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GAUGE-INVARIANT YANG-MILLS FIELDS AND THE ROLE OF LORENTZ GAUGE CONDITION

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During the last few years much interest was paid to the problem of studying the structure of gauge theories in terms of gauge-invariant (G.I.) objects [1,2]. Our work is devoted to the same problem. We shall start with a short analysis of the Abelian case.

Let us consider two classes of G.I. vector fields

$$B_{\mu}(x|\xi) = A_{\mu}(x) - \partial_{\mu} \int_{\xi}^{x} dz^{\nu} A_{\nu}(z), \tag{1}$$

and

$$B_{\mu}(x|f) = A_{\mu}(x) - \partial_{\mu} \int dy\, f^{\nu}(x-y) A_{\nu}(y), \tag{2}$$

where ξ is a fixed point in the Minkowski space and f^{ν} is a real function that satisfies the condition $\partial^{\nu} f_{\nu}(z) = \delta(z)$.

In the case of a straight-line integration contour in (1) the field $B_{\mu}(x|\xi)$ coincides with the field taken in the Fock gauge [3]*: $(x-\xi)^{\mu} A^{F}_{\mu}(x) = 0$. Due to this fact we shall call these fields the fields of the Fock class. The fields (2) were introduced by Dirac [5] (the Dirac class fields) and studied in refs. [1,2]. It is important to emphasize that the fields (2) coincide with the fields A_{μ} taken in the gauge $f^{\mu}(p) A_{\mu}(p) = 0$.

The field $B_{\mu}(x|\xi)$ (1) in case of the straight line integration contour can be expressed through the tension tensor $F_{\mu\nu}$ by the inversion formula [3,4]

$$B_{\mu}(x|\xi) = \int_{0}^{1} d\alpha\, \alpha (x-\xi)^{\nu} F_{\nu\mu}(\xi + \alpha(x-\xi)). \tag{3}$$

The inversion formula connecting the fields with $F_{\mu\nu}$ has been derived by us for the Dirac class in [6]

$$B_{\mu}(x|f) = \int dy\, f^{\nu}(x-y) F_{\mu\nu}(y). \tag{4}$$

In was shown that the fields (1) and (2) considered on the equations of motion satisfy the Lorentz gauge condition $\partial^{\mu} B_{\mu} = 0$. This condition according to Dirac terminology [7] appears here as

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* This gauge condition has also been considered in [4].

a secondary constraint. Let us discuss some possible choices of the functions $f_\mu(x)$ in (2). In ref./2/ the next form of f has been considered

$$f^{(0)}_\mu(x) = \frac{1}{2} n_\mu \int_{-\infty}^{\infty} da\, \epsilon(a)\, \delta(x - an), \qquad (5)$$

where n_μ is some fixed 4-vector. In p-space formula (5) looks like $f^{(0)}_\mu(p) = -i\,\mathrm{V.P.}\, n_\mu/(np)$. It is obvious that the function $f_\mu(p)$ can be taken in another form: $f^{\pm}_\mu(p) = -i n_\mu/((np) \pm i\epsilon)$, i.e., in the configurational representation

$$f^{\pm}_\mu(x) = \mp n_\mu \int_{-\infty}^{\infty} da\, \theta(\mp a)\, \delta(x - an). \qquad (6)$$

The G.I. field (2) with the choice of f_μ in the form of (5) or (6) coincides with the field A_μ taken in the axial gauge $(nA) = 0$.

With another choice of $f^L_\mu(p) = -i\,\mathrm{V.P.}\, p_\mu/p^2$ the fields (2) coincide with the fields A_μ taken in the Lorentz gauge $p^\mu A_\mu(p) = 0$. The corresponding inversion formula has the form $B^L_\mu(p) = -i\,\mathrm{V.P.}\, \frac{p_\nu}{p^2} F_{\mu\nu}(p)$.

One can get an impression that the right-hand side of this equation is equal to zero due to the equations of motion. But this is not so due to the singular nature of the denominator.

Let us mention that in the Fock class (1) the point ξ is taken to be fixed, that leads to the breakdown of the translation invariance for vector and spinor fields (if we shall consider in (1) ξ as a function of x, i.e., $\xi = \xi(x)$, then the gauge invariance of the vector and spinor field would be broken).

Moreover, the spinor field transforms under the local gauge transformations in a global way by the factor $\exp(ig\Lambda(x/\xi))$. These difficulties can be avoided as $\xi_\mu \to \infty$.

In the present paper we suggest a new way of introduction of the G.I. fields, that is free of these difficulties and allows one to construct such a G.I. spinor propagator $G^{\mathrm{str.line}}(s,y)$ in which the integration is performed along only one distinct contour - a piece of the straight line (str. line) between x and y.

Let us introduce the G.I. field

$$B_\mu(x|\xi;f) = A_\mu(x) - \frac{\partial}{\partial x^\mu}\Big[\int_{\xi(x)}^{x} dz^\nu A_\nu(z) + \int d\tau\, f^\nu(\xi(x) - \tau)\, A_\nu(\tau)\Big], \qquad (7)$$

where for the function $f^\nu(\xi(x) - \tau)$ the same conditions as for the function f^ν in the Dirac class are valid.

The inversion formula for the field (7) has the form

$$B_\mu(x|\xi;f) = \int_0^1 da\, a\,(x-\xi)^\nu F_{\mu\nu}(\xi + a(x-\xi)) + \frac{\partial \xi^\rho(x)}{\partial x^\mu}\Big\{\int_0^1 da\,(1-a)(x-\xi)^\nu F_{\rho\nu}(\xi + a(x-\xi)) + \int dy\, f^\nu(x-y)\, F_{\rho\nu}(y)\Big\}. \qquad (8)$$

The G.I. spinor field corresponding to (7) is introduced with the help of the phase transition $\psi(x) \to \psi(x|\xi;f) = \exp(ig\Lambda(x|\xi;f))\,\psi(x)$, where by $\Lambda(x|\xi;f)$ we denote an expression that appears in square brackets in (7).

For the construction of $G^{\mathrm{str.line}}(x,y)$ let us choose $\xi = (x+y)/2$,

$$f^\nu(\xi - \tau) = -(x-y)^\nu/2 \cdot \int_0^\infty da\, \delta((x+y)/2 + a(x-y)/2 - \tau)\,^*.$$

In this case the G.I. spinor propagator is defined as

$$G^{\mathrm{str.l.}}(x,y) = i\langle 0|\, T\,\psi(x) \exp\Big[ig\int_x^y dz^\nu A_\nu(z)\Big]\, \bar\psi(y)\,|0\rangle, \qquad (9)$$

where the integration is performed over the piece of the straight line that connects the points x and y.

We find that Schwinger's equation for the G.I. Green function has the form

$$\Big[i\gamma^\mu\Big(\frac{\partial}{\partial x^\mu} - g\frac{\delta}{\delta J^\mu(x)} - g\Big[\frac{\partial}{\partial x^\mu}\int_x^y dz^\nu \frac{\delta}{\delta J^\nu(z)}\Big] - igu^\mu(x) - ig\Big[\frac{\partial}{\partial x^\mu}\int_x^y dz^\nu u_\nu(z)\Big]\Big) - m\Big]\, G^{\mathrm{str.l.}}(x,y|J) = -\delta(x-y). \qquad (10)$$

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* With this choice of ξ and f the vector field B_μ coincides with the field A_μ taken in the gauge $(z - \frac{x+y}{2})^\mu A_\mu(z) = 0$, which has been used in/8/ while deriving the dynamical equations for two-particle wave functions.

The second Schwinger equation for $u_\mu(x)$ coincides with the usual one. To the propagator (9) there corresponds the next G.I. mass operator:

$$M(x,y|u) = ig\gamma^\mu \int dy' \langle 0|T\psi(x)\exp[ig\int_x^{y'} dz^\nu A_\nu(z)]\bar{\psi}(y') \times$$

$$\times [u_\mu(x) + \frac{\partial}{\partial x^\mu}\int_x^y dz^\nu u_\nu(z)]|0\rangle G^{-1\,str.\ell.}(y',y|u). \qquad (11)$$

From (10) it follows that in the infrared limit a Fourier transform of (9) would contain a singularity of a simple pole $(m-\hat{p})^{-1}$. So a singularity of the branching point which occurs for the standard propagator in some gauges does not appear here. This result has been obtained by us previously in [9] with the help of another method, namely the method of the functional integration.

We define the generalization of the fields of the Fock class for the nonabelian case by the next formula:

$$B_\mu(x|\xi) = A_\mu(x) - \partial_\mu \int_\xi^x dz^\nu A_\nu(z) - ig\int_0^1 d\alpha\, \alpha[A_\mu(z(\alpha)), A_\nu(z(\alpha))], \qquad (12)$$

where $z(\alpha) = \xi + \alpha(x-\xi)$, $0 \le \alpha \le 1$. The inversion formula for the fields (12) coincides with (3) up to the substitution of QED $F_{\mu\nu}$ to $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} - ig[A_\mu, A_\nu]$.

In papers [1] the G.I. tension tensor $\mathcal{F}_{\mu\nu}(x|C) = U^+(x|C) \times F_{\mu\nu}(x)U(x|C)$ was constructed with the help of the matrix

$$U(x|C) = \mathcal{P}\exp[-ig\int_{-\infty}^x dz^\nu A_\nu(z)]. \qquad (13)$$

But the G.I. vector fields themselves were not considered there. To introduce them we shall perform the gauge transformation $A_\mu \to \overset{\omega}{A}_\mu = \omega A_\mu \omega^{-1} + \frac{i}{g}(\partial_\mu \omega)\omega^{-1}$ with $\omega(x) = U^+(x|C)$.

The field (12) transforms into the G.I. field $\mathcal{B}_\mu(x|\xi;C)$ connected with $\mathcal{F}_{\mu\nu}(x|C)$ by

$$\mathcal{B}_\mu(x|\xi;C) = \int_0^1 d\alpha\, \alpha(x-\xi)^\nu \mathcal{F}_{\mu\nu}(\xi+\alpha(x-\xi)). \qquad (14)$$

In terms of the Mandelstam contour derivatives defined by the relation $\bar{\partial}_\mu U(x|C) = \lim_{\Delta x\to 0}[U(x+\Delta x|C') - U(x|C)]/\Delta x$, where

the contours C and C′ differ from each other by Δx only, the equality $\bar{\partial}_\rho \mathcal{F}_{\mu\nu} + \bar{\partial}_\mu \mathcal{F}_{\nu\rho} + \bar{\partial}_\nu \mathcal{F}_{\rho\nu} = 0$ takes place. With the help of this equality it can be shown that the next formula

$$\mathcal{F}_{\mu\nu}(x|C) = \bar{\partial}_\nu \mathcal{B}_\mu(x|\xi;C) - \bar{\partial}_\mu \mathcal{B}_\nu(x|\xi;C) \qquad (15)$$

is valid.

Thus, we see that the connection of the G.I. strength tensor $\mathcal{F}_{\mu\nu}(x|C)$ and the G.I. vector fields $\mathcal{B}_\mu(x|\xi;C)$ is analogous to the well-known relation in the abelian case up to a substitution of usual derivatives by the Mandelstam contour derivatives.

It is easy to find by making use of equations of motion $\bar{\partial}^\nu \mathcal{F}_{\mu\nu}(x|C) = 0$ that the G.I. field $\mathcal{B}_\mu(x|\xi;C)$ satisfies the condition $\bar{\partial}^\mu \mathcal{B}_\mu(x|\xi;C) = 0$ appearing as a generalisation of the Lorentz gauge condition for a non-Abelian case and has a sense of the secondary constraint.

Now we shall give a generalization of the fields of the Dirac class for a non-Abelian case. We define the field

$$B_\mu(x|f) = A_\mu(x) - \int dy\, f^\nu(x-y) D_\mu A_\nu(y), \qquad (16)$$

where $D_\mu = \frac{\partial}{\partial y^\mu} - ig[A_\mu, \dots]$ is a usual covariant derivative and the function $f^\nu(x-y)$ satisfies the same conditions as in an Abelian case. The inversion formula for the field (16) has the form of (4). By analogy with the previous case let us perform with the help of the gauge transformation $\omega(x) = U^+(x|C)$ the transition to the G.I. variables $\mathcal{B}_\mu(x|f;C)$ and $\mathcal{F}_{\mu\nu}(x|C)$ that are connected by

$$\mathcal{B}_\mu(x|f;C) = \int dy\, f^\nu(x-y)\mathcal{F}_{\mu\nu}(y). \qquad (17)$$

The strength tensor $\mathcal{F}_{\mu\nu}(x|C)$ is expressed through $\mathcal{B}_\mu(x|f;C)$ by the formula analogous to (15). As in the case of the fields (12) it can be shown that the relation $\bar{\partial}^\mu \mathcal{B}_\mu(x|f;C)$ takes place as the secondyry constraint.

The local gauge transition of spinor fields that is consistent with the gauge transitions of the vector fields with $\omega(x) = U^+(x|C)$ leads to the G.I. spinor variables $\Psi(x|C) = \Gamma(U^+(x|C))\psi(x)$, where Γ connects the adjoint and fundamental representations.

It is clear from (14) and (17) that the G.I. fields $\mathcal{B}_\mu(x|\xi;C)$ and $\mathcal{B}_\mu(x|f;C)$ obey the conditions $(x-\xi)^\mu \mathcal{B}_\mu(x|\xi;C) = 0$ and $f^\nu(p)\mathcal{B}_\nu(p|f;C) = 0$ respectively (for f^ν taken along the fixed vector). But for the arbitrary choice of the contour C they do not coincide with the ordinary Yang-Mills fields A_μ in these gauges, and therefore the corresponding Green functions are not connected by the usual reduction formulae with S-matrix.

We shall consider the cases when the G.I. fields $\mathcal{B}_\mu$ in some definite gauges coincide with the usual fields. Let us start with the Fock gauge. We shall perform in (12) the gauge transition with $\omega(\mathbf{x}) = V^+(\mathbf{x}\,|\,\xi)$, where $V(\mathbf{x}\,|\,\xi) = \mathcal{P}\exp[-ig\int_0^1 d\alpha\,(x-\xi)^\nu A_\nu(\xi+\alpha(x-\xi))]$.

In contrast with (13) we have chosen in $V(\mathbf{x}|\xi)$ instead of the infinite contour C of an arbitrary form the piece of the straight line that connects the points ξ and $\mathbf{x}$. In this case we get the fields $\mathcal{B}_\mu(x|\xi)$ that under the gauge transformations transform into $\mathcal{B}^\omega_\mu(x\,|\,\xi) = \omega(\xi)\,\mathcal{B}_\mu(x|\xi)\,\omega^{-1}(\xi)$. We have used that $V^\omega(x|\xi) = \omega(\xi)\,V(x\,|\,\xi)\,\omega^{-1}(x)$. In the Fock gauge $(x-\xi)^\mu A_\mu(x) = 0$ the fields $\mathcal{B}_\mu(\mathbf{x}|\xi)$ coincide with the fields $A_\mu(\mathbf{x})$ because in this case $V(x|\xi) = 1$ and the second and third terms in the right-hand side of (12) disappear. Thus under the local gauge transformations the field $\mathcal{B}_\mu(\mathbf{x}|\xi)$ transforms in a local way only. It is obvious that in the limit $\xi \to \infty$ the gauge invariance restors.

Let us consider the axial gauge. We introduce the matrix

$$V(\mathbf{x}\,|\,\mathbf{f}^{\pm}) = \mathcal{P}(\bar{\mathcal{P}})\exp[-ig\int dy\, f_\nu(x-y)\, A^\nu(y)]\,. \tag{18}$$

where $f^{\pm}_\nu$ is defined by (6), and $\mathcal{P}(\bar{\mathcal{P}})$ are the symbols of α-ordering (anti-α-ordering). Under the gauge transformations $V(x\,|f^{\pm})$ transforms in the next way $V^\omega(\mathbf{x}|f^{\pm}) = \omega(x)\,V(x|f^{\pm})$. The unitary matrix (18) is a particular case of (13). Thus the construction of the G.I. variables is analogous to that we have considered before. The fields $\mathcal{B}_\mu(\mathbf{x}|f^{\pm})$ would coincide with the usual fields taken in the axial gauge. The inversion formula for them is

$$\mathcal{B}_\mu(\mathbf{x}\,|\,\mathbf{f}^{\pm}) = \mp n^\nu \int_{-\infty}^{\infty} d\alpha\,\theta(\mp\alpha)\,\mathcal{F}_{\mu\nu}(\mathbf{x}-\alpha n\,|\,f^{\pm}). \tag{19}$$

In order to generalize the fields (7) to the non-Abelian case with the choice of f_ν in the form $f_\nu(\xi-z) = (x-\xi)_\nu \times \int_0^\infty d\alpha\,\delta(\xi+\alpha(\xi-\mathbf{x})-z)$ we introduce the matrix

$$V(\mathbf{x})|\xi(\mathbf{x})) = \mathcal{P}\exp[-ig\int_{-\infty}^{1} d\alpha\,(\mathbf{x}-\xi)^\nu A_\nu(\xi+\alpha(x-\xi))]$$

that is a particular case of (13), Thus the construction of the non-Abelian G.I. fields is performed following the general approach considered before.

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Скачков Н.Б., Соловцов И.Л., Шевченко О.Ю. E2-85-430
Калибровочно-инвариантные поля Янга - Миллса
и роль условий Лоренца

Построен новый класс калибровочно-инвариантных /К.И./ полей. Получены формулы обращения, выражающие К.И. векторные поля через К.И. тензоры напряженности. Показано, что для введенных К.И. полей в качестве вторичной связи выполняется условие Лоренца и что эти поля в определенных калибровках совпадают с обычными. Получены уравнения Дайсона - Швингера для К.И. спинорного пропагатора. Показано, что в случае КЭД он имеет в инфракрасной области особенность в виде простого полюса $(\hat{p}-m)^{-1}$.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Skachkov N.B., Solovtsov I.L., Shevchenko O.Yu. E2-85-430
Gauge-Invariant Yang-Mills Fields
and the Role of Lorentz Gauge Condition

A new class of gauge-invariant /G.I./ fields is constructed. The inversion formulae that express these fields through the G.I. strength tensor are obtained. It is shown that for the G.I. fields the Lorentz gauge condition appears as the secondary constraint. These fields coincide with the usual ones in some definite gauges. The Dyson-Schwinger equations for the G.I. spinor propagator are derived. It is found that in QED this propagator has a simple pole singularity $(\hat{p}-m)^{-1}$ in the infrared limit.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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