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THE MAGNETIC MOMENT OF AN ELECTRON BETWEEN MIRRORS IN A HOMOGENEOUS MAGNETIC FIELD

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1. INTRODUCTION

The measurement of the anomalous magnetic moment of the electron is one of the basic tests of quantum electrodynamics (QED).

As is well known the free Dirac equation gives a gyromagnetic factor $g_e = 2$ for the electron and the anomaly $a_e=(g_e-2)/2$ arises if one includes radiation corrections. Its measurement in the latest Geonium spectroscopy experiment [1] gives

$$a_e(\text{exp.}) = 1\,159\,652\,193\,(4) \times 10^{-12}. \qquad (1)$$

The current theoretical value is [2]

$$a_e\,(\text{theor.}) = 1\,159\,652\,460\,(127)\,(43) \times 10^{-12} \qquad (2)$$

where the first error arises from the uncertainty in the fine structure constant α and the second is due to theory. The difference of about 2 standard deviations between (1) and (2) is an open question and has stimulated a more detailed study of various higher order corrections to a_e.

In this paper we consider the magnetic moment a_e of an electron moving in a homogeneous magnetic field H between two parallel, infinitely large superconducting mirrors with distance a between them. This is a simple model for the experimental situation in the Geonium spectroscopy experiment [1]. There are two sources for additional contributions to the magnetic moment. The first is the magnetic field which gives corrections of the order [3]

$$a_e(\text{magn.}) \sim \alpha \left(\frac{eH}{m_e^2}\right)^2. \qquad (3)$$

For the used in the experiment magnetic field of some ten kG they are completely negligible.

The second source is the mirrors which modify the photon-propagator entering the radiation corrections. To have an image from the order of these contributions we consider the dimensional constants H, a, and m_e which are present in this case. In the following we use

$$\delta = \frac{1}{a\,m_e} \qquad (4)$$

and

$$\lambda = \frac{\Omega}{\omega} = \frac{a e H}{\pi m_e} \tag{5}$$

as dimensionless combinations of them. Here $\Omega = \frac{eH}{m_e}$ is the cyclotron frequency of the electron in the magnetic field and $\omega = \pi/a$ is the lowest eigen-frequency of a photon between the mirrors. In the experiment /1/ the cyclotron frequencies used are Ω = 51, 89, 142 GHz. The electron moves there inside a cavity ("Penning" trap) of about 1 cm diameter, which gives $\omega \sim$ 90 GHz so that the parameter λ is of an order of one. The other parameter, δ, is very small and for a = 1 cm equals $2.4 \cdot 10^{-10}$.

From these parameters we expect for $\delta \to 0$ an apparatus-dependent contribution

$$a_e\,(\text{app.}) = \alpha \delta f(\lambda), \tag{6}$$

where $f(\lambda)$ is sone function to be calculated. If this function is of an order of one we get a_e (app.) $\sim 10^{-12}$ which is by two orders smaller than the uncertainty in (2) and of the same order as the uncertainty in (1) and as weak and hadronic contributions (they are listed in ref. /2/, for example). So it is of interest to calculate the apparatus-dependent corrections to a_e in order to exclude them from the explanation of the difference between (1) and (2). Furthermore if the function $f(\lambda)$ in (6) is not much smaller than one, these corrections could be measurable.

Here, a remark is in order concerning the role of the magnetic field. At the first sight having in mind the estimation (3) it seems to be negligible so that the apparatus-dependent contribution can be calculated for a free electron. This was done already in ref. /4/ However, in the experiment /1/ mainly the magnetic field ensures that the electron moves inside a cavity ("Penning" trap) whereby the radius $R = 1/\sqrt{eH}$ of the electron orbit is much smaller than the diameter of the cavity. So what is measured is the magnetic moment in a stationary electron state. Now the question arising is: Will this be the same magnetic moment as that of a free electron. The answer we get is no. Namely, from our calculations it turns out that the function $f(\lambda)$ in (6) depends on a nontrivial manner on both H and a. Therefore, it is necessary to calculate the apparatus-dependent contribution to a_e for a stationary electron-state in the magnetic field.

We use the following model. The magnetic field H is directed along the third axis and homogeneous. The mirrors are oriented perpendicular to the first axis, intersecting them at the points $x^1 = \pm \frac{a}{2}$. They are assumed to be plane, infinitely large and superconducting. So we have for the electromagnetic

field the usual superconductor boundary conditions and can use the representation of the photon-propagator developed earlier in ref. /5/. For the electron in a homogeneous magnetic field conserved quantities are (besides the energy, the spin projection to the magnetic field and others the x^1-and x^2 -coordinates of the centre of motion (see, e.g., ref. /7/). We use a representation of the electron state which is the eigenfunction of the operator for the x^1-coordinate of the centre of motion with zero eigenvalue. This state is localized in the x^1-direction and therefore, the electron moves in some region in the middle between the mirrors. In the x^2- as well as in the x^3-directions the position of the electron is not fixed but in those directions the mirrors are infinitely large.

As concerns the dimensionless parameters, δ eq. (4), and λ, eq. (5), we assume λ to be of an order of one and δ to be small. We calculate the leading contribution for $\delta \to 0$.

The paper is organized as follows. In the next section we introduce the formalism for handling the electron in a homogeneous magnetic field. The third section contains the calculation of the mirror-dependent contribution to the magnetic moment. The results are discussed in the last section.

2. THE ELECTRON IN A HOMOGENEOUS MAGNETIC FIELD

In this section we introduce the formalism needed for handling the electron in a homogeneous magnetic field. It is well known and can be found in standard textbooks*. However, the representation used in this paper is not standard so we will explain it here in some detail.

The starting point is the generalized Dirac equation (i.e., including radiative corrections) in an external field $A_\mu^{ext}(x)$ /6/. The corresponding action reads at the one-loop level

$$S = \int d^4x\, d^4y\, \overline{\psi}(x)\, [\,(i\hat{\partial}_x - m + e\hat{A}^{ext}(x))\,\delta(x-y) - \hat{\Sigma}(x,y)]\,\psi(y). \tag{7}$$

Here $\hat{\Sigma}(x,y)$ is the self-energy operator of the electron

$$\hat{\Sigma}(x,y) = -ie^2 \gamma^\mu S^c(x,y) \gamma^\nu D^c_{\mu\nu}(x,y), \tag{8}$$

$S^c(x,y)$ is the electron propagator in the external field $A_\mu^{ext}(x)$ obeying the equation

$$[\,i\hat{\partial}_x - m + e\hat{A}^{ext}(x)\,]S^c(x,y) = -\delta(x-y), \tag{9}$$

and $D^c_{\mu\nu}(x,y)$ is the photon propagator which will be specified in

*A detailed representation is given in the book of Sokolov and Ternov /8/.

the next section. Terms containing the vacuum-polarization are suppressed in eq. (7) and will not be discussed in this paper. The potential $A^{ext}_\mu(x)$ for the magnetic field H is chosen in the form $A^{ext}(x)^\mu = H\delta_{\mu 2} x^1$.

The self-energy operator $\hat\Sigma(x,y)$ contains ultraviolet divergences. They are not influenced by the mirrors. We assume that they are removed in a standard manner.

To handle the action (7) it is convenient to take Fourier transform with respect to x^α $(\alpha = 0,2,3)$ and to expand the x^1-dependence in the Hermite polynomials. For this reason we expand

$$\psi(x) = \int \frac{d^3 p_\alpha}{(2\pi)^3} \sqrt{eH} \sum_{n\geq 0} e^{-ip_\alpha x^\alpha} u_n(\eta^p_x)\, \psi_n(p_\alpha) \tag{10}$$

with $\eta^p_x = \sqrt{eH}\, x^1 + \sqrt{eH}^{-1} p_2$ and $u_n(\eta) = (2^n n! \sqrt{\pi})^{-1/2} H_n(\eta) \exp(-\frac{1}{2}\eta^2)$.

$H_n(\eta)$ are the Hermite polynomials. The functions $u_n(\eta)$ are orthogonal $\int_{-\infty}^{\infty} u_s(\eta)\, u_n(\eta)\, d\eta = \delta_{s,n}$

In terms of $\psi_n(p_\alpha)$ the action (7) reads

$$S = \int \frac{d^3 p_\alpha}{(2\pi)^3} \sqrt{eH} \sum_{s,n\geq 0} \overline{\psi_s(p_\alpha)}\, [\hat K_{s,n}(p_\alpha) - \hat\Sigma_{s,n}(p_\alpha)]\, \psi_n(p_\alpha). \tag{11}$$

where

$$K_{s,n}(p_\alpha) = (\check p - m)\,\delta_{s,n} + \sum_{\mu=\pm 1} i\mu\gamma_{(\mu)} \sqrt{2eH(s + \tfrac{1+\mu}{2})}\; \delta_{s,n-\mu} \tag{12}$$

is the kernel of the free action and

$$\hat\Sigma_{s,n}(p_\alpha) = \int d^3 z_\alpha\, e^{ip_\alpha z^\alpha} \sqrt{eH} \int dx^1 \int dy^1 u_s(\eta^p_x)\, u_n(\eta^p_y)\, \hat\Sigma(x,y)\,\big|_{x^\alpha - y^\alpha = z^\alpha} \tag{13}$$

is the self-energy operator in this representation. Here the following abbreviations are used

$$\check p = p_0\gamma^0 + p_3\gamma^3, \quad \gamma_{(\mu)} = \frac{1}{2}(\gamma^1 + i\mu\gamma^2) \quad (\mu = \pm 1).$$

Furthermore we have assumed that $\hat\Sigma(x,y)$ is translational invariant in the 0,2,3-directions. In fact, translational invariance is broken by the mirrors as well as by $A^{ext}_\mu(x)$ in the first direction only.

The wave function of the free electron in the magnetic field (i.e., without radiative corrections) obeys the equation

$$\sum_{n\geq 0} \hat K_{s,n}(p_\alpha)\, \psi_n(p_\alpha) = 0.$$

It has solutions for $p_0 = \epsilon\sqrt{m^2 + p_3^2 + 2eHN}$. These are the well-known energy levels of the electron in a homogeneous magnetic field whereby N denotes the number of the state and ϵ distinguish between electrons and positrons. For every N and ϵ, this equation has two solutions with different spin projection $(\nu = \pm 1)$ on the magnetic field. For $\epsilon = +1$ they are

$$\psi_s^{\nu=+1}(p_\alpha) = \begin{pmatrix} (p_0+m)\,\delta_{s,N-1} \\ 0 \\ -p_3\,\delta_{s,N-1} \\ i\sqrt{2eHN}\,\delta_{s,N} \end{pmatrix} \frac{1}{\mathfrak{N}}, \quad \psi_s^{\nu=-1}(p_\alpha) = \begin{pmatrix} 0 \\ (p_0+m)\,\delta_{s,N} \\ -i\sqrt{2eHN}\,\delta_{s,N-1} \\ p_3\,\delta_{s,N} \end{pmatrix} \frac{1}{\mathfrak{N}}, \tag{14}$$

$$\mathfrak{N} = \sqrt{2p_0(p_0+m)}.$$

Later on we need the following formulae, which can be checked easily

$$\begin{aligned} \overline{\psi_s^\nu}(1+\mu\sigma^{12})\psi_t^{\nu'} &= \delta_{\nu\nu'}\Big(\frac{m}{p_0} + \mu\nu\Big)\,\delta_{s,N-\frac{1+\mu}{2}}\,\delta_{t,N-\frac{1+\mu}{2}} \\ \overline{\psi_s^\nu}\gamma^0(1+\mu\sigma^{12})\psi_t^{\nu'} &= \delta_{\nu\nu'}\Big(1 + \frac{m}{p_0}\mu\nu\Big)\,\delta_{s,N-\frac{1+\mu}{2}}\,\delta_{t,N-\frac{1+\mu}{2}} \\ \overline{\psi_s^\nu}\gamma_{(\mu)}\psi_t^{\nu'} &= \delta_{\nu\nu'}\big(i\sqrt{2eHN}/2p_0\big)\,\mu\,\delta_{s,N-\frac{1+\mu}{2}}\,\delta_{t,N-\frac{1-\mu}{2}}. \end{aligned} \tag{15}$$

They are valid for $p_3 = 0$ and $\mu = \pm 1$. The conjugated spinor is $\overline{\psi} = \psi^{*T}\gamma^0$.

The solutions $\psi(x)$ given by eqs. (14) and (10) describe an electron state with fixed energy $p_0 = \sqrt{m^2 + p_3^2 + 2eHN}$, spin projection $\nu = \pm 1$ to the magnetic field, impulse component p_3, and x^1-coordinate of the centre of motion whose eigenvalue is p_2/eH. In all of the following considerations we set $p_2 = p_3 = 0$ and get the representation of the electron state which has been explained in the introduction.

For the calculation of the self-energy operator of the electron we need the electron-propagator in the magnetic field. It is, as usual in field theory, the inverse kernel of the free action. We define $S^c_{s,t}(p_\alpha)$ by

$$\sum_{n\geq 0} \hat K_{s,n}(p_\alpha)\, S^c_{n,t}(p_\alpha) = -\delta_{s,t}. \tag{16}$$

This equation can be solved exactly with the result

$$S^{c}_{s,t}(p_\alpha) = \sum_{\mu=\pm 1} \frac{(\tilde{p}+m)\frac{1+\mu\sigma^{12}}{2}\delta_{s,t} + i\mu\gamma_{(\mu)}\sqrt{2eH(s+\frac{1+\mu}{2})}\,\delta_{s,t-\mu}}{-p_0^2 + p_3^2 + m^2 + 2eH(n+\frac{1+\mu}{2}) - i\epsilon} \quad (17)$$

Here $\sigma^{12} = i\gamma^1\gamma^2$. The $i\epsilon$ -prescription in the denominator is taken in such a way that $S^c_{s,t}(p_\alpha)$ is the causal propagator and for vanishing magnetic field coincides with the usual causal propagator. The corresponding representation in coordinate space is

$$S^c(x,y) = \int \frac{d^3p_\alpha}{(2\pi)^3}\sqrt{eH}\sum_{s,t>0} e^{-ip_\alpha(x^\alpha - y^\alpha)} u_s(\eta^p_x)\,u_t(\eta^p_y)\,S^c_{s,t}(p_\alpha)\,. \quad (18)$$

Obviously, $S^c(x,y)$ given by (18), satisfies eq.(9).

Now we consider the corrections to the energy coming from the self-energy operator. For this reason we use the action (11). The corresponding equation of motion for $\psi_s(p_\alpha)$ is

$$\sum_{t\geq 0}[\hat{K}_{s,t}(p_\alpha) - \hat{\Sigma}_{s,t}(p_\alpha)]\psi_t(p_\alpha) = 0. \quad (19)$$

In the sense of standard perturbation theory for quantum mechanical systems (see ref./7/ for example) the correction to the free value for p_0 is given by the perturbation $\Sigma_{s,t}(p_\alpha)$ taken in the unperturbed states $\psi^\nu_s(p_\alpha)$ eq.(14):

$$\Delta p_0 = \sum_{s,t\geq 0} \overline{\psi^\nu_s(p_\alpha)}\,\hat{\Sigma}_{s,t}(p_\alpha)\,\psi^\nu_t(p_\alpha)\,\Big|_{\substack{p_2 = p_3 = 0 \\ p_0 = \sqrt{m^2 + 2eHN}}} \quad (20)$$

The unperturbed energy levels are degenerated with respect to the spin projection and therefore one has to be carefull with perturbation theory. However, the self-energy operator $\hat{\Sigma}_{s,t}(p_\alpha)$ does not mix states with different spin-projection/1/* and therefore trouble does not occur.

The correction Δp_0 to the energy is a function of the number N of the electron state and of the spin projection ν. So, it can be expressed in the form

$$\Delta p_0 = C + \mu_B H \nu a_e\,, \quad (21)$$

where $\mu_B = e/2m_e$ is the Bohr magneton, a_e is the anomaly of the magnetic moment, and C is independent of ν.

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* This can be seen explicitly in the next section, where only averages of the form (15) occur.

3. THE MIRROR-DEPENDENT CONTRIBUTION TO THE MAGNETIC MOMENT

As we have seen in the foregoing section the corrections to the energy levels are given by eq.(20), and especially the corrections to the anomaly of the magnetic moment by eq.(21). They can be written in the form

$$a_e = \frac{1}{2}\sum_{\nu=\pm 1}\frac{\nu}{\mu_B H}\sum_{s,t\geq 0}\overline{\psi^\nu_s}\,\hat{\Sigma}_{s,t}\,\psi^\nu_t\,. \quad (22)$$

Now we must fix the photon-propagator, entering a_e, eq.(22), via the electron self-energy, eq.(8). As is explained in the introduction we have for the electromagnetic field superconductor boundary conditions at $x^1 = \pm a/2$ and use the representation of the photon-propagator with such boundary conditions given in ref./5/ It reads

$$^sD^c_{\mu\nu}(x,y) = D^c_{\mu\nu}(x-y) + \tilde{D}^c_{\mu\nu}(x,y), \quad (23)$$

where $^sD^c_{\mu\nu}(x,y)$ is the full photon-propagator consisting of the free (i.e., without boundary conditions) part

$$D^c_{\mu\nu}(x-y) = g_{\mu\nu}\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{-k^2 - i\epsilon}$$

and of the mirror-contribution

$$\tilde{D}^c_{\mu\nu}(x,y) = \int\frac{d^3k_\alpha}{(2\pi)^3}(k)_{\mu\nu}\sum_{\sigma,\sigma'=\pm 1}\frac{\sigma\sigma'}{4\Gamma\sin\Gamma a} \times \exp[ik_\alpha(x^\alpha - y^\alpha) + i\Gamma(|x^1 - \tfrac{a}{2}\sigma| + |y^1 - \tfrac{a}{2}\sigma'| - \tfrac{a}{2}(1+\sigma\sigma'))]. \quad (24)$$

Here the following notation is used. The index α takes values $\alpha = 0,2,3$, Γ is given by $\Gamma = \sqrt{k_0^2 - k_2^2 - k_3^2 + i\epsilon}$, where the $i\epsilon$ -prescription ensures that $\mathrm{Im}\Gamma > 0$. It originates from the usual $i\epsilon$ - prescription of the causal propagator. The vector-structure is given by

$$(k)_{\mu\nu} = \begin{cases} g_{\mu\nu} - \dfrac{k_\mu k_\nu}{\Gamma^2} & \text{for } \mu,\nu = 0,2,3 \\ 0 & \text{for } \mu = 1 \text{ or } \nu = 1\,. \end{cases} \quad (25)$$

The free part $D^c_{\mu\nu}(x-y)$ of the photon-propagator, if inserting them into eq.(22) gives rise to corrections independent of the mirrors, whereas the mirror-contribution $\tilde{D}^c_{\mu\nu}(x,y)$ to $^sD^c_{\mu\nu}(x,y)$

gives rise to mirror-dependent contributions a_e(mirr.). This is that we are interested in. The corresponding part of the self-energy-operator $\hat{\Sigma}_{s,t}$ eq.(13), is

$$\Sigma_{s,t} = -ie^2 \int d^3z_\alpha \sqrt{eH} \int dx^1 dy^1 e^{ip_\alpha z^\alpha} u_s(\eta_x^p) u_t(\eta_y^p) \times$$

$$\times \tilde{D}^c_{\mu\nu}(x,y) \gamma^\mu S^c(x,y) \gamma^\nu \Big|_{x^\alpha - y^\alpha = z^\alpha}. \tag{26}$$

At this place it is useful to replace the vector-structure $(k)_{\mu\nu}$, eq.(25), of $\tilde{D}^c_{\mu\nu}(x,y)$, eq.(24), by

$$(\tilde{k})_{\mu\nu} = g_{\mu\nu} + g_{\mu 1} g_{\nu 1}(1 + \mathrm{sgn}(x^1 - \frac{a}{2}\sigma)\,\mathrm{sgn}(y^1 - \frac{a}{2}\sigma')). \tag{27}$$

The propagator occurring in this way is denote by $\tilde{\tilde{D}}^c_{\mu\nu}(x,y)$. We state, that the difference between $(k)_{\mu\nu}$, eq.(25), and $(\tilde{k})_{\mu\nu}$, eq.(27), being inserted into the mirror-dependent part of the photon-propagator, gives no contribution to the magnetic moment. The proof of this statement will not be given here because it is rather lengthing than different.

Therefore, inserting $(\tilde{k})_{\mu\nu}$, eq.(27), instead of $(k)_{\mu\nu}$, eq.(25), into $\tilde{D}^c_{\mu\nu}(x,y)$, eq.(24), we get the mirror-dependent contribution to the electron self-energy in the form

$$\Sigma_{s,t}(p_\alpha)' = -ie^2 \int \frac{d^3k_\alpha}{(2\pi)^3} \sum_{\sigma,\sigma'=\pm 1} \frac{\sigma\sigma' e^{-i\Gamma a \frac{1+\sigma\sigma'}{2}}}{4\Gamma \sin\Gamma a} \sum_{n,n'\geq 0} \gamma^\mu S^c_{u,u'}(p_\alpha + k_\alpha) \gamma^\nu$$

$$\times [(g_{\mu\nu} + g_{\mu 1} g_{\nu 1}) F^\sigma_{s,n}(k) F^{\sigma'}_{t,n'}(k) + g_{\mu 1} g_{\nu 1} G^\sigma_{s,n}(k) G^{\sigma'}_{t,n}(k)] \tag{28}$$

with the notation

$$F^\sigma_{s,n}(k) = \sqrt{eH} \int_{-\infty}^{\infty} dx\, u_s(\sqrt{eH}\,x)\, u_n(\sqrt{eH}\,x - \sqrt{eH}^{-1} k_2) \exp(i\Gamma|x - \frac{a}{2}\sigma|) \tag{29}$$

and

$$G^\sigma_{s,n}(k) = \sqrt{eH} \int_{-\infty}^{\infty} dx\, u_s(\sqrt{eH}\,x)\, u_n(\sqrt{eH}\,x - \sqrt{eH}^{-1} k_2)\, \mathrm{sgn}(x - \frac{a}{2}\sigma) \exp(i\Gamma|x - \frac{a}{2}\sigma|). \tag{30}$$

For handling the γ-matrices we use the following formulae ($\mu = \pm 1$):

$$g_{\alpha\beta} \gamma^\alpha (p_0 \gamma^0 + m) \frac{1 + \mu\sigma^{12}}{2} \gamma^\beta = \sum_{\mu'=\pm 1} 2(m - p_0\gamma^0 \frac{1-\mu\mu'}{2}) \frac{1+\mu'\sigma^{12}}{2}$$

$$g_{\alpha 1} g_{\beta 1} \gamma^\alpha (p_0 \gamma^0 + m) \frac{1 + \mu\sigma^{12}}{2} \gamma^\beta = \sum_{\mu'=\pm 1} (p_0\gamma^0 - m) \frac{1-\mu\mu'}{2} \frac{1+\mu'\sigma^{12}}{2}$$

$$g_{\alpha\beta} \gamma^\alpha \mu \gamma_{(\mu)} \gamma^\beta = \sum_{\mu'=\pm 1} (\mu + \mu') \gamma_{(\mu')}$$

$$g_{\alpha 1} g_{\beta 1} \gamma^\alpha \mu \gamma_{(\mu)} \gamma^\beta = \sum_{\mu'=\pm 1} \frac{\mu - \mu'}{2} \gamma_{(\mu')} \tag{31}$$

which can be checked by the standard rules for manipulations with γ-matrices.

Here a remark is in order concerning the spin structure. As it can be seen from eq.(17) all γ-matrices occurring in the self-energy operator $\hat{\Sigma}_{s,t}$ eq.(28), have a structure given by eq.(31) and therefore, having in mind the averages (15), do not mix states with different spin projections ν.

Next we insert $\hat{\Sigma}_{s,t}$ eq.(28), into a_e, eq.(22). Using formulae (31) and (15) we get the following expression for the apparatus-dependent part of the anomalous magnetic moment:

$$a_e(\mathrm{app.}) = \frac{-ie^2 m}{\mu_0 H} \int \frac{d^3k_\alpha}{(2\pi)^3} \sum_{\sigma,\sigma'=\pm 1} \frac{\sigma\sigma' e^{-i\Gamma a \frac{1+\sigma\sigma'}{2}}}{4\Gamma \sin\Gamma a}$$

$$\sum_{\mu,\mu'=\pm 1} \sum_{n,n'\geq 0} [\frac{\mu+\mu'}{2} F^\sigma_{N-\frac{1+\mu'}{2}, n-\frac{1+\mu}{2}}(k) F^{\sigma'}_{N-\frac{1+\mu'}{2}, n-\frac{1+\mu}{2}}(k)$$

$$+ \frac{k_0}{p_0} \frac{\mu-\mu'}{2} \frac{1}{2} (F^\sigma_{N-\frac{1+\mu'}{2}, n-\frac{1+\mu}{2}}(k) F^{\sigma'}_{N-\frac{1+\mu'}{2}, n-\frac{1+\mu}{2}}(k) - G^\sigma_{N-\frac{1+\mu'}{2}, n-\frac{1+\mu}{2}}(k) \times \tag{32}$$

$$\times G^{\sigma'}_{N-\frac{1+\mu'}{2}, n-\frac{1+\mu}{2}}(k))] (-(p_0 + k_0)^2 + k_3^2 + m^2 + 2eHn - i\epsilon)^{-1}.$$

Here we have used the invariance of the integral over k_3 with respect to $k_3 \to -k_3$. This leads to the vanishing of terms linear in k_3 in the numerator of the electron propagator because the other k_3-dependent quantities as Γ and the denominator depend on k_3^2 (for $p_3 = 0$, as we have).

Expression (32) is the complete apparatus-dependent contribution and we are interested in the leading behaviour for $\delta \to 0$. For this reason we first substitute $k_\alpha \to k_\alpha / a$ in the integral in the right-hand side of eq.(32). Thereby the functions $F^\sigma_{s,n}(k)$, eq.(29), and $G^\sigma_{s,n}(k)$, eq.(30), becomes functions of the combinations of parameters $(\delta/2\pi\lambda) = (a^2 2eH)^{-1}$. For $(\delta/2\pi\lambda) \ll 1$ we get

$$G^\sigma_{s,n}(k) = \sigma F^\sigma_{s,n}(k),$$

$$F^\sigma_{s,n}(k) = \exp(-i\Gamma \frac{a}{2}) \{ \delta_{s,n} + \sqrt{\frac{\delta}{2\pi\lambda}} [(k_2 - i\sigma\Gamma) \sqrt{s}\, \delta_{s,n+1}$$

$$-(k_2+i\sigma\Gamma)\sqrt{n}\,\delta_{s,n-1}\;]-\frac{1}{2}\frac{\delta}{2\pi\lambda}[((2s+1)(k_2^2+\Gamma^2)+$$

$$+2ik_2\sigma\Gamma)\,\delta_{s,n}-(k_2-i\sigma\Gamma)^2\sqrt{s(s-1)}\;\delta_{s,n+2} \tag{33}$$

$$-(k_2+i\sigma\Gamma)^2\sqrt{n(n-1)}\;\delta_{s,n-2}\;]+O(\delta^{3/2})],$$

where contributions of an order of $\exp(-2\pi\lambda/\delta)$ and smaller are suppressed. One gets formula (33) simply by a formal expansion of the integrands in eqs.(29) and (30) into powers of $\sqrt{\delta/2\pi\lambda}$. In general, this is not correct because the integrands are not analytic in $\sqrt{\delta/2\pi\lambda}$. However, it can be shown that this leads to additional contributions of an order of $\exp(-2\pi\lambda/\delta)$ which are completely negligible to us. The reason is that the integrands are nonanalytic for $x=\frac{a}{2}\sigma$, i.e., on the surface of the mirrors where the electron wave function is exponentially small due to the assumption of the size $R=\sqrt{eH}^{\,-1}$ of the electron orbit to be much smaller than a.

Using eq.(33) we rewrite eq.(32) in the form

$$a_e\,(\text{app.}) = -ie^2\frac{\delta}{2\pi^3\lambda}\int\frac{d^3k_a}{(2\pi)^3}\sum_{\sigma,\sigma'=\pm1}\frac{\sigma\sigma' e^{i\Gamma\frac{1-\sigma\sigma'}{2}}}{4\Gamma\sin\Gamma}$$

$$\sum_{n\geq 0}[\,2(k_2^2+\Gamma^2)\delta_{N,n}-(k_2^2-\sigma\sigma'\Gamma^2)(\delta_{N,n+1}+\delta_{N,n-1})$$

$$+k_0\,2\pi\lambda\sqrt{1-2\pi\lambda\delta N}^{\,-1}\,\frac{1-\sigma\sigma'}{2}\,(\delta_{N,n-1}-\delta_{N,n+1}))] \tag{34}$$

$$\left(-2k_0\frac{\sqrt{1+2\pi\lambda\delta N}}{2\pi\lambda}+n-N-\frac{\delta}{2\pi\lambda}(k_0^2-k_3^2)-i\epsilon\right)^{-1}.$$

As the next step we rotate the k_0-integration by means of $k_0\to k_4=ik_0$. This is just the usual Wick-rotation in accordance with the $i\epsilon$-prescription in Γ and in the electron propagator. However, because we consider an on-shell matrix-element (i.e., in the physical region, $p_0=\sqrt{m^2+2eHN}>m$) from the Wick-rotation there arises an additional term. It comes from the pole of the electron propagator at

$$k_0=-\frac{1}{\delta}(\sqrt{1+2\pi\lambda\delta N}-\sqrt{1+2\pi\lambda\delta n+\delta^2k_3^2}) \tag{35}$$

lying below the real axis in the complex k_0-plane. This contribution is present for $n=0,1,\ldots,N-1$ and $k_3^2\leq(2\pi\lambda/\delta)(N-n)$. Denoting the rotated contribution by $a_e^{(1)}(\text{app.})$ and that from the pole by $a_e^{(2)}(\text{app.})$ we have

$$a_e\,(\text{app.}) = a_e^{(1)}\,(\text{app.})+a_e^{(2)}\,(\text{app.})\,. \tag{36}$$

First we calculate $a_e^{(1)}(\text{app.})$ and consider there the terms containing $\delta_{N,n\mp1}$ in the numerator of the electron propagator. Because of $n-N=\pm1$ in this case the denominator of the electron-propagator can be expanded in powers of δ. Using than the symmetry with respect to the substitution $k_4\to-k_4$ all terms of an order of δ vanish. So, there remains the contribution containing $\delta_{N,n}$. Here, the denominator cannot be expanded directly. We proceed as follows. Introduce polar coordinates

$$k_4=k\cos\phi\cos\theta\,,\quad k_3=k\sin\phi\cos\theta\,,\quad k_2=k\sin\theta$$

with $\phi=0,\ldots2\pi$ and $\theta=-\frac{\pi}{2}\ldots\frac{\pi}{2}$ and using $\Gamma=i\gamma,\gamma=\sqrt{k_4^2+k_2^2+k_3^2}=k$ we get after the integration over the angles

$$a_e^{(1)}(\text{app.}) = e^2\frac{\delta}{\pi^2\lambda^2}\int_0^\infty\frac{dk\,k^2}{(2\pi)^3}\sum_{\sigma,\sigma'=\pm1}\frac{\sigma\sigma' e^{-k\frac{1-\sigma\sigma'}{2}}}{4\,\mathrm{sh}\,k}$$

$$\times\frac{2\pi^2}{\sqrt{4(1+2\pi\lambda\delta N)+\delta^2k^2\cos^2\theta}}\,.$$

Now it is possible to set $\delta=0$ in the denominator and we arrive at

$$a_e^{(1)}\,(\text{app.}) = d\frac{\delta}{\lambda^2}\,\frac{3\,\zeta(3)}{4\pi^2}\,. \tag{37}$$

It remains to calculate $a_e^{(2)}(\text{app.})$. The k_0-integration gives the residium in the pole and we get

$$a_e^{(2)}(\text{app.}) = -e^2\frac{\delta}{2\pi^3\lambda^3}\int\frac{dk_2dk_3}{(2\pi)^3}\sum_{\sigma,\sigma'=\pm1}\frac{\sigma\sigma' e^{i\Gamma\frac{1-\sigma\sigma'}{2}}}{4\Gamma\sin\Gamma}$$

$$\frac{2\pi\lambda}{-2\sqrt{1+2\pi\lambda\delta(N-1)+\delta^2k_3^2}}[-k_2^2+\sigma\sigma'\Gamma^2-k_0\,2\pi\lambda\sqrt{1-2\pi\lambda\delta N}^{\,-1}\,\frac{1-\sigma\sigma'}{2}], \tag{38}$$

with

$$k_0=\delta^{-1}[\sqrt{1+2\pi\lambda\delta(N-1)+\delta^2\Gamma_3^2}-\sqrt{1+2\pi\lambda\delta N}]$$

and $k_3^2 \leq 2\pi\lambda/\delta$. In order to obtain the leading term for $\delta \to 0$ we set $\delta = 0$ in the integrand in the right-hand side of eq.(38) whereby we have $k_0 = -\pi\lambda$ and $k_3^2 \leq \infty$. Introducing polar coordinates $k_2 = k\cos\phi$, $k_3 = k\sin\phi$ we get after the integration over the angle ϕ

$$a_e^{(2)}(\text{app.}) = -\alpha^2 \frac{\delta}{2\pi^3\lambda^3} \int_0^\infty \frac{dkk}{2\pi} \sum_{\sigma,\sigma'=\pm 1} \frac{\sigma\sigma' e^{i\Gamma\frac{1-\sigma\sigma'}{2}}}{4\Gamma \sin\Gamma} \lambda\pi\left(k^2\left(\frac{1}{2}+\sigma\sigma'\right)-\pi^2\lambda^2\right) \qquad (39)$$

with $\Gamma = \sqrt{\pi^2\lambda^2 - k^2 + i\epsilon}$. Due to the $i\epsilon$-prescription, for $k^2 \leq \pi^2\lambda^2$, Γ takes values above the real axis and for $k^2 > \pi^2\lambda^2$ we have $\Gamma = i\sqrt{k^2 - \pi^2\lambda^2}$. Integrating over Γ instead **k** we rewrite eq.(39) in the form

$$a_e^{(2)}(\text{app.}) = \alpha \frac{\delta}{8\pi^2\lambda^2} \int_{i\infty}^{\pi\lambda} d\Gamma \frac{-\pi^2\lambda^2 + 3\Gamma^2 - e^{i\Gamma}(3\pi^2\lambda^2 - \Gamma^2)}{\sin(\Gamma + i\epsilon)} . \qquad (40)$$

Since we are interested in the corrections to the energy levels rather than in the line width, we have to calculate the real part of this integral. Assuming $\lambda\pi \neq 1,3,5,\ldots$, (the discussion for $\lambda\pi = 1,3,5,\ldots$ is given in the next section) we can integrate by parts and get

$$a_e^{(2)}(\text{app.}) =$$

$$= \alpha\delta \left\{ \frac{1}{\lambda^2} \frac{29\zeta(3)}{16\pi^2} + \frac{1}{4}\ln\left(2\cos^2\frac{\lambda\pi}{2}\right) - \frac{1}{4\lambda^2\pi^2} \int_0^{\lambda\pi} d\Gamma\cdot\Gamma\left(3\ln\left|\operatorname{tg}\frac{\Gamma}{2}\right| + \ln|\sin\Gamma|\right) \right\}.$$

Together with eq.(37) this gives the complete mirror-dependent contribution in the leading order for $\delta \to 0$ to the anomalous magnetic moment. It can be written finally in the form

$$a_e(\text{app.}) = \alpha\delta \left\{ \frac{1}{\lambda^2} \frac{41\zeta(3)}{16\pi^2} + \frac{1}{4}\ln\left(2\cos^2\frac{\lambda\pi}{2}\right) - \frac{1}{4}\int_0^1 dz\cdot z\left(3\ln\left|\operatorname{tg}\frac{\lambda\pi z}{2}\right| + \ln|\sin\lambda\pi z|\right) \right\}. \qquad (41)$$

4. DISCUSSION

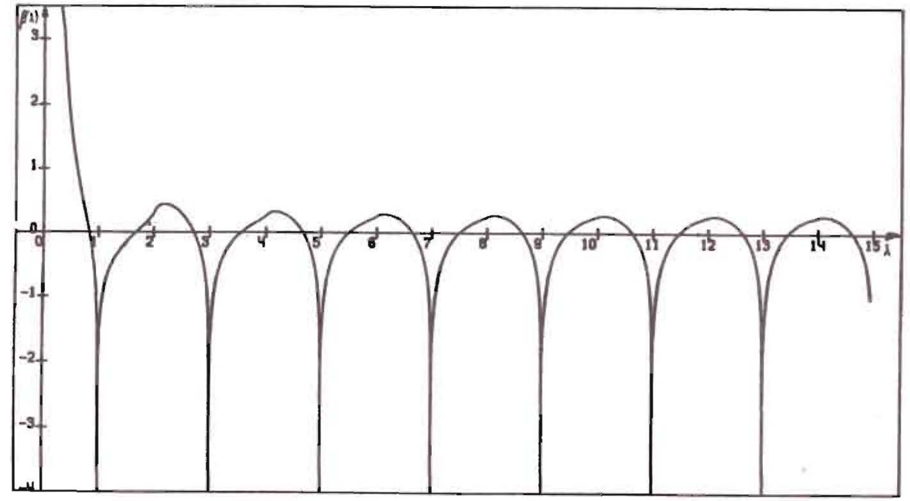
The apparatus-dependent contribution to the anomalous magnetic moment of the electron has in the leading order for $\delta \to 0$ the form

$$a_e(\text{app.}) = \alpha\delta f(\lambda), \qquad (6)$$

whereby the function $f(\lambda)$, calculated within the model as explained in the introduction, has the form (cf. eq.(41))

$$f(\lambda) = \frac{1}{\lambda^2}\frac{41\zeta(3)}{16\pi^2} + \frac{1}{4}\ln\left(2\cos^2\frac{\lambda\pi}{2}\right) - \frac{1}{4}\int_0^1 dz\, z\left(3\ln\left|\operatorname{tg}\frac{\lambda\pi z}{2}\right| + \ln|\sin\lambda\pi z|\right). \qquad (42)$$

This function is shown in the Figure.



Expression (42) for the function $f(\lambda)$ is valid in the following region of parameters. From the expansion of the functions $F^\sigma_{s,n}$, eq.(30), and $G^\sigma_{s,n}$, eq.(31), in powers of $\delta/2\pi\lambda$ we get the restriction

$$\frac{\delta}{2\pi\lambda} \equiv \frac{1}{a^2 2eH} \sim \frac{R^2}{a^2} \ll 1, \qquad (43)$$

where $R = 1/\sqrt{eH}$ is the radius of the electron orbit. This means that the electron is far from the mirrors. The expansion of the electron propagator in section 3 is possible for

$$2\pi\nu\delta N \equiv \frac{2eHN}{m_e^2} \ll 1, \qquad (44)$$

where N is the number of the electron state and means that the electron is nonrelativistic. In the geonium-spectroscopy experiment these restrictions are fulfilled. As can be seen from Eq.(42), the function $f(\lambda)$ diverges logarithmically for $\lambda = 1,3,5,\ldots$. This comes from the fact that for such values of λ the self-energy operator $\Sigma_{s,r}(p_\alpha)$, eq.(13), taken in the states $\psi_s^\nu(p_\alpha)$ eq.(14) (cf. eq.(20), is not small (actually in-finite) and, therefore, eq.(19) cannot be solved in perturbation theory as we have done. We expect that the nonperturbative solution of eq.(19) is finite for $\lambda = 1,3,5,\ldots$ too so that really no divergences occur. However, it remains the fact that the function $f(\lambda)$ shows a resonant behaviour. Having in mind that λ, eq.(5), is the ratio of the cyclotron frequency and the lowest eigenfrequency of stationary photon states between the mirrors we conclude that there appear some kinds of resonances between the cyclotron radiation and stationary photon states.

From our calculations it turns out that the apparatus-dependent contribution to a_e is quite sensitive to changes of the magnetic field H entering into a_e(app.) eq.(6), through the parameter $\lambda \equiv aeH/\pi m_e$, see the Figure. The order of magnitude of a_e(app.) in the experimental region ($a \sim 1$ cm, $H \sim 18\ldots51$ kG, so that $\lambda \sim 0.6\ldots1.6$, $\delta \sim 10^{-10}$) is $a_e(\text{app.}) \sim 10^{-12}$*, which is too small for the explanation of the difference between the current experimental and theoretical values of a_e. However, due to the high experimental precision the apparatus-dependent contribution seems to be measurable by changing the magnetic field H or by using cavities with different diameter.

The most interesting feature is the existence of the above mentioned resonances between the cyclotron radiation and stationary photon states between the mirrors, which take place at $\lambda = 1,3,5,\ldots$. Here, a further theoretical work is necessary and it would be interesting to observe the resonances experimentally.

A remark is in order concerning the geometry. In the present paper, the calculations are performed for the most simple case of two plane parallel mirrors. We think that this case shows up all interesting new features: the dependence of a_e on the magnetic field and the appearance of resonances. One can expect that these will be present in more realistic geometries too, although the precise form of the function $f(\lambda)$ will be changed. In particular, one would expect that the resonances are connected with the zeros of the Bessel functions for a cylindrical cavity.

Finally, we would like to point out that a further work (in particular, calculations for the realistic geometry) is necessary to confirm the conclusion that the difference between the current experimental and theoretical values of a_e cannot be explained by the apparatus-dependent effects.

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* Near the resonances higher values are possible.

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Бордаг М. E2-85-409

Магнитный момент электрона в однородном магнитном поле между зеркалами

Вычисляется аномальный магнитный момент электрона a_e, находящегося в однородном магнитном поле H между двумя параллельными зеркалами, отстоящими на расстоянии a друг от друга. Зеркала вместе с магнитным полем могут служить простой моделью при учете вклада прибора при измерении a_e в эксперименте с геониумом. Оказывается, что зависящий от прибора вклад в a_e нетривиальным образом зависит от обеих величин: H и a. В частности, для таких значений H и a, когда отношение циклотронной частоты электрона к низшей частоте фотонов между зеркалами принимает значения 1,3,5,..., теория возмущений по константе тонкой структуры неприменима и возникают резонансы между циклотронным излучением и стационарными фотонными состояниями между зеркалами.

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Bordag M. E2-85-409

The Magnetic Moment of an Electron between Mirrors in a Homogeneous Magnetic Field

The anomaly a_e of the electron magnetic moment is calculated for stationary electron states in a homogeneous magnetic field H between two parallel mirrors with distance a between them in order to have a simple model for the apparatus-dependent contribution to the measurement of a_e in the Geonium spectroscopy experiment. It turns out that the mirror-dependent contribution to a_e depends in a nontrivial manner on both H and a. Especially for such values of H and a that the quotient of the cyclotron frequency of the electron and the lowest eigenfrequency of the photon between the mirrors take values 1,3,5,..., perturbation theory in the fine structure constant α breaks down and resonances between the cyclotron radiation and the stationary photon states between the mirrors appear.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1985