{J}_2 C] + \mathcal{J}_1\cdot[[\mathcal{J}_2 C]D], \qquad (B.11)$$

$$[[D, \mathcal{J}_1\cdot\mathcal{J}_2]C] = [[D\mathcal{J}_1]\cdot\mathcal{J}_2 + \mathcal{J}_1\cdot[D\mathcal{J}_2], C] =$$

$$= [[D\mathcal{J}_1]C]\cdot\mathcal{J}_2 + [D\mathcal{J}_1]\cdot[\mathcal{J}_2 C] + [\mathcal{J}_1 C]\cdot[D\mathcal{J}_2] + \mathcal{J}_1\cdot[[D\mathcal{J}_2]C]$$

but together they are. (B.12)

to 5) To pass from eq. (B.5a) to (B.5b) the general identity (E.1) of Appendix E is used. Then we insert

$$C = c_f\varkappa_f + c_{ai}\varkappa_{ai}, \quad D = d_g\varkappa_g + d_{bj}\varkappa_{bj}, \quad \mathcal{J} = j_0\mu_0 + j_a\mu_a + j_{hk}\mu_{hk} \qquad (B.13)$$

and obtain

$$\sum_{\text{antisymm.}}(C,D,\mathcal{J}) = c_{ai}d_{bj}j_{hk}\sum(\varkappa_{ai},\varkappa_{bj},\mu_{hk}) =$$

$$= c_{ai}d_{bj}j_{hk}\,2\delta_{ab}(e_i,e_j,\mu_{hk}) = 2c_{ai}d_{bj}\delta_{ab}(e_i,e_j,\mathcal{J}) \qquad (B.14)$$

with the use of the relations

$$\sum_{\text{antisymm.}} (\varkappa_{ai}, \varkappa_{bj}, \mu_{hk}) = \sum (\lambda_a e_i, \lambda_b e_j, \varkappa_h e_k) =$$

$$= \{\lambda_a \lambda_b \varkappa_h\}(e_i, e_j, e_k) \tag{B.15}$$

and

$$\{\lambda_a \lambda_b \varkappa_h\} = 2\delta_{ab}\varkappa_h . \tag{B.16}$$

The trace is reduced to

$$Tr[CD] = 2 c_{ai} d_{bj} \delta_{ab} [e_i e_j]. \tag{B.17}$$

The automorphism transformation (B.5a) is the sum of transformations (B.3) and (B.4). It leaves unchanged any real matrix J $(\delta J = 0)$ since in that case we have the associative situation and the Jacobi identity

$$[[CD]J] + [[JC]D] + [[DJ]C] = 0 \tag{B.18}$$

is valid;

$$[Tr[CD], J] = 0 \qquad Tr[CD] = \mu_0 a \tag{B.19}$$

where a is an imaginary octonion, $J = c_0\mu_0 + c_a\mu_a$.

Appendix C. Properties of the scalar product. At first in the algebra $\mathcal{M}_3^8$ of all the 3x3 matrices with octonionic entries we introduce the trace [5]

$$Tr\, X = \sum_i x_{ii} , \qquad Tr(XY) = \sum_{ij} x_{ij} y_{ji} \tag{C.1}$$

and the scalar product

$$(X,Y)e_0 = Re\, Tr(XY) = Re \sum_{ij} x_{ij} y_{ji} = \sum Re(x_{ij}y_{ji}) = \sum Re(y_{ji}x_{ij})$$

$$= Re\, Tr(YX) = (Y,X)e_0 . \tag{C.2}$$

Here x_{ij} and y_{ij} are octonions, matrix elements of the 3x3 matrices X and Y. Re refers to the octonions. Using the fact that the associator of octonions is always a pure imaginary octonion, i.e.,

$$Re\,(a,b,c) = 0 \tag{C.3}$$

we obtain

$$(XY,Z)e_0 = Re\, Tr((XY)Z) =$$

$$= Re \sum (x_{ij}y_{jk})z_{ki} = \sum Re((x_{ij}y_{jk})z_{ki}) = \tag{C.4a}$$

$$= \sum Re[x_{ij}(y_{jk}z_{ki}) + (x_{ij}, y_{jk}, z_{ki})] = \sum Re(x_{ij}(y_{jk}z_{ki})) =$$

$$= (X, YZ)e_0 = (YZ, X)e_0 , \tag{C.4b}$$

$$(C.4a) = \sum Re(z_{ki}(x_{ij}y_{jk})) =$$

$$= \sum Re[(z_{ki}x_{ij})y_{jk} - (z_{ki}, x_{ij}, y_{jk})] = \sum Re((z_{ki}x_{ij})y_{jk}) =$$

$$= (ZX, Y)e_0 . \tag{C.5}$$

Thus, for the scalar product in the algebra $\mathcal{M}_3^8$ the properties are stated [5]

$$(X,Y) = (Y,X) \tag{C.6}$$

$(XY,Z) = (X,YZ)$ is cyclically symmetric in X,Y,Z. (C.7)

Let us turn to the algebra $\mathcal{M}_3^{8+}$ with the Jordan product [5] $X \cdot Y = \frac{1}{2}(XY+YX)$. Now the scalar product has the properties

$$(X,Y)e_0 = Tr(X \cdot Y), \tag{C.8}$$

$$(X \cdot Y, Z)e_0 = Re\, Tr((X \cdot Y)Z) = Tr((X \cdot Y) \cdot Z) = Tr(X \cdot (Y \cdot Z)) =$$

is symmetric in X,Y,Z

$$= Tr(X \cdot Y \cdot Z) = (X, Y, Z). \tag{C.9}$$

In the last line merely the notations are introduced, which stress the symmetry. The symmetry means

$$(X \cdot Y, Z) = (Y \cdot Z, X) = (Z \cdot X, Y) =$$

$$= (Z, X \cdot Y) = (X, Y \cdot Z) = (Y, Z \cdot X) \tag{C.10}$$

and it is shown as follows

$$(X \cdot Y, Z) = \tfrac{1}{2}(XY, Z) + \tfrac{1}{2}(YX, Z) = \tfrac{1}{2}(X, YZ) + \tfrac{1}{2}(Y, XZ) =$$

$$= \tfrac{1}{2}(X, YZ) + \tfrac{1}{2}(X, ZY) \qquad \big((Y, XZ) = (Z, YX) = (X, ZY)\big)$$

$$= (X, Y \cdot Z) = (Y \cdot Z, X). \tag{C.11}$$

Appendix D. The principle of triality. The transformations of the octonions, which conserve the norm $\bar{x}x$, form the group SO(8). They consist of: 1) the group G_2 of automorphisms of octonions, 2) the transformations: infinitesimal

$$x \to x' = x + \alpha x + \dots \qquad \alpha \text{ is pure imaginary} \tag{D.1a}$$

$$x \to x' = x + x\alpha + \dots \qquad \alpha \text{ is pure imaginary} \tag{D.1b}$$

or finite

$$x \to x' = \bar{\alpha} x \qquad |\alpha| = 1 \tag{D.2a}$$

$$x \to x' = x\bar{\alpha} \qquad |\alpha| = 1 \tag{D.2b}$$

It is clear that the norm $\bar{x}x$ remains invariant under transformations (D.1) and (D.2):

$$\bar{x}'x' = (\bar{x}a)(\bar{a}x) = (a\bar{x})(x\bar{a}) = (\bar{a}a)(\bar{x}x) = (\bar{x}x). \tag{D.3}$$

Transformations (D.1) and (D.2) are not automorphism transformations. However they obey the more general principle, the principle of triality,

$$\delta_1(xy) = (\delta_2 x)y + x(\delta_3 y), \tag{D.4}$$

$$(xy)' = x''y'''. \tag{D.5}$$

Infinitesimal triality transformations follow from the alternativity

$$(d, x, y) = -(x, d, y), \tag{D.6}$$

$$(x, y, d) = -(x, d, y). \tag{D.7}$$

Hence we obtain immediately

$$d(xy) = (dx + xd)y - x(dy), \tag{D.8}$$

$$(xy)d = -(xd)y + x(dy + yd), \tag{D.9}$$

$$d(xy) + (xy)d = (dx)y + x(yd), \tag{D.10}$$

$$\begin{aligned} d(xy) - (xy)d &= (dx + 2xd)y - x(2dy + yd) = \\ &= (dx - xd)y + x(dy - yd) + 3(x, d, y). \end{aligned} \tag{D.11}$$

Finite triality transformations follow from the Moufang identities

$$(axa)y = a(x(ay)), \tag{D.12}$$

$$x(aya) = ((xa)y)a, \tag{D.13}$$

$$(ax)(ya) = a(xy)a. \tag{D.14}$$

Replacing $y \to \bar{a}y$ in the first identity and $x \to x\bar{a}$ in the second one, we obtain the desired finite triality relations $(|a| = 1)$

$$a(xy) = (axa)(\bar{a}y), \tag{D.15}$$

$$(xy)a = (x\bar{a})(aya), \tag{D.16}$$

$$a(xy)a = (ax)(ya). \tag{D.17}$$

These finite triality transformations generate automorphism transformations in $\mathfrak{M}_3^{8+}$, namely, those SO(8) transformations, which leave invariant the diagonal elements

$$X' = \begin{bmatrix} 1 & & \\ & a & \\ & & \bar{a} \end{bmatrix} \begin{bmatrix} \xi & z & \bar{y} \\ \bar{z} & \eta & x \\ y & \bar{x} & \zeta \end{bmatrix} \begin{bmatrix} 1 & & \\ & \bar{a} & \\ & & a \end{bmatrix} = \begin{bmatrix} \xi & z\bar{a} & \bar{y}a \\ a\bar{z} & \eta & axa \\ \bar{a}y & \bar{a}\bar{x}\bar{a} & \zeta \end{bmatrix}, \quad |a| = 1, \tag{D.18a}$$

$$X' = \begin{bmatrix} \bar{b} & & \\ & 1 & \\ & & b \end{bmatrix} \begin{bmatrix} \xi & z & \bar{y} \\ \bar{z} & \eta & x \\ y & \bar{x} & \zeta \end{bmatrix} \begin{bmatrix} b & & \\ & 1 & \\ & & \bar{b} \end{bmatrix} = \begin{bmatrix} \xi & \bar{b}z & \bar{b}\bar{y}\bar{b} \\ \bar{z}b & \eta & x\bar{b} \\ byb & b\bar{x} & \zeta \end{bmatrix}, \quad |b| = 1, \tag{D.18b}$$

$$X' = \begin{bmatrix} c & & \\ & \bar{c} & \\ & & 1 \end{bmatrix} \begin{bmatrix} \xi & z & \bar{y} \\ \bar{z} & \eta & x \\ y & \bar{x} & \zeta \end{bmatrix} \begin{bmatrix} \bar{c} & & \\ & c & \\ & & 1 \end{bmatrix} = \begin{bmatrix} \xi & czc & c\bar{y} \\ \bar{c}\bar{z}\bar{c} & \eta & \bar{c}x \\ y\bar{c} & \bar{x}c & \zeta \end{bmatrix}, \quad |c| = 1. \tag{D.18c}$$

One can observe the cyclic symmetry: $x \to y \to z \to x$, $a \to b \to c \to a$.
In all these cases the automorphism property

$$(X \cdot Y)' = X' \cdot Y' \tag{D.19}$$

is valid. This can be checked if we write the product $X \cdot Y$ of matrices (1) explicitly and use the Moufang identities (D.12)-(D.14) [20] or the triality relations (D.15)-(D.17). Therefore these SO(8) transformations belong to the F_4 group. For a more general formulation of triality transformations see refs.[13,15,17,18].

Any matrix of $\mathfrak{M}_3^{8+}$ can be diagonalized by some F_4 transformation [5]. In fact, some sequence of triality transformations and real rotations (R) can be used, for example, (we follow ref.[18])

$$\begin{bmatrix} \xi & z & \bar{y} \\ \bar{z} & \eta & x \\ y & \bar{x} & \zeta \end{bmatrix} \xrightarrow[\substack{\text{with} \\ a = z/|z|}]{\text{(D.18a)}} \begin{bmatrix} \xi & |z| & \bar{y}_1 \\ |z| & \eta & x_1 \\ y_1 & \bar{x}_1 & \zeta \end{bmatrix} \longrightarrow$$

$$\longrightarrow \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{R}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi & |z| & \bar{y}_1 \\ |z| & \eta & x_1 \\ y_1 & \bar{x}_1 & \zeta \end{bmatrix} \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{R^T}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \xi_2 & 0 & \bar{y}_2 \\ 0 & \eta_2 & x_2 \\ y_2 & \bar{x}_2 & \zeta \end{bmatrix} \xrightarrow[\substack{\text{with} \\ c = x_2/|x_2|}]{\text{(D.18c)}} \begin{bmatrix} \xi_2 & 0 & \bar{y}_3 \\ 0 & \eta_2 & |x_2| \\ y_3 & |x_2| & \zeta \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \xi_2 & 0 & \bar{y}_3 \\ 0 & \eta_2 & |x_2| \\ y_3 & |x_2| & \zeta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_{22} & r_{32} \\ 0 & r_{23} & r_{33} \end{bmatrix} = \begin{bmatrix} \xi_2 & r_{23}\bar{y}_3 & r_{33}\bar{y}_3 \\ r_{23}y_3 & \eta_4 & 0 \\ r_{33}y_3 & 0 & \zeta_4 \end{bmatrix} \rightarrow$$

$$\xrightarrow[\substack{\text{with} \\ a = y_3/|y_3|}]{\text{(D.18a)}} \begin{bmatrix} \xi_2 & r_{23}|y_3| & r_{33}|y_3| \\ r_{23}|y_3| & \eta_4 & 0 \\ r_{33}|y_3| & 0 & \zeta_4 \end{bmatrix} \tag{D.20}$$

and the latter symmetric real matrix can be always diagonalized by some real rotation.

Appendix E. Identities with commutators and anticommutators. For all nonassociative quantities there are the identities

$$[A[BC]]+[C[AB]]+[B[CA]]=-(A,B,C)-(C,A,B)-(B,C,A)+ \\ +(C,B,A)+(B,A,C)+(A,C,B), \qquad (E.1)$$

$$[A[BC]]-\{C\{AB\}\}+\{B\{CA\}\}=-(A,B,C)+(C,A,B)-(B,C,A)+ \\ +(C,B,A)-(B,A,C)+(A,C,B), \qquad (E.2)$$

$$[A\{BC\}]+[C\{AB\}]+[B\{CA\}]=-(A,B,C)-(C,A,B)-(B,C,A)- \\ -(C,B,A)-(B,A,C)-(A,C,B), \qquad (E.3)$$

$$[A\{BC\}]-\{C[AB]\}+\{B[CA]\}=-(A,B,C)+(C,A,B)-(B,C,A)- \\ -(C,B,A)+(B,A,C)-(A,C,B), \qquad (E.4)$$

$$\{A[BC]\}+\{C[AB]\}+\{B[CA]\}= \\ =2\sum_{\substack{\text{antisymm.}\\ \text{over } A,B,C}} A(BC)+(A,B,C)+(C,A,B)+(B,C,A)-(C,B,A)-(B,A,C)-(A,C,B)= \\ =2\sum_{\substack{\text{antisymm.}\\ \text{over } A,B,C}} (AB)C-(A,B,C)-(C,A,B)-(B,C,A)+(C,B,A)+(B,A,C)+(A,C,B), \qquad (E.5)$$

$$\{A\{BC\}\}+\{C\{AB\}\}+\{B\{CA\}\}= \\ =2\sum_{\substack{\text{symm.}\\ \text{over } A,B,C}} A(BC)+(A,B,C)+(C,A,B)+(B,C,A)+(C,B,A)+(B,A,C)+(A,C,B)= \\ =2\sum_{\substack{\text{symm.}\\ \text{over } A,B,C}} (AB)C-(A,B,C)-(C,A,B)-(B,C,A)-(C,B,A)-(B,A,C)-(A,C,B). \qquad (E.6)$$

For octonions with the use of alternativity the identities take form

$$[a[bc]]+[c[ab]]+[b[ca]]=-6(a,b,c), \qquad (E.7)$$

$$[a[bc]]-\{c\{ab\}\}+\{b\{ca\}\}=-2(a,b,c), \qquad (E.8)$$

$$[a\{bc\}]+[c\{ab\}]+[b\{ca\}]=0, \qquad (E.9)$$

$$[a\{bc\}]-\{c[ab]\}+\{b[ca]\}=0, \qquad (E.10)$$

$$\{a[bc]\}+\{c[ab]\}+\{b[ca]\}=2\sum_{\text{antisymm.}} a(bc)+6(a,b,c)= \\ =2\sum_{\text{antisymm.}}(ab)c-6(a,b,c), \qquad (E.11)$$

$$\{a\{bc\}\}+\{c\{ab\}\}+\{b\{ca\}\}=2\sum_{\text{symm.}} a(bc)=2\sum_{\text{symm.}}(ab)c. \qquad (E.12)$$

Appendix F. One more matrix representation. We can start with matrices (2). The trace is generalized to be 72

$$\mathbb{T}r(\lambda_i e_\mu F)=Tr(\lambda_i\,Re(e_\mu F)),$$
$$\mathbb{T}r(\lambda_i e_\mu(\lambda_j e_\nu F))=(\lambda_j^{\ell})_{ik}(e_\nu^{\ell})_{\mu\lambda}\mathbb{T}r(\lambda_k e_\lambda F),$$
$$\mathbb{T}r(\lambda_i e_\mu(F\lambda_j e_\nu))=(\lambda_j^{r})_{ik}(e_\nu^{r})_{\mu\lambda}\mathbb{T}r(\lambda_k e_\lambda F).$$

Now the left and right representatives do not coincide. The left and right representatives of the Gell-Mann matrices λ commute mutually due to associativity. However, the left and right representatives of octonions do not commute (see, e.g., refs./22,23/). The left representatives are multiplied in the same order as origilal quantities, but the right ones in the inverse order. For an explicit form of e_μ^{ℓ} and e_μ^{r} see refs./22,23/. The 9x9 matrices λ_j^{ℓ} and λ_j^{r} can be easily found by using the Gell-Mann multiplication table /21/

$$(\lambda_j^{\ell,r})_{ik}=\delta_{i0}\delta_{jk}+\tfrac{2}{3}\delta_{ij}\delta_{0k}+d_{ijk}\pm if_{ijk}$$

(the last two terms do not contribute to 0-row and 0-column),

$$\lambda_j^{\ell,r}=\left(\begin{array}{c|c} 0 & \delta_{jk} \\ \hline \tfrac{2}{3}\delta_{ij} & d_{ijk}\pm if_{ijk} \end{array}\right), \quad \lambda_0^{\ell}=\lambda_0^{r}=\begin{pmatrix}1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}.$$

With the Jordan product we have

$$Tr(\lambda_i e_\mu\cdot(\mu_A\cdot F))=\mu_A^{s}\,\mathbb{T}r(\lambda_i e_\mu\cdot F),$$
$$Tr(\lambda_i e_\mu\cdot(\mu_A\cdot(\mu_B\cdot F)))=\mu_A^{s}\mu_B^{s}\,\mathbb{T}r(\lambda_i e_\mu\cdot F),$$
$$\mu_A^{s}=\tfrac{1}{2}(\mu_A^{\ell}+\mu_A^{r}).$$

μ_A^{s} correspond to $\mu_A^{\ell}=\mu_A^{r}$ of n. 8. One can introduce also

$$\mu_A^{a}=\tfrac{1}{2}(\mu_A^{\ell}-\mu_A^{r}).$$

Then

$$\mathbb{T}r(\lambda_i e_\mu\cdot(\mu_a\cdot\mu_b)\cdot F)=(\mu_a\cdot\mu_b)^{s}\,\mathbb{T}r(\lambda_i e_\mu\cdot F),$$
$$(\mu_a\cdot\mu_b)^{s}=\tfrac{1}{2}(\mu_a^{\ell}\cdot\mu_b^{\ell}+\mu_a^{r}\cdot\mu_b^{r})=\mu_a^{s}\cdot\mu_b^{s}+\mu_a^{a}\cdot\mu_b^{a}\left(=\tfrac{1}{3}\delta_{ab}+d_{abc}\mu_c^{s}\right),$$
$$\mathbb{T}r(\lambda_i e_\mu\cdot(\mu_a\cdot\mu_{gj})\cdot F)=(\mu_a\cdot\mu_{gj})^{s}\,\mathbb{T}(\lambda_i e_\mu\cdot F),$$
$$(\mu_a\cdot\mu_{gj})^{s}=\tfrac{1}{2}(\mu_a^{\ell}\cdot\mu_{gj}^{\ell}+\mu_a^{r}\cdot\mu_{gj}^{r})=\mu_a^{s}\cdot\mu_{gj}^{s}+\mu_a^{a}\cdot\mu_{gj}^{a},$$
$$\mathbb{T}r(\lambda_i e_\mu\cdot(\mu_{gj}\cdot\mu_{hk})\cdot F)=(\mu_{gj}\cdot\mu_{hk})^{s}\,\mathbb{T}r(\lambda_i e_\mu\cdot F),$$
$$(\mu_{gj}\cdot\mu_{hk})^{s}=\tfrac{1}{2}(\mu_{gj}^{\ell}\cdot\mu_{hk}^{\ell}+\mu_{gj}^{r}\cdot\mu_{hk}^{r})+\tfrac{1}{4}([\varkappa_g^{\ell}\varkappa_h^{\ell}]+[\varkappa_g^{r}\varkappa_h^{r}])[e_j^{\ell}e_k^{r}]= \\ =\mu_{gj}^{s}\cdot\mu_{hk}^{s}+\mu_{gj}^{a}\cdot\mu_{hk}^{a}+\tfrac{1}{4}([\varkappa_g^{\ell}\varkappa_h^{\ell}]+[\varkappa_g^{r}\varkappa_h^{r}])[e_j^{\ell}e_k^{r}].$$

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Полубаринов И.В. E2-85-506

Исключительная квантовая механика

Дано введение в исключительную квантовую механику Йордана, Неймана и Вигнера. Определены наблюдаемые, собственные состояния, вероятности переходов, уравнения движения. Изложены основные свойства лежащей в основе исключительной алгебры Йордана, алгебры эрмитовых матриц 3x3 с октонионными элементами. Определены представления в терминах матриц с вещественными элементами.

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Polubarinov I.V. E2-85-506

Exceptional Quantum Mechanics

An introduction into the exceptional quantum mechanics by Jordan, von Neumann and Wigner is given. Observables, eigenstates, transition probabilities, equations of motion are defined. Main properties of the underlying exceptional Jordan algebra, algebra of Hermitian 3x3 matrices with octonionic entries, are exposed. Matrix representations in terms of usual matrices with real entries are defined.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1985