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ON EXCEPTIONAL SOLUTIONS OF THE BETHE-ANSATZ EQUATIONS

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1. The algebraic system of the Bethe-ansatz equations

$$\left(\lambda_j - is\right)^{N}\prod_{\substack{k=1\\k\neq j}}^{M}\left(\lambda_j-\lambda_k+i\right)=\left(\lambda_j+is\right)^{N}\prod_{\substack{k=1\\k\neq j}}^{M}\left(\lambda_j-\lambda_k-i\right), \quad j=1,\dots,M; \quad M\leq Ns \tag{1}$$

arises when diagonalizing the Hamiltonian of the isotropic (or else XXX) Heisenberg model which describes the interaction of N spin- s particles on the one-dimensional periodic chain. For the $s=1/2$ case, the Hamiltonian has the form

$$H_{1/2} = -\frac{1}{2}\sum_{n=1}^{N}\left(\sum_{a=1}^{3}\sigma_n^a\sigma_{n+1}^a - 1\right), \quad \sigma_{N+1}^a \equiv \sigma_1^a, \tag{2}$$

where σ_n^a are the Pauli matrices. The spin – s generalization [1] respecting exact integrability of model (2) results in the Hamiltonian [2,3]

$$H_s = 2\sum_{n=1}^{N}\sum_{j=0}^{2s-1}\left(\sum_{k=j+1}^{2s}\frac{1}{k}\right)\prod_{\substack{m=0\\m\neq j}}^{2s}\left[\left(x_j-x_m\right)^{-1}\left(\sum_{a=1}^{3}S_n^a S_{n+1}^a - x_m\right)\right], \tag{3}$$

$$x_m = \frac{1}{2}m(m+1) - s(s+1).$$

Here S_n^a are the SU(2) generators in the spin – s representation, which act in the space $V_n^s \simeq \mathbb{C}^{2s+1}$ associated with site n of the spin chain.'

The sets $\{\lambda_j\}$ of complex numbers $\lambda_1,\dots,\lambda_M$ satisfying (1) parametrize the eigenvectors of Hamiltonian (3) in the space $W_s = \prod_{n=1}^{N}\otimes V_n^s$. Both alternative versions of the Bethe ansatz - the coordinate one [4,5] (for spin 1/2; for $s>\frac{1}{2}$ see below) and the algebraic one [6,7] - lead to a common conclusion: any solution $\{\lambda_j\}$ of system (1) with λ_j being pairwise different specifies the "Bethe vector" $|\{\lambda_j\}\rangle$ in the space W_s , which is an eigenvector of the

Hamiltonian H_s. One implies, of course, that $|\{\lambda_j\}\rangle \neq 0$. Such is the general case.

The aim of the present paper is to look for exceptions to this rule. We call a solution set $\{\lambda_j\}$ of (1) exceptional if it

a) generates the vanishing vector $|\{\lambda_j\}\rangle = 0$ even though consists only of distinct λ_j's, or

b) contains two or more equal roots $\lambda_j = \lambda_k$ but, nevertheless, defines nonzero eigenvector of the Hamiltonian H_S.

Whereas the exceptional solutions of the type a) could be anticipated from the general formulas[8] for the norms of Bethe vectors, the examples of the b)-type solutions were unknown so far. In this article we present a nonzero Bethe vector for the case $S=1$, $N=4$, $M=4$ related to the exceptional solution $\{\lambda_j\} = \{0, 0, i, -i\}$ and point out some other candidates for exceptional solutions.

Our investigation was to a large extent prompted by the completeness problem of the Bethe ansatz. There is a general belief that the solutions of (1) parametrize the whole set of eigenvectors of the Hamiltonian H_S. All exceptional solutions we have found explicitly in the present paper really do complete the set of Bethe vectors in each individual case. Thus, we obtain one more indication in favor of the hypothesis of completeness. However, up to now no comprehensive proof of this hypothesis is obtained in model (3) and other integrable models. The evaluation of the string-configurations number in refs.[4,7,9,10] is actually, as shown in[11], an estimate of the number of Bethe vectors to a leading order in $N \to \infty$.

2. The algebraic construction of Bethe vectors[6,7] is based on the use of the monodromy matrix T_λ,

$$T_\lambda = \begin{pmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{pmatrix} = L_N(\lambda) L_{N-1}(\lambda) \ldots L_1(\lambda), \tag{4}$$

$$L_n(\lambda) = \begin{pmatrix} \lambda + i S_n^3 & i S_n^- \\ i S_n^+ & \lambda - i S_n^3 \end{pmatrix}, \quad S_n^\pm = S_n^1 \pm i S_n^2. \tag{5}$$

The elements of T_λ are operators in W_S. The commutation relations for these operators are well known[7], for example

$$(\lambda-\mu) A_\lambda B_\mu = (\lambda-\mu-i) B_\mu A_\lambda + i B_\lambda A_\mu, \tag{6}$$

$$(\lambda-\mu) D_\lambda B_\mu = (\lambda-\mu+i) B_\mu D_\lambda - i B_\lambda D_\mu, \tag{6'}$$

$$B_\lambda B_\mu = B_\mu B_\lambda, \quad \tau_\lambda \tau_\mu = \tau_\mu \tau_\lambda, \tag{7}$$

where $\tau_\lambda = tr\, T_\lambda = A_\lambda + D_\lambda$.

The Bethe vector corresponding to a solution $\{\lambda_j\}$ of system (1) is given by

$$|\{\lambda_j\}\rangle = B_{\lambda_1} \ldots B_{\lambda_M} |0\rangle, \tag{8}$$

where the vector $|0\rangle \in W_S$ (pseudovacuum) is defined by $S_n^+|0\rangle = 0$, $S_n^3|0\rangle = s|0\rangle$ for each n, $\langle 0|0\rangle = 1$. Vector (8) proves to be an eigenvector of the Hamiltonian H_S due to the following property. For any complex λ and $\{\lambda_j\}$ obeying (1), vector (8) is an eigenvector of τ_λ,

$$\tau_\lambda |\{\lambda_j\}\rangle = \Lambda(\lambda, \{\lambda_j\}) |\{\lambda_j\}\rangle, \tag{9}$$

$$\Lambda(\lambda, \{\lambda_j\}) = (\lambda - is)^N \prod_{j=1}^{M} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j} + (\lambda + is)^N \prod_{j=1}^{M} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j}, \tag{10}$$

whereas H_S is proportional to its logarithmic derivative $\frac{d}{d\lambda} \ln \tau_\lambda \big|_{\lambda = is}$.

The proof[6] of the key equality (9) is based on the commutation relations (6),(7), and essentially uses the fact that all λ_j in the set $\{\lambda_j\}$ are distinct. In the presence of equal roots we would have to use, for instance, formula (6) with $\lambda = \mu$. However, the resulting trivial identity yields no information on how A_μ commutes with B_μ.

As shown in ref.[12], the information we need can be obtained by differentiating eqs. (6) with respect to μ. This leads to

$$A_\mu B_\mu = B_\mu A_\mu + i B'_\mu A_\mu - i B_\mu A'_\mu \tag{11}$$

and other similar relations allowing us to embed the solutions with equal roots into the general scheme of algebraic Bethe ansatz. For example, in the case of two equal numbers in a set

$$\{\lambda_j\} = \{\mu, \mu, \lambda_3, \ldots, \lambda_M\}, \quad \mu \neq \lambda_j \neq \lambda_k, \tag{12}$$

the condition for vector (8) to obey eq. (9) will be a system of M equations for $M-1$ unknown quantities:

$$(\mu - is)^N \prod_{k=3}^{M} (\mu - \lambda_k + i) = -(\mu + is)^N \prod_{k=3}^{M} (\mu - \lambda_k - i); \tag{13}$$

$$(\lambda_j - is)^N(\lambda_j-\mu+i)^2\prod_{\substack{k=3\\k\neq j}}^{M}(\lambda_j-\lambda_k+i) = (\lambda_j+is)^N(\lambda_j-\mu-i)^2\prod_{\substack{k=3\\k\neq j}}^{M}(\lambda_j-\lambda_k-i), \quad j=3,\dots,M; \tag{14}$$

$$N\left[(\mu-is)^{N-1}\prod_{k=3}^{M}(\mu-\lambda_k+i)+(\mu+is)^{N-1}\prod_{k=3}^{M}(\mu-\lambda_k-i)\right]+4i(\mu+is)^N\prod_{k=3}^{M}(\mu-\lambda_k-i)$$
$$+\,i\sum_{j=3}^{M}\frac{1}{\mu-\lambda_j}\left[(\mu+is)^N\prod_{\substack{k=3\\k\neq j}}^{M}(\mu-\lambda_k-i)-(\mu-is)^N\prod_{\substack{k=3\\k\neq j}}^{M}(\mu-\lambda_k+i)\right]=0. \tag{15}$$

Eqs. (13), (14) are nothing but the standard system (1) for the particular set (12), whereas eq. (15) is an additional equation due to the double root μ. Note that if $\mu-\lambda_j\neq i$, $j=3,\dots,M$, eq.(15) reduces to

$$\sum_{j=3}^{M}\frac{1}{(\mu-\lambda_j)^2+1}=\frac{Ns}{\mu^2+s^2}-2. \tag{16}$$

3. Equations analogous to (13)-(15) obtained in ref./12/ for the repulsive case of nonlinear Schrödinger model turned out to be unsolvable. By contrast, system (13)-(15) possesses a lot of solutions. Specifically, for integer $S\geqslant 1$, even $N\geqslant 4$, and $M=2S+2$ there exist exceptional solutions

$$\{\lambda_j\}=\{0,\,0,\,i,\,-i,\,2i,\,-2i,\,\dots,\,is,\,-is\}. \tag{17}$$

In the "string" terminology/7,10/, a solution of that type consists of a 1-string and a perfect $(2S+1)$-string with centers at zero. Solutions of a more general form are also possible. Let a set

$$\{\lambda_j\}=\{0,\,0,\,i,\,-i,\,\lambda_5,\dots,\lambda_M\} \tag{18}$$

obey eqs. (13), (14), i.e., ordinary Bethe-ansatz equations (1). Then this set satisfies eq. (15) too if

$$\prod_{j=5}^{M}(\lambda_j+i)=(-)^N\prod_{j=5}^{M}(\lambda_j-i). \tag{19}$$

With N even, this equality is fulfilled for any self-conjugate $(\{\bar\lambda_j\}=\{\lambda_j\})$ and simultaneously symmetric $(\{-\lambda_j\}=\{\lambda_j\})$ set $\{\lambda_5,\dots,\lambda_M\}$. Such sets consist of pairs $\{a,-a\}$, $\{ib,-ib\}$ and (or) quartets $\{c+id,\ c-id,\ -c+id,\ -c-id\}$ with real a, b, c, d. Solutions of that type do really exist. For example ($S=1$, $N=M=6$):

$$\left\{0,0,i,-i,\ i\frac{\sqrt5+\sqrt2}{\sqrt3},\ -i\frac{\sqrt5+\sqrt2}{\sqrt3}\right\}\ \text{and}\ \left\{0,0,i,-i,\ i\frac{\sqrt5-\sqrt2}{\sqrt3},\ -i\frac{\sqrt5-\sqrt2}{\sqrt3}\right\}. \tag{20}$$

In particular, for $N\to\infty$, $M\lesssim NS$ there are solutions of form (18) that describe low-lying excitations in the physically interesting antiferromagnetic case/7,10/.

Generalization of eqs. (13)-(15) to the case of several double roots, or three or more equal roots, is performed along the same line. However, we have not succeeded in finding exceptional solutions in these sectors. In particular, for $S>1$ additional equations similar to (15) do not allow the configuration with $2S-1$ double roots, which includes perfect strings of lengths $(2S-1)$ and $(2S+1)$, respectively, with centers at zero. This implies that in the $(2S-1)$-string, imaginary roots will actually deviate from their nominal positions.

One more possible generalization is the XXZ model of spin S /13,14/, where the Bethe-ansatz equations read/14/

$$\sin^N(\lambda_j-is\gamma)\prod_{\substack{k=1\\k\neq j}}^{M}\sin(\lambda_j-\lambda_k+i\gamma)=\sin^N(\lambda_j+is\gamma)\prod_{\substack{k=1\\k\neq j}}^{M}\sin(\lambda_j-\lambda_k-i\gamma) \tag{21}$$

An analysis of the appropriate system analogous to (13)-(15) discovers the exceptional solutions

$$\{\lambda_j\}=\{0,0,i\gamma,-i\gamma,2i\gamma,-2i\gamma,\dots,is\gamma,-is\gamma\} \tag{22}$$

for integer $S\geqslant 1$ and even $N\geqslant 4$. Furthermore, if a set

$$\{\lambda_j\}=\{0,0,i\gamma,-i\gamma,\lambda_5,\dots,\lambda_M\} \tag{23}$$

satisfies (21) and the subsidiary condition

$$\prod_{j=5}^{M}\sin(\lambda_j+i\gamma)=(-)^N\prod_{j=5}^{M}\sin(\lambda_j-i\gamma), \tag{24}$$

then this set is also an exceptional solution.

4. Let us return to solution (17) and consider the corresponding vector

$$B_0 B_0 B_i B_{-i} \dots B_{is} B_{-is} |0\rangle . \qquad (25)$$

It follows from general formulas of /8/ that vectors (8) related to solution sets $\{\lambda_j\}$ with $\lambda_j - \lambda_k = i$ for some j and k, may happen to be zero. A direct check in the simplest case $S=1$, $\mathcal{N}=4$ shows that vector (25) actually equals zero. This is by no means due to coincidence of roots because even the vector $B_0 B_i B_{-i}|0\rangle$ is equal to zero for $S=1$, $\mathcal{N}=4$. Also, for example, the vector $B_{i/2} B_{-i/2}|0\rangle$ ($s=\frac{1}{2}$, $\mathcal{N}$ arbitrary) associated with the solution $\{\frac{i}{2},-\frac{i}{2}\}$ of system (1) is equal to zero. We may say that solution (17) appears to be twice exceptional, both in the a) and b) sense.

Can one now conclude that solution (17) defines no Bethe vector ? We are to demonstrate that such a conclusion would be premature. One cannot exclude beforehand that vector (8) vanishes for some $\{\lambda_j\} = \{\lambda_j^\circ\}$ merely due to the fact that the expression $B_{\lambda_1}\dots B_{\lambda_M}|0\rangle$ itself implicitly contains a numerical factor which equals zero at $\{\lambda_j\}=\{\lambda_j^\circ\}$. After dividing the initial vector by this factor we can, in principle, succeed in extracting a nonzero Bethe vector which corresponds to the exceptional solution $\{\lambda_j^\circ\}$.

We describe this procedure in more rigorous terms. Consider vectors (8) for arbitrary sets of complex numbers $\{\lambda_j\}$, not necessarily being solutions of the Bethe-ansatz equations (1). Then

$$\tau_\lambda |\{\lambda_j\}\rangle = \Lambda(\lambda,\{\lambda_j\})|\{\lambda_j\}\rangle + |\lambda,\{\lambda_j\}\rangle , \qquad (26)$$

where we pick out explicitly an "eigenpart" with the Λ factor given by (10). If $\{\lambda_j\}$ is a solution of system (1) or its modifications, then $|\lambda,\{\lambda_j\}\rangle = 0$ $\forall\lambda$. Now let $\{\lambda_j^\circ\}$ be an exceptional solution obeying $|\{\lambda_j^\circ\}\rangle = 0$. Then choose in the space $\mathbb{C}^M$ of parameters $\{\lambda_j\}$ a sufficiently small neighbourhood of $\{\lambda_j^\circ\}$ and denote by U that (open) domain within it where $|\{\lambda_j\}\rangle \neq 0$. Define in U a normalized vector

$$|\{\lambda_j\}\rangle_* = \frac{|\{\lambda_j\}\rangle}{\sqrt{\langle\{\lambda_j\}|\{\lambda_j\}\rangle}} . \qquad (27)$$

Now, if there exists a limit trajectory $\{\lambda_j\} \longrightarrow \{\lambda_j^\circ\}$ in U such that

$$\frac{|\lambda,\{\lambda_j\}\rangle}{\sqrt{\langle\{\lambda_j\}|\{\lambda_j\}\rangle}} \longrightarrow 0 \qquad (28)$$

for any complex λ, and if a limit

$$|\{\lambda_j\}\rangle_* \longrightarrow |\{\lambda_j^\circ\}\rangle_* \qquad (29)$$

exists, then, evidently,

$$\tau_\lambda |\{\lambda_j^\circ\}\rangle_* = \Lambda(\lambda,\{\lambda_j^\circ\})|\{\lambda_j^\circ\}\rangle_* , \qquad (30)$$

and we end up with a nonzero Bethe vector $|\{\lambda_j^\circ\}\rangle_*$, ${}_*\langle\{\lambda_j^\circ\}|\{\lambda_j^\circ\}\rangle_* = 1$, corresponding to exceptional solution $\{\lambda_j^\circ\}$.

The method just described allows us to determine, in principle, whether there actually exists a nonzero Bethe vector for a given solution of eqs. (1). For instance, the solution $\{\frac{i}{2},-\frac{i}{2}\}$ ($S=\frac{1}{2}$, $\mathcal{N}$ even) admits a trajectory

$$\left\{\frac{i}{2}+\varepsilon,\ -\frac{i}{2}+\varepsilon+2i\varepsilon^{\mathcal{N}}\right\} \xrightarrow[\varepsilon\to 0]{} \left\{\frac{i}{2},-\frac{i}{2}\right\} \qquad (31)$$

which satisfies conditions (28), (29) and generates a certain Bethe vector. For $\mathcal{N}=4$ it looks like

$$\left|\frac{i}{2},-\frac{i}{2}\right\rangle_* = \frac{1}{2}\left(\sigma_1^-\sigma_2^- - \sigma_2^-\sigma_3^- + \sigma_3^-\sigma_4^- - \sigma_1^-\sigma_4^-\right)|0\rangle , \quad \sigma_n^- = \frac{\sigma_n^1 - i\sigma_n^2}{2} . \qquad (32)$$

By contrast, for N odd one can show that no proper trajectories and hence no Bethe vectors exist related to the exceptional solution $\{\frac{i}{2},-\frac{i}{2}\}$.

5. The limit procedure described above relies solely on the algebraic properties of the elements of T-matrix. So, this approach is entirely formulated in terms of algebraic Bethe's ansatz. In practice, however, the use of this procedure entails extremely cumbersome calculations, and the analysis of even the simplest examples from series (17) proves hardly manageable. Therefore, we think it expedient to give one more, purely practical, recipe for constructing Bethe vectors associated with exceptional solutions, which is based on the coordinate version of Bethe's ansatz. Some statements in the present and following sections have the status of hypotheses (these will be mentioned apart in each individual case).

The coordinate Bethe ansatz for $S=\frac{1}{2}$ produces an explicit expression/4,5/ for the Bethe vector related to a solution $\{\lambda_j\}$ of eqs. (1). We propose a generalization of the corresponding formula to arbitrary spin S :

$$|\{\lambda_j\}\rangle = \sum_{(n_j)} \left(\prod_{n=1}^{N} \frac{1}{m_n!} \right) \sum_R C_R \prod_{j=1}^{M} \left(\mu_{R_j}^{n_j - 1} S^-_{n_j} \right) |0\rangle . \tag{33}$$

Here n_j , $1 \le n_1 \le \ldots \le n_M \le N$, are the numbers of lowered spins in the chain; $m_n = \sum_{j=1}^{M} \delta_{n n_j}$ is the number of times the lowering operator S^-_n has been applied at site n . In other words, a number n occurs m_n times in the set (n_j). Obviously, $0 \le m_n \le 2S$. The sum over R includes all permutations from the S_M group,

$$(1, \ldots, M) \xrightarrow{R} (R_1, \ldots, R_M) . \tag{34}$$

The quantities μ_j and C_R depend on the solution set $\{\lambda_j\}$:

$$\mu_j = \frac{\lambda_j - iS}{\lambda_j + iS} , \tag{35}$$

$$\frac{C_{(R_1, \ldots, R_{j-2}, R_j, R_{j-1}, R_{j+1}, \ldots, R_M)}}{C_{(R_1, \ldots, R_M)}} = \frac{\lambda_{R_j} - \lambda_{R_{j-1}} - i}{\lambda_{R_j} - \lambda_{R_{j-1}} + i} . \tag{36}$$

Fixing the value of C_I (I = identity permutation) determines the overall normalization of the vector $|\{\lambda_j\}\rangle$.

Eq. (33) differs from the corresponding $S = \frac{1}{2}$ relation in that μ_j depends on S explicitly, and the $1/m_n!$ factors appear for $m_n > 1$. We have verified that, in accordance with the general scheme of the coordinate Bethe ansatz [4,5], vectors (33) diagonalize the operator of cyclic permutations on the chain as well as spin operators $\vec{S}^2 = \sum_{n=1}^{N} \sum_{a=1}^{3} S^a_n S^a_n$ (the total spin squared) and $S_3 = \sum_{n=1}^{N} S^3_n$ (its projection), being annihilated by the operator $S_+ = \sum_{n=1}^{N} S^+_n$. However, we have traced vectors (33) being eigenvectors of the Hamiltonian H_S only for $S = 1$. In the coordinate (for $S = \frac{1}{2}$) and algebraic (for arbitrary S) Bethe's ansatz, the aforementioned spin restrictions do fix Bethe vectors unambiguously as eigenvectors of the Hamiltonian. It is this analogy that supports our belief that we have found the correct representation (33) for Bethe vectors.

Another our hypothesis concerns the correspondence between the coordinate and algebraic formulations of Bethe's ansatz. We suppose that choosing C_I in the form

$$C_I = \prod_{j=1}^{M} \left[i (\lambda_j + iS)^{N-1} \prod_{k=j+1}^{M} \frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k} \right] \tag{37}$$

leads to the equality

$$|\{\lambda_j\}\rangle = |\{\lambda_j\}\rangle \tag{38}$$

of the coordinate (33) and algebraic (8) Bethe vectors. Moreover, we think this identity holds irrespective of whether a set of complex numbers $\{\lambda_j\}$ is a solution set of eqs. (1). The validity of this hypothesis would imply that for arbitrary set $\{\lambda_j\}$ the coordinate formula (33) is simply an explicit form of the operator expression (8) having nothing to do with the problem of diagonalizing the Hamiltonian.

Proportionality of the algebraic and coordinate Bethe vectors associated with the solution sets $\{\lambda_j\}$ of the Bethe-ansatz equations has already been noticed in the literature [15]. Conjecture (38) about the identical (not only on solutions of eqs. (1)) coincidence of these vectors summarizes empirical facts at our disposal. We have directly confirmed its validity for $N=3$, $M=3$, S arbitrary, as well as in some other particular cases.

6. Returning again to the problem of exceptional solutions of eqs. (1), we notice that since in the coordinate Bethe ansatz one deals with ratios like (35) and (36), it is only natural to rewrite equations (1) themselves also in the form of ratios:

$$\left(\frac{\lambda_j - iS}{\lambda_j + iS} \right)^{N} \prod_{\substack{k=1 \\ k \ne j}}^{M} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} = 1 . \tag{39}$$

Now we are in a position to describe a practical recipe of how to find the limit procedure of sect. 4. In the space $\mathbb{C}^M$ of (arbitrary complex) parameters $\{\lambda_j\}$ we choose a single-parameter trajectory $\{\lambda_j\} \xrightarrow[\varepsilon \to 0]{} \{\lambda_j^\circ\}$ such that the l.h.s. of (39) tends to unity as $\varepsilon \to 0$. On this trajectory one has to construct a vector $|\{\lambda_j\}\rangle$ according to the coordinate formula (33), pick up the leading power of ε as an overall factor and, after dividing the vector by it, take the limit $\varepsilon \to 0$. Finally, one must substitute the resulting vector into eq. (9) to check whether this is an eigenvector of τ_λ with the proper eigenvalue (10).

This procedure requires neither evaluating norms of vectors nor further tedious calculations with them. It readily yields a limit trajectory (if any)and, for relatively small M and N , produces candidate Bethe vectors rapidly enough, leaving deciding vote to the direct test. It is by this method that we have found explicitly the Bethe vector for exceptional solution $\{\lambda_j^\circ\} = \{0, 0, i, -i\}$, $S=1$, $N=4$. The limit trajectory has been chosen as follows,

$$\{\lambda_j\} = \{\varepsilon + \varepsilon^2, \ \varepsilon - \tfrac{3}{4}\varepsilon^2, \ i+\varepsilon, \ -i+\varepsilon\} \xrightarrow[\varepsilon \to 0]{} \{\lambda_j^\circ\}. \tag{40}$$

The explicit form of the corresponding normalized vector is

$$|0, 0, i, -i\rangle_* = \frac{1}{8\sqrt{3}} \Big(|1122\rangle - |2233\rangle + |3344\rangle - |1144\rangle - |1123\rangle \tag{41}$$

$$+ |2234\rangle - |1334\rangle + |1244\rangle + |1134\rangle - |1224\rangle + |1233\rangle - |2344\rangle \Big),$$

where $|1123\rangle = S_1^- S_1^- S_2^- S_3^- |0\rangle$, and so on. The eigenvalue of the operator τ_λ on this vector equals

$$\Lambda = 2(\lambda^2+1)(\lambda^2+3). \tag{42}$$

in agreement with the general formula (10).

To our knowledge, vector (41) is the first and as yet the only example of a Bethe vector associated with a solution set involving equal roots. The problem of constructing the explicit form of Bethe vectors for other exceptional solutions listed in this paper remains still unsolved.

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Об исключительных решениях уравнений анзаца Бете

Исследованы свойства некоторых специальных решений уравнений анзаца Бете. Найдены примеры нетривиальных бетевских векторов, соответствующих решениям с совпадающими значениями спектральных параметров. Дана интерпретация этих случаев в контексте алгебраического и координатного анцаза Бете. Предложено обобщение координатного анзаца на случай XXX-модели произвольного спина.

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On Exceptional Solutions of the Bethe-Ansatz Equations

Some special solutions of the Bethe-ansatz equations are investigated. Examples of nontrivial Bethe vectors associated with coinciding spectral parameters are found. These cases are interpreted in the framework of both algebraic and coordinate Bethe ansatz. A generalization of the coordinate ansatz to the XXX chain with arbitrary spin is proposed.

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