сообщения
объединенного
института
ядерных
исследований
дубна

E2-85-468

J.Dittrich,* P.Exner

NONEXISTENCE OF FINITE-ENERGY SOLUTIONS IN SOME GAUGE MODELS

* Nuclear Physics Institute, Czechoslovak Academy of Sciences, 250 68 Řež near Prauge, Czechoslovakia

1985

1. Introduction

Various static finite-energy solutions in gauge theories are known : vortices in the 2+1 dimensional Ginzburg-Landau model, magnetic monopoles in the 3+1 dimensional Yang-Mills-Higgs theories or instantons in the 4-dimensional Euclidean Yang-Mills theory. The following features manifested in these examples are necessary for the existence of nontrivial solutions/1-3/:

(i) both the gauge and scalar fields are present for the spatial dimension $d = 2,3$,

(ii) Higgs potential (self-interaction of the scalar field) is present for $d = 2$.

These properties have been proved for time-independent fields in the gauge $A_0 = 0$, i.e., for d-dimensional Euclidean theory/1/. In a d+1 dimensional Minkowskian theory, this gauge cannot be used for fields with nonzero electric-like components F_{0j} . The aim of the present letter is to show that the mentioned results hold under more general assumptions, namely for static fields with nonvanishing F_{0j}.

On the other hand, for static fields with $A_0 = 0$ in higher dimensions one can derive that/1/

(iii) finite-energy solutions are essentially pure Yang-Mills field for $d = 4$,

(iv) no nontrivial finite-energy solutions exist for $d > 4$.

We recover these properties only if $F_{0j} = 0$ and $D_0\phi = 0$ is assumed. Let us remark that the case of pure Yang-Mills field (including $F_{0j} \neq 0$) was treated in Ref.4 by means of a scale transformation the use of which requires a further justification (compare to Refs.5 and 6).

2. Preliminaries

We consider the system of Yang-Mills fields A_μ^a and a Higgs field ϕ in d+1 dimensional Minkowski space-time. The Greek indices μ,ν,ρ run over $0,1,\dots,d$, Roman indices $j,k=1,\dots,d$. The values of the field ϕ belong to a finite-dimensional Hilbert space L (i.e., $L=\mathbb{R}^n$ or $\mathbb{C}^n$) which is the carrier space of a unitary representation of the gauge group G . The latter is determined by infinitesimal generators T_a , $a=1,\dots,r$, which satisfy the condition *)

$$\mathrm{Tr}\, T_a T_b = 2\delta_{ab} .$$

We use the notation

$$A_\mu = iT_a A_\mu^a$$

and similarly for the field tensor

$$F_{\mu\nu} = iT_a F_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] .$$

Since the Yang-Mills fields A_μ^a are real-valued, A_μ and $F_{\mu\nu}$ are anti-hermitean operators in L , i.e., skew-symmetric matrices if we express them by means of an orthonormal basis in L . Using the covariant derivatives

$$D_\mu\phi = \partial_\mu\phi - A_\mu\phi ,$$

one can write Lagrangian of the Yang-Mills-Higgs field as

$$\mathcal{L} = \frac{1}{8}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi, D^\mu\phi) - V(|\phi|^2) . \qquad (1)$$

Here the trace and the norm $|\phi| = (\phi,\phi)^{1/2}$ refer to the scalar product $(.,.)$ of L , and the Higgs potential is assumed to fulfil

$$V(|\phi|^2) \geq 0 . \qquad (2)$$

The field equations read

$$D_\mu D^\mu\phi = -2V'(|\phi|^2) \qquad (3a)$$

*) This can be achieved for the adjoint representation of any connected semisimple compact Lie group G – cf., e.g., Ref.12.

and

$$\partial^\mu F_{\mu\nu} - [A^\mu, F_{\mu\nu}] = -\sum_a iT_a \,\mathrm{Re}(\phi, iT_a D_\nu\phi) . \qquad (3b)$$

Finally, the energy-momentum tensor of the Yang-Mills-Higgs field is

$$\mathcal{T}_{\mu\nu} = \frac{1}{2}\mathrm{Tr}\, F_{\mu\rho}F_\nu{}^\rho + \mathrm{Re}(D_\mu\phi, D_\nu\phi) - g_{\mu\nu}\mathcal{L} . \qquad (4)$$

3. The results and proofs

Our results are summarized in the following theorem which represents a straightforward generalization to Corollary 2.3 of Ref.1 .

Theorem. Let (A_μ,ϕ) be a solution to Yang-Mills-Higgs equations (3) with a finite energy and the energy-momentum tensor (4) independent of time, then the following assertions are valid :

(a) If $d>1$, $D_0\phi=0$ and $F_{\mu\nu}=0$, then $|\phi|=\mathrm{const}$ and $V(|\phi|^2)=0$. In particular, for a pure scalar field ($A_\mu=0$) in $d>1$ we have $\phi=\mathrm{const}$; this is known as Derrick theorem /5,6/.

(b) If $d=1$, $|\phi|=\mathrm{const}$, $V(|\phi|^2)=0$ and $D_0\phi=0$, then $F_{\mu\nu}=0$.

(c) If $d=2$ and $V(|\phi|^2)=0$, then $F_{\mu\nu}=0$, $|\phi|=\mathrm{const}$ and $D_\mu\phi=0$.

(d) If $d=3$, $|\phi|=\mathrm{const}$ and $V(|\phi|^2)=0$, then $D_\mu\phi=0$ and $F_{\mu\nu}=0$.

(e) If $d=4$, $F_{0j}=0$ for $j=1,\dots,d$ and $D_0\phi=0$, then $D_\mu\phi=0$ and $V(|\phi|^2)=0$.

(f) If $d>4$, $F_{0j}=0$ for $j=1,\dots,d$ and $D_0\phi=0$, then $F_{\mu\nu}=0$, $D_\mu\phi=0$ and $V(|\phi|^2)=0$.

Proof : The argument follows the same line as in Ref.1 . Due to the assumption, $\partial_j\mathcal{T}_{\mu j}=0$, so a formal integration by parts yields

$$\int_{\mathbb{R}^d} \mathcal{T}_{\mu j}(x)\, d^d x = 0 . \qquad (5)$$

For a finite-energy solution, this relation can be obtained rigorously by introducing a smooth cut-off ; removing of it is justified by the inequality $|\mathcal{T}_{\mu\nu}| \leq \mathcal{T}_{00}$ which verifies directly /1/. Summation over $\mu=j=1,\dots,d$ in (5) leads to the equality

$$\left(\frac{d}{2}-1\right)\sum_{a,j}\|F_{0j}^a\|^2 + \left(1-\frac{d}{4}\right)\sum_{a,j,k}\|F_{jk}^a\|^2 + \frac{d}{2}\|D_0\phi\|^2 = \qquad (6)$$
$$= \left(\frac{d}{2}-1\right)\sum_j\|D_j\phi\|^2 + d\,\|V(|\phi|^2)^{1/2}\|^2 ,$$

which is crucial for the proof ; here $\|.\|$ denotes the norm in $L^2(\mathbb{R}^d)$

or $L^2(\mathbb{R}^d;L)$. Now we can check the assertions (a)-(f) :

(a) Under the assumptions $F_{\mu\nu}=0$ and $D_0\phi=0$, the relation (6) gives $V(|\phi|^2)=0$. Since $F_{\mu\nu}=0$, we can choose a gauge in which $A_\mu=0$; this commonly used fact is justified by means of the non-abelian Stokes theorem [2,7-11]. Then $\partial_0\phi=D_0\phi=0$. If $d>2$, $\partial_j\phi=D_j\phi=0$ follows from (6) so $\phi=\text{const}$. If $d=2$, the argument is slightly more complicated. Since $V(|\phi|^2)=0$ and V is non-negative, we have $V'(|\phi|^2)=0$. The equation (3a) for the Higgs field reduces then in the chosen gauge to Laplace equation

$$\Delta\phi=0 \quad . \tag{7}$$

However, ϕ is assumed to have a finite energy so $\partial_j\phi\in L^2(\mathbb{R}^2;L)$; this fact together with (7) gives again $\phi=\text{const}$ as one can check easily using the mean-value theorem for harmonic functions. The gauge group acts unitarily on ϕ , thus $|\phi|$ is preserved when we return to the original gauge.

(b) and (d). If $|\phi|=\text{const}$, one obtains

$$0=\frac{1}{2}\partial_\mu\partial^\mu(\phi,\phi)=\text{Re}(D_\mu D^\mu\phi,\phi)+(D_\mu\phi,D^\mu\phi) \tag{8}$$

using the antihermiticity of A_μ . The assumption $V(|\phi|^2)=0$ yields $D_\mu D^\mu\phi=0$ according to Eq.(3a) in the same way as above. The relation (8) then gives

$$|D_0\phi|^2=\sum_j|D_j\phi|^2 \tag{9}$$

so (6) can be rewritten as

$$(\frac{d}{2}-1)\sum_{a,j}\|F^a_{0j}\|^2+(1-\frac{d}{4})\sum_{a,j,k}\|F^a_{jk}\|^2+\|D_0\phi\|^2=0 \quad . \tag{10}$$

If $d=1$, the components F_{jk} are absent and the last relation gives $F_{0j}=0$ for $D_0\phi=0$. If $d=3$, we get $F_{\mu\nu}=0$ and $D_0\phi=0$ according to (10). In that case, $D_j\phi=0$ follows from (9).

(c) For $d=2$ and $V(|\phi|^2)=0$, the relation (6) yields $F_{jk}=0$ and $D_0\phi=0$. The Stokes-theorem argument shows that we can choose a gauge in which $A_j=0$. As in the case (a), we have $\partial_j\phi\in L^2(\mathbb{R}^2;L)$ and $\Delta\phi=0$, hence ϕ is a function of time only, $\phi=\varphi(t)$. Due to the assumption of energy finiteness, $F^a_{j0}=\partial_j A^a_0$ belongs to $L^2(\mathbb{R}^2)$. Since the field equation (3b) reduces to $\Delta A_0=0$ for $\nu=0$, the component $A_0=a_0(t)$ is also independent of space coordinates and $F_{j0}=0$. Together we get $F_{\mu\nu}=0$ and $D_\mu\phi=0$, so it is possible to choose a new gauge in which $A_\mu=0$ and $\phi=\text{const}$; then $|\phi|=\text{const}$ in any gauge.

(e) and (f) follow immediately from (6). ∎

References

1 Jaffe A., Taubes C., Vortices and Monopoles, Birkhäuser, Boston 1980 , section II.2 .
2 Goddard P., Olive D.I., Rep.Progr.Phys., 1978, v.41, pp.1357-1437.
3 Krasnikov N.V., Phys.Lett.B, 1977, v.72, pp.455-456.
4 Coleman S., in:"New Phenomena in Subnuclear Physics" (A.Zichici, ed.), Plenum Press, New York 1977 ; pp.297-421.
5 Derrick G.H., J.Math.Phys., 1964, v.5, pp.1252-1254.
6 Requardt M., Commun.Math.Phys., 1981, v.80, pp.369-379.
7 Aref'eva I.Ya., Teor.Mat.Fiz., 1980, v.43, pp.111-116.
8 Bralić N.E., Phys.Rev.D, 1980, v.22, pp.3090-3103.
9 Iyanaga M., J.Math.Phys., 1981, v.22, pp.2713-2715.
10 Fishbane P.M., Gasiorowicz S., Kaus P., Phys.Rev.D, 1981, v.24, pp.2324-2329.
11 Diósi L., Phys.Rev.D, 1983, v.27, pp.2552-2555.
12 Zhelobenko D.P., Compact Lie Groups and Their Representations, Nauka, Moscow 1970 (in Russian) , § 90 .

Received by Publishing Department
on June 19, 1985.

Диттрих Я., Экснер П. E2-85-468

Несуществование решений с конечной энергией в некоторых калибровочных моделях

Теорема, запрещающая существование некоторых статических решений с конечной энергией для классической модели Янга-Миллса-Хиггса в зависимости от пространственной размерности, обобщается таким образом, чтобы ее можно было применить к полям с ненулевыми компонентами электрического типа F_{0j}.

Работа выполнена в Лаботаории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1985

Dittrich J, Exner P. E2-85-468

Nonexistence of Finite-Energy Solutions in Some Gauge Models

A theorem which forbids the existence of certain nontrivial static finite-energy solutions to Yang-Mills-Higgs equations in dependence on the spatial dimension is generalized to include fields with nonvanishing electric-like components F_{0j}.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1985