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исследований
дубна

E2-85-408

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ON THE APPARATUS-DEPENDENCE OF THE ANOMALOUS MAGNETIC MOMENT OF THE ELECTRON

Submitted to "Physics Letters"

1985

1. The precise measurement of the anomalous magnetic moment of the electron is one of the basic tests of quantum electrodynamics. The current experimental value is [1]

$$a_e(\text{exper.}) = 1\ 159\ 652\ 193\ (4)\cdot 10^{-12} \quad (1)$$

and the current theoretical value is [2]

$$a_e(\text{theor.}) = 1\ 159\ 652\ 460\ (127)\ (43)\cdot 10^{-12}. \quad (2)$$

The experimental value (1) is measured in the geonium-spectroscopy experiment[3] where the electron is confined inside a metal cavity by a special configuration of magnetic and electric fields ("Penning" trap). In this paper we consider the problem of the dependence of the anomalous magnetic moment a_e on the cavity (apparatus-dependence). This dependence originates from the modification of the photon propagator due to the boundary conditions imposed by the walls of the cavity. (One of the most well-known effects related to this modification is the Casimir effect). In refs.[4,5,6] these corrections to a_e have been calculated for a free electron between plane parallel mirrors; it has been found that they give a contribution of the order 10^{-12}, which is comparable with the experimental precision as well as with weak and hadronic contributions (the latter are listed, e.g., in ref.[2]).

However, in the geonium-spectroscopy experiment[3], the electron is confined inside the cavity mainly by a homogeneous magnetic field H and what is measured is the magnetic moment of an electron in a stationary orbit but not in a free state. The presence of the magnetic field gives rise to the following contributions. First, there are "pure" (i.e., without reference to the boundary conditions) magnetic contributions to the magnetic moment of the electron, which lead effectively to the correction of the order

$$a_e\,(mag.) \sim \alpha\,(eH/m_e^2)^2. \quad (3)$$

For the magnetic field of some ten kG used in geonium this is completely negligible.

Second, the magnetic field H determines the possible electron states in geonium and the apparatus-dependent contribution can be dependent on H in this way as well.

The main purpose of the present paper is to show that this dependence of a_e on H is very essential; in particular, there exist resonances between the cyclotron radiation and stationary states of photons between the mirrors.We have three dimensional parameters in this problem (H, m_e, and a - the diameter of the cavity). As dimensionless combinations of them, we choose

$$\delta = 1/a m_e \tag{4}$$

and

$$\lambda = \Omega/\omega \equiv a e H/\pi m_e , \tag{5}$$

where $\Omega = eH/m_e$ is the cyclotron frequency of the electron in the magnetic field and $\omega = \pi/a$ is the characteristics photon frequency in the cavity. In the geonium-spectroscopy experiment the cyclotron frequencies used are $\Omega \sim 100\,\mathrm{GHz}$. The diameter of the cavity is $a \sim 1\,\mathrm{cm}$, so that $\omega \sim 90\,\mathrm{GHz}$. Therefore, the first parameter, δ, is very small ($\delta \sim 10^{-10}$) whereas the second, λ, is of an order of one. Now, having in mind the smallness of δ the apparatus-dependent corrections $a_e(app.)$ to a_e can be expressed in the form

$$a_e(app.) = \alpha \delta f(\lambda) \tag{6}$$

up to contributions of an order of δ^2. Here α is the fine structure constant and $f(\lambda)$ is some function containing the dependence on the magnetic field.

We calculate the function $f(\lambda)$ in the most simple model, namely for that of an electron moving in a homogeneous magnetic field H between two plane parallel mirrors. Thereby, the magnetic field is assumed to be directed along the third axis and we consider the mirrors to be ideal (superconducting) and infinitely large. They are oriented perpendicular to the first axis, intersecting it at $x^1 = \pm a/2$.

2. In order to calculate the function $f(\lambda)$ in eq. (6) we have to evaluate the radiative corrections to the energy levels of an electron in a homogeneous magnetic field between mirrors. The appropriate starting point for this is the generalized Dirac

equation /8/ (i.e., including radiative corrections) in an external field $A^{ext}_\mu(x)$. At the one-loop level, it reads

$$[i\hat\partial_x - m + e\hat A^{ext}(x)]\psi(x) = \int d^4y\, \hat\Sigma(x,y)\psi(y). \tag{7}$$

Here

$$\hat\Sigma(x,y) = -ie^2\gamma^\mu S^c(x,y)\gamma^\nu\, {}^sD^c_{\mu\nu}(x,y) \tag{8}$$

is the electron self-energy operator, $S^c(x,y)$ is the electron propagator in the magnetic field obeying

$$[i\hat\partial_x - m + e\hat A^{ext}(x)]S^c(x,y) = -\delta(x-y) \tag{9}$$

and $A^{ext}_\mu(x)$ is the potential of the magnetic field H. We choose it in the form

$$A^{ext}_\mu(x) = H\delta_{\mu 2}x^1 \tag{10}$$

The electron propagator in the external field $S^c(x,y)$, eq. (10), can be constructed in a usual way (see, e.g., ref. /7/)

$$S^c(x,y) = \int dE \sum_n e^{-iE(x^0-y^0)} \frac{\psi_n(x)\overline{\psi_n(y)}}{E - E_n(1+i0)} ,$$

where $\psi_n(x)$ is a complete set of solutions of the Dirac equation $[i\hat\partial_x - m + e\hat A^{ext}(x)]\psi_n(x) = 0$ with energy E_n. Finally, ${}^sD^c_{\mu\nu}(x,y)$ is the photon propagator (in the covariant gauge) between the mirrors. It is derived and discussed extensively in ref. /9/ and reads

$${}^sD^c_{\mu\nu}(x,y) = D^c_{\mu\nu}(x-y) + \tilde D^c_{\mu\nu}(x,y), \tag{11}$$

where $D^c_{\mu\nu}(x-y)$ is the usual free space propagator and $\tilde D^c_{\mu\nu}(x,y)$ is the mirror-depending part:

$$\tilde D^c_{\mu\nu}(x,y) = \int \frac{d^3K_\alpha}{(2\pi)^3}(K)_{\mu\nu}\frac{-i}{2\Gamma}\sum_{i,j=1}^{2} h^{-1}_{ij}\cdot \exp\left[-iK_\alpha(x^\alpha - y^\alpha) + i\Gamma(|x^1-a_i| + |y^1-a_j|)\right]. \tag{12}$$

Here the following notations are used: $\Gamma = \sqrt{K_0^2 - K_2^2 - K_3^2 + i\varepsilon}$,

$(\varepsilon > 0)$, $\alpha = 0, 2, 3$, $a_1 = -a/2$, $a_2 = a/2$, $(K)_{\mu\nu} = g_{\mu\nu} - K_\mu K_\nu / K^2$ for $\mu, \nu = 0, 2, 3$, $(K)_{\mu\nu} = 0$ for $\mu, \nu = 1$, $h^{-1}_{11} = h^{-1}_{22} = i\exp(-i\Gamma a)/2\sin\Gamma a$, and $h^{-1}_{12} = h^{-1}_{21} = -i/2\sin\Gamma a$.

From eq. (7) we get the energy levels of the electron in the following way. Let $\Psi_0(x)$ be a solution of the Dirac equation without radiative corrections in the external field $A^{ext}_\mu(x)$, eq. (10),

$$[i\hat{\partial}_x - m + e\hat{A}^{ext}(x)]\Psi_0(x) = 0 \qquad (13)$$

with fixed values of: i) energy $E_N = \sqrt{m_e^2 + p_3^2 + 2eHN}$, where N is the number of the state, ii) third projection of the momentum, p_3, iii) spin projection, $\nu = \pm 1$ on the direction of the magnetic field, and iv) x^1 - coordinate of the centre of motion with zero eigenvalue[1]. It reads

$$\Psi_0(x) = \int \frac{dp_2\, dp_3}{(2\pi)^2} \sqrt{eH} \sum_{n \geq 0} e^{-ip_\alpha x^\alpha} u_n(\eta) \psi_n^\nu, \quad (\alpha = 0, 2, 3),$$

$$\psi_n^{\nu=-1} = \frac{1}{\mathcal{N}} \begin{pmatrix} (p_0 + m)\delta_{n,N-1} \\ 0 \\ -p_3 \cdot \delta_{n,N-1} \\ i\sqrt{2eHN}\,\delta_{n,N} \end{pmatrix}, \quad \psi_n^{\nu=-1} = \frac{1}{\mathcal{N}} \begin{pmatrix} 0 \\ (p_0 + m)\delta_{n,N} \\ -i\sqrt{2eHN}\,\delta_{n,N-1} \\ p_3\,\delta_{n,N} \end{pmatrix}$$

with $p_0 = E_N$, $\mathcal{N} = \sqrt{2p_0(p_0 + m)}$, $\eta = \sqrt{eH}x^1 - p_2/\sqrt{eH}$, and $u_n(\eta) = (2^n n! \sqrt{\pi})^{-1/2} H_n(\eta) \exp(-\eta^2/2)$, where $H_n(\eta)$ are the Hermite polynomials. This solution describes an electron moving circularly in the middle of the region between the mirrors.

The corrections ΔE to the energy E_N of the electron state $\Psi_0(x)$, resulting from the self-energy operator $\hat{\Sigma}(x, y)$ in eq. (7) is given by

$$\Delta E = \int d^4x\, d^4y\, \overline{\Psi_0(x)}\, \hat{\Sigma}(x, y)\, \Psi_0(y). \qquad (14)$$

[1] The coordinates x^1 and x^2 of the centre of motion are both conserved quantities (i.e., they commute with the Hamiltonian) but do not commute with each other. So, only one of them may have a fixed value. Details can be found, e.g., in ref. /10/.

This is the perturbative solution of eq. (7), $\hat{\Sigma}(x, y)$ being considered as perturbation. Now, the correction ΔE to the energy depends on the spin projection $\nu = \pm 1$. The contribution proportional to ν is connected with the deviation of the gyromagnetic ratio g_e from 2, i.e., with the anomalous magnetic moment $a_e = (g_e - 2)/2$. We express ΔE, eq. (14), in the form

$$\Delta E = G + \mu_B H \nu a_e, \qquad (15)$$

where μ_B is the Bohr magneton, H is the magnetic field, and G is independent of ν. Actually, eq. (15) can be viewed as a definition of a_e. Using eqs. (14) and (15) we express a_e in the form

$$a_e = \frac{1}{2} \sum_{\nu = \pm 1} \frac{\nu}{\mu_B H} \int d^4x\, d^4y\, \overline{\Psi_0(x)}\, \hat{\Sigma}(x, y)\, \Psi_0(y). \qquad (16)$$

The mirror independent part of the photon propagator (see eq. (11)), being inserted into eqs. (8), (16) gives rise, at small H, to the standard Schwinger contribution to the apparatus-independent part of the anomalous magnetic moment (as well as to the "pure" magnetic corrections, eq. (3)).

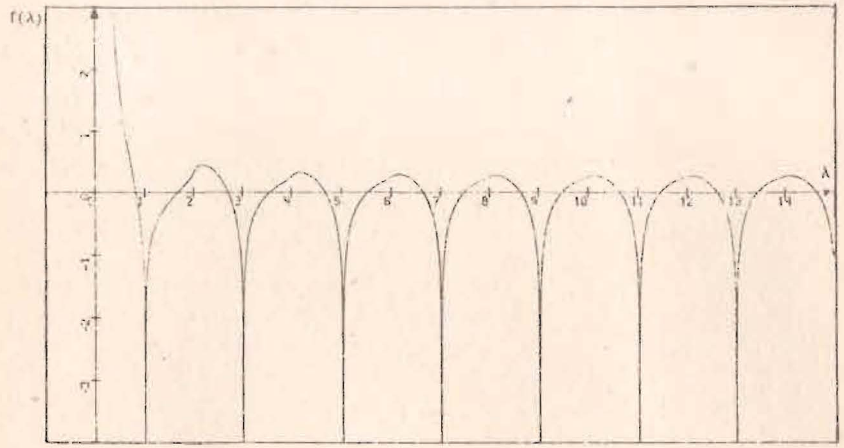
However, we are interested here in the mirror-dependent contributions to a_e and, therefore, insert into eq. (16) the mirror-depending part $\tilde{D}^c_{\mu\nu}(x, y)$, eq. (12), of the photon propagator ${}^S D^c_{\mu\nu}(x, y)$. The further calculations are performed under the following restrictions. First, we calculate the leading contribution for $\delta \to 0$. Second, we assume $\sqrt{\delta/2\pi\lambda} \sim R/a \ll 1$ where $R = \sqrt{eH}^{-1}$ is the radius of the electron orbit in the magnetic field. The third restriction is that $2\pi\lambda\delta N \equiv 2eHN/m_e^2 \ll 1$, which means that the electron is nonrelativistic. In the geonium-spectroscopy experiment all these restrictions are fulfilled. An additional restriction to the validity of our calculations is $\lambda \neq 1, 3, 5, \ldots$, see below.

Under these restrictions we have calculated the function $f(\lambda)$ entering into $a_e(app.)$, eq. (6). The calculations are rather lengthy and their details will be published elsewhere. The result is

$$f(\lambda) = \frac{1}{\lambda^2} \frac{41\zeta(3)}{16\pi^2} + \frac{1}{4}\ln\left(2\cos^2\frac{\lambda\pi}{2}\right) \qquad (17)$$

$$- \frac{1}{4}\int_0^1 dz\, z \left(3\ln\left|tg\frac{\lambda\pi z}{2}\right| + \ln|\sin\lambda\pi z|\right).$$

Here $\zeta(3)$ is the Riemann zeta function at 3. The behaviour of $f(\lambda)$ is shown in the Figure. For $\lambda = 1, 3, 5, \ldots$ it diverges logarithmically. This comes from the fact that for such values of λ the self-energy operator $\hat{\Sigma}(x, y)$ taken in the states $\psi_0(x)$ (see eq. (16)) is not small (actually infinite) and, therefore, eq. (7) cannot be solved in perturbation theory as we have done. We expect that the nonperturbative solution of eq. (7) is finite for $\lambda = 1, 3, 5, \ldots$ so that really no divergences occur. However, it remains the fact that the function $f(\lambda)$ shows a resonant behaviour. Having in mind that λ, eq. (5), is the ratio of the cyclotron frequency and the lowest eigenfrequency of stationary photon states between the mirrors we conclude that there appear some kinds of resonances between the cyclotron radiation and stationary photon states.



3. From our calculations it turns out that the apparatus-dependent contribution to a_e is quite sensitive to changes of the magnetic field H entering into $a_e(app.)$, eq. (6), through the parameter $\lambda = aeH/\pi m_e$, see the Figure. The order of magnitude of $a_e(app.)$ in the experimental region ($a \sim 1cm$, $H = 18, \ldots, 51 KG$, so that $\lambda \sim 0,6, \ldots, 1,6$; $\delta \sim 10^{-10}$) is $a_e(app.) \sim 10^{-12}$, which is too small for the explanation of the difference between the current experimental and theoretical values of a_e. However, due to the high experimental precision the apparatus-dependent contribution seems to be measurable by changing the magnetic field H or by using cavities with different diameter.

The most interesting feature is the existence of the above-mentioned resonances between the cyclotron radiation and stationary photon states between the mirrors, which take place at $\lambda = 1, 3, 5 \ldots$. Here, a further theoretical work is necessary and it would be interesting to observe the resonances experimentally.

A remark is in order concerning the geometry. In the present paper, the calculations are performed for the most simple case of two plane parallel mirrors. We think that this case shows up all interesting essentially new features: the dependence of a_e on the magnetic field and the appearance of the resonances. One can expect that these will be present in more realistic geometries too, although the precise form of the function $f(\lambda)$ will be changed. In particular, one would expect that the resonances are connected with the zeros of the Bessel functinns for a cylindrical cavity.

Finally, we would like to point out that a further work (in particular, calculations for the realistic geometry) is necessary to confirm the conclusion that the difference between the current experimental and theoretical values of a_e cannot be explained by the apparatus-dependent effects.

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Received by Publishing Department
on May 30, 1985.

Бордаг М. E2-85-408
О зависимости от прибора
аномального магнитного момента электрона

Вычисляется вклад прибора в аномальный магнитный момент электрона a_e в магнитном поле H, имеющемся в эксперименте по измерению a_e. В качестве модели для прибора взяты два параллельных плоских зеркала, расположенные на расстоянии a друг от друга. Оказывается, что a_e сильно зависит от обеих величин: H и a. В частности, для $\lambda \equiv aeH/\pi m_e = 1,3,5,...$ возникают резонансы между циклотронным излучением и стационарными состояниями фотонов между зеркалами.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Bordag M. E2-85-408
On the Apparatus-Dependence
of the Anomalous Magnetic Moment of the Electron

We show that the apparatus-dependent contribution to the anomalous magnetic moment of the electron measured in the geonium-spectroscopy experiment depends quite sensitively on the magnetic field H used in the experiment. Especially, for $\lambda \equiv aeH/\pi m_e = 1,3,5,...$, where a is the distance between the two parallel mirrors used as a model for the apparatus, resonances appear. The possibility of the experimental observation of these phenomena is discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985