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IRREDUCIBLE *-REPRESENTATIONS OF LIE SUPERALGEBRAS B(0,n) WITH FINITE-DEGENERATED VACUUM. Results for B(0,1)

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1. Introduction and summary of results

In the first part of this study [1] referred hereafter to as I, we described a general method for constructing irreducible $*$-representations of Lie superalgebras B(0,n) starting with given families $\{\Omega^{(n)}\}$ of linear representations of these algebras in terms of linear differential operators on $\mathbb{C}^N$-values C^∞-functions. In the present paper, which is a direct continuation of I, the construction is performed explicitly for the family $\{\Omega^{(1)}\}=\{\Omega_\varkappa : \varkappa\in\mathbb{R}\}$ of linear representations of B(0,1) on the vector space $C^\infty(\mathbb{R}^+)\otimes\mathbb{C}^2$. The notation and basic definitions introduced in the first part are used without reproducing them here.

The results can be summarized as follows:

(i) A one-parameter family Π of non-equivalent irreducible $*$-representations of B(0,1) in terms of unbounded operators on $L^2(\mathbb{R}^+)\otimes\mathbb{C}^2$ was obtained. Each representation $\pi=\pi_\varkappa\in\Pi$ equals $\Omega_\varkappa\restriction\mathcal{D}_\varkappa$ for some $\varkappa\in(-1/2,\infty)\setminus\{0\}$, $\mathcal{D}_\varkappa$ being a $\Omega_\varkappa$-invariant subspace of $C^\infty(\mathbb{R}^+)\otimes\mathbb{C}^2$, $\overline{\mathcal{D}_\varkappa}=L^2(\mathbb{R}^+)\otimes\mathbb{C}^2$; in addition, $\pi_\varkappa$ has non-degenerated vacuum.

(ii) The family Π is complete in the following sense: if $\varkappa'\in\mathbb{R}\setminus\{0\}$, $\mathcal{D}'$ is a subspace of $C^\infty(\mathbb{R}^+)\otimes\mathbb{C}^2$ whose intersection with the vacuum subspace is non-trivial and if $\pi'=\Omega_{\varkappa'}\restriction\mathcal{D}'$ is an irreducible $*$-representation of B(0,1), then π' is equivalent to some $\pi\in\Pi$.

(iii) For each $\pi\in\Pi$ and all elements $z\in B(0,1)$ satisfying $z=z^*$ the operators $\pi(z)$ are essentially self-adjoint on $\mathcal{D}$. Particularly, this holds for $z=ix_{jk}, \bar{\epsilon}y_k$, where x_{jk}, $y_k, k=\pm1$, form the Racah basis of B(0,1).

(iv) If π is restricted to the even subalgebra $sp(2,\mathbb{R})\sim sl(2,\mathbb{R})$ of the unique real form osp(1,2) of B(0,1), a skew-symmetric representation τ of $sl(2,\mathbb{R})$ is obtained that equals direct sum of two irreducible skew-symmetric representations of $sl(2,\mathbb{R})$ on $L^2(\mathbb{R}^+)$. Each of these representations is integrable to a unitary irreducible representation of the universal covering group of $SL(2,\mathbb{R})$.

The problem was recently considered (starting with a family of linear representations of B(0,1) equivalent to our $\{\Omega^{(1)}\}$) by Mukunda et al.[2]. These authors stressed the importance of specifying carefully domains of unbounded Hilbert-space operators which arise from "formal" differential operators $\Omega(z)$, $z\in B(0,1)$. They also corrected some erroneous conclusions of an earlier study [3].

The "Schrödinger description" of Ref.2 is in fact identical with our family Π and so are the representations of osp(1,2) used by D'Hoker nad Vinet in their study of dynamical symmetries of Dirac monopole /4/. On the other hand, the results (ii) – (iv) are new, as well as the approach we used. Its advantages become apparent especially when the case n=2 (and possibly n>2) is considered for which the subalgebra of sp(2n,$\mathbb{C}$) that leaves invariant the vacuum subspace is non-trivial (cf.I Lemma 3.1 and Proposition 3.4). Construction of irreducible $*$-representations of B(0,2) based on this approach is in progress.

2. Specific features of the case n=1

The operators $\Omega^{(1)}(z)\equiv\Omega_{\varkappa}(z)$, $z\in B(0,1)$, $\varkappa\in\mathbb{R}$, are ordinary differential operators on $C^{\infty}(\mathbb{R}^{+})\otimes\mathbb{C}^{2}$. The explicit formulae for

$$\tilde{X}_{jk}:=\Omega_{\varkappa}(x_{jk}),\ \tilde{Y}_{1}:=\Omega_{\varkappa}(y_{1})$$

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$$\tilde{X}_{-1-1}=ir^{2},\ \tilde{X}_{1-1}=r\frac{d}{dr}+\frac{1}{2},\ \tilde{X}_{11}=i\left(-\frac{d^{2}}{dr^{2}}+\frac{\varkappa^{2}}{r^{2}}-\frac{\varkappa}{r^{2}}\otimes\sigma_{3}\right)$$
$$\tilde{Y}_{-1}=\epsilon r\otimes\sigma_{2},\ \tilde{Y}_{1}=\bar{\epsilon}\left(\frac{d}{dr}\otimes\sigma_{2}-i\frac{\varkappa}{r}\otimes\sigma_{1}\right). \tag{1}$$

Here $\epsilon:=\exp(i\pi/4)$ and $\sigma_1,\sigma_2,\sigma_3$ are the Pauli matrices. The single Casimir element of B(0,1)

$$k_{2}:=2x_{1-1}^{2}-\{x_{11},x_{-1-1}\}+[y_{1},y_{-1}]$$

is represented by

$$\tilde{K}_{2}:=\Omega_{\varkappa}(k_{2})=(2\varkappa^{2}-\tfrac{1}{2})\ I\ .$$

Let us denote the components of any $\phi\in C^{\infty}(\mathbb{R}^{+})\otimes\mathbb{C}^{2}$ by $(\phi)_{\alpha}$, $\alpha=\pm1$:

$$\phi=\{(\phi)_{+},(\phi)_{-}\},\ (\phi)_{\alpha}\in C^{\infty}(\mathbb{R}^{+}),$$

and introduce matrices

$$\sigma^{(\alpha)}:=\tfrac{1}{2}\,(\sigma_{1}-i\sigma_{2}),\ \alpha=\pm1.$$

Then one has

$$(\sigma^{(\alpha)}\phi)_{\beta}=\delta_{\alpha+\beta}(\phi)_{\alpha},\quad(\sigma_{3}\phi)_{\beta}=\beta(\phi)_{\beta}\ ,\quad\alpha,\beta=\pm1\ . \tag{2}$$

As has been shown in I, the relevant operators are

$$\tilde{a}:=\Omega_{\varkappa}(a_{1})=2^{-1/2}(\tilde{Y}_{1}-i\tilde{Y}_{-1}),\ \tilde{n}:=\Omega_{\varkappa}(b_{1-1})=\tfrac{1}{2}\{\tilde{a},\tilde{a}^{*}\};$$

substituting from (1) yields

$$\tilde{a}=\epsilon 2^{-1/2}\sum_{\alpha=\pm1}\alpha\left(\frac{d}{dr}+r-\frac{\alpha\varkappa}{r}\right)\otimes\sigma^{(\alpha)}, \tag{3}$$

$$\tilde{n}=\tfrac{1}{2}\left(-\frac{d^{2}}{dr^{2}}+r^{2}+\frac{\varkappa^{2}}{r^{2}}-\frac{\varkappa}{r^{2}}\otimes\sigma_{3}\right). \tag{4}$$

The condition $\tilde{a}\Psi=0$, which determines the vacuum subspace $V_{\Omega_{\varkappa}}\equiv V_{\varkappa}$, then becomes

$$\left(\frac{d}{dr}+r-\frac{\alpha\varkappa}{r}\right)(\Psi)_{\alpha}=0,\ \alpha=\pm1.$$

The solution $\exp(-r^{2}/2)r^{\alpha\varkappa}$ belongs to $C^{\infty}(\mathbb{R}^{+})$ for both $\alpha=\pm1$ and thus

$$V_{\varkappa}=\{\Psi_{\varkappa}^{(+)},\Psi_{\varkappa}^{(-)}\}_{\text{lin}},\ (\Psi_{\varkappa}^{(\beta)})_{\alpha}(r):=\delta_{\alpha-\beta}\exp(-r^{2}/2)r^{\alpha\varkappa}, \tag{5}$$

i.e., $\dim V_{\varkappa}=2$. Hence, the finite degeneracy of vacuum (FDV condition) is automatically fulfilled.

Of the conditions (I-3.3,3.4a,b) which determine the vector Ψ_{λ} only the first one, i.e., $\tilde{n}\Psi_{\lambda}=\nu\Psi_{\lambda}$, makes sense for n=1. By using (4), one finds

$$\tilde{n}\Psi_{\varkappa}^{(\beta)}=(\beta\varkappa+\tfrac{1}{2})\Psi_{\varkappa}^{(\beta)} \tag{6}$$

and thus a vector $\Psi\in V_{\varkappa}$ fulfils $\tilde{n}\Psi=\nu\Psi$ for some $\nu\geq0$ iff $\Psi=\Psi_{\varkappa}^{(+)}$, $\varkappa\geq-\frac{1}{2}$ or $\Psi=\Psi_{\varkappa}^{(-)}$, $\varkappa\leq\frac{1}{2}$. As no highest weight occurs for n=1 and by Eq.(6) the eigenvalue ν depends on $\varkappa$ only, we hereafter write $\Psi_{\varkappa}$ instead of Ψ_{λ}. Consequently, the necessary conditions of I-Sect.3 are simplified as follows:

<u>2.1 Proposition</u>: <u>Suppose that for a given $\varkappa\in\mathbb{R}$ there is a $\Omega_{\varkappa}$-invariant subspace $\mathcal{D}_{\varkappa}\subset C^{\infty}(\mathbb{R}^{+})\otimes\mathbb{C}^{2}$ fulfilling $\mathcal{D}_{\varkappa}\cap V_{\varkappa}\neq\{0\}$ and let $\Omega_{\varkappa}\restriction\mathcal{D}_{\varkappa}$ be a $*$-representation. Then</u>

$$\mathcal{D}_{\varkappa}=\mathcal{D}(\Psi_{\varkappa})=\{(\tilde{a}^{*})^{k}\Psi_{\varkappa}:\ k=0,1,\ldots\}_{\text{lin}} \tag{7a}$$

where

$$\Psi_{\varkappa}:=\begin{cases}\Psi_{\varkappa}^{(+)} & \varkappa\in(-\frac{1}{2},\infty)\setminus\{0\}\\ \Psi_{\varkappa}^{(-)} & \varkappa\in(-\infty,\frac{1}{2})\setminus\{0\}.\end{cases} \tag{7b}$$

Proof: With the help of $\{\tilde{a},\tilde{a}^{*}\}=2\tilde{n}$ one easily verifies by induction that for each monomial $\tilde M$ in $\tilde a, \tilde a^\#$ holds $\tilde M \psi_\varkappa \in \{(a^\#)^k \psi_\varkappa : k=0,1,..\}_{lin}$. The reason for excluding the values $\varkappa=0$ and $\varkappa=\mp\frac{1}{2}$ for $\psi_\varkappa=\psi_\varkappa^{(\pm)}$ is as follows: for $\varkappa=0$ the operators $\tilde a, \tilde a^\#$ equal direct sum of two identical Schrödinger representations of the canonical commutation relations, which is the case we are not interested in (see the discussion in I-Sect.2). If $\varkappa=\mp 1/2$, then by Eq.(6) one has for any scalar product on $\mathcal{D}$ under which $\Omega_\varkappa \restriction \mathcal{D}$ becomes a $*$-representation: $\tilde a^\# \psi_\varkappa^{(\pm)}=0$ and, in view of $\tilde a \psi_\varkappa^{(\pm)}=0$, the representation would be trivial. ∎

3. Construction of irreducible $*$-representations

According to Eq.(7a), the sought domain $\mathcal{D}_\varkappa$ is spanned by functions

$$\phi_k := (a^\#)^k \psi_\varkappa . \tag{8}$$

For getting the functional dependence $r \mapsto \phi_k(r)$ notice that Eq.(3) yields for any $\phi = \{(\phi)_+, (\phi)_-\} \in C^\infty(\mathbb{R}^+) \otimes \mathbb{C}^2$

$$(\tilde a^\# \phi)_+ = -\bar\epsilon 2^{-1/2}\left(\frac{d}{dr} - r + \frac{\varkappa}{r}\right)(\phi)_-$$
$$(\tilde a^\# \phi)_- = \bar\epsilon 2^{-1/2}\left(\frac{d}{dr} - r - \frac{\varkappa}{r}\right)(\phi)_+ . \tag{9a}$$

On the other hand, by applying functional relations for the Laguerre polynomials (Ref.6, §8.971) to functions

$$f_k^{(\alpha)}(r) := c_k^{(\alpha)} r^{\alpha+1/2} \exp(-r^2/2) L_k^{(\alpha)}(r^2), \quad c_k^{(\alpha)} := \left(\frac{2k!}{\Gamma(\alpha+k+1)}\right)^{1/2}, \tag{10}$$

one finds

$$\left(\frac{d}{dr} - \frac{\alpha+1/2}{r} - r\right) f_k^{(\alpha)} = -2(\alpha+k+1)^{1/2} f_k^{(\alpha+1)}$$
$$\left(\frac{d}{dr} + \frac{\alpha+1/2}{r} - r\right) f_k^{(\alpha+1)} = 2(k+1)^{1/2} f_{k+1}^{(\alpha)} . \tag{9b}$$

Consider the first of alternatives (7b): $\psi_\varkappa = \psi_\varkappa^{(+)}, \varkappa \in (-\frac{1}{2}, \infty) \setminus \{0\}$. Comparing (5) to (10) gives

$$\psi_\varkappa \equiv \phi_0 = \left(\frac{\Gamma(\varkappa+1/2)}{2}\right)^{1/2} F_0, \quad F_0 := \{f_0^{(\varkappa-1/2)}, 0\};$$

Eqs.(9a,b) then yield by induction

$$\phi_k = (\bar\epsilon\sqrt{2})^k \left(\frac{1}{2}\Gamma\left(\varkappa + \frac{1}{2} + \left[\frac{k+1}{2}\right]\right)\left[\frac{k}{2}\right]!\right)^{1/2} F_k \;{}^{+)}, \quad k=0,1,\ldots , \tag{11}$$

+) For any $x \in \mathbb{R}$ $[x]$ denotes the largest integer that does not exceed x.

with

$$F_{2k} \equiv F_{2k}^{(\varkappa)} := \{f_k^{(\varkappa-1/2)}, 0\}, \quad F_{2k+1} \equiv F_{2k+1}^{(\varkappa)} := \{0, -f_k^{(\varkappa+1/2)}\}. \tag{12}$$

Clearly, the functions F_k also span $\mathcal{D}(\psi_\varkappa^{(+)})$ and we shall see when considering the $*$-condition that working with F_k instead of ϕ_k has some technical advantages.

By $\tilde a^\# \phi_k = \phi_{k+1}$ and (11) one has

$$\tilde a^\# F_k = \bar\epsilon d_{k+1} F_{k+1} \tag{13a}$$

with

$$d_k \equiv d_k(\varkappa) := \left(k + (1-(-1)^k)\varkappa\right)^{1/2} . \tag{13b}$$

For $\tilde a \phi_k$ we get by induction with the help of $\{\tilde a, \tilde a^\#\} = 2\tilde n$, $[\tilde n, (\tilde a^\#)^l] = l(\tilde a^\#)^l$ and Eq.(6)

$$\tilde a \phi_{2k} = 2k\phi_{2k-1}, \quad \tilde a \phi_{2k+1} = (2k+2\varkappa+1)\phi_{2k},$$

whence

$$\tilde a F_k = \epsilon d_k F_{k-1} . \tag{13c}$$

Now when the action of operators $\tilde a$, $\tilde a^\#$ on the vectors spanning the domain $\mathcal{D}(\psi_\varkappa^{(+)})$ is known, algebraic irreducibility of $\Omega_\varkappa \restriction \mathcal{D}(\psi_\varkappa^{(+)})$ can easily be proven. We have to verify that to each $\phi = \alpha_0 F_0 + \ldots + \alpha_K F_K$, $\alpha_K \neq 0$, there exist operators $T, S \in \mathcal{U}(\tilde a, \tilde a^\#)$ such that $\phi = TF_0$, $F_0 = S\phi$. Existence of T directly follows by Eqs.(8,11); further (13c) yields $\tilde a^K \phi = \alpha_K \epsilon^K d_K d_{K-1} \ldots d_1 F_0$ and thus $S \sim \tilde a^K$ (notice that $d_k \neq 0$, k=1,2,..).

Next we have to introduce a scalar product $(.,.)$ on $\mathcal{D}(\psi_\varkappa^{(+)})$ such that the condition (I-3.9) holds. As has been argued in I-Sect.3, such a scalar product must fulfil

$$(F_k, F_l) = t_k \delta_{k-l}, \quad t_k > 0, \quad k,l=0,1,\ldots .$$

Now the condition (I-3.9) is equivalent to

$$(\tilde a F_k, F_l) = (F_k, \tilde a^\# F_l), \quad k,l=0,1,\ldots . \tag{14}$$

By Eqs.(13) these conditions become

$$d_k t_{k-1} \delta_{k-1-l} = d_k t_k \delta_{k-1-l}, \text{ i.e. } t_{k-1} = t_k, \quad k=1,2,\ldots .$$

We thus see that Eq.(14) is satisfied iff there is a positive t such that

$$(F_k, F_l) = (F_k, F_l)_t := t\delta_{k-l}, \quad k,l=0,1,\ldots . \tag{15}$$

Let $\mathcal{H}_t$ be the Hilbert space obtained by completing $\mathcal{D}(\psi^{(+)}_\varkappa)$ under $(.,.)_t$ and let $\pi^{(t)}_\varkappa$ be the representation $\Omega_\varkappa \restriction \mathcal{D}(\psi^{(+)}_\varkappa)$ regarded as a Hilbert-space representation on $\mathcal{H}_t$; especially we set $\pi_\varkappa = \pi^{(1)}_\varkappa$. Clearly, one has

$$\pi^{(t)}_\varkappa = V_t \pi_\varkappa V_t^{-1}, \quad t>0,$$

V_t being the unitary map of $\mathcal{H}_1$ onto $\mathcal{H}_t$ given by

$$V_t F_k := t^{-1/2} F_k .$$

Hence, it is sufficient to consider the case $t=1$ only. The functions $f^{(\alpha)}_k$, $k=0,1,\ldots$, form an orthonormal basis in $L^2(\mathbb{R}^+)$ for each $\alpha>-1$, which implies that $\{F_k\}^{\infty}_{k=0}$ is an orthonormal basis in $L^2(\mathbb{R}^+)\otimes \mathbb{C}^2$ for each $\varkappa>-1/2$. Thus $\mathcal{H}_1$ can be chosen as $L^2(\mathbb{R}^+)\otimes\mathbb{C}^2$, this choice being unique up to unitary maps.

The above considerations concerning the choice $\psi_\varkappa := \psi^{(+)}_\varkappa$ can be concluded as follows: for each $\varkappa>-1/2, \varkappa\neq 0$, the linear representation $\Omega_\varkappa$ yields an irreducible representation $\pi_\varkappa$ on $L^2(\mathbb{R}^+)\otimes\mathbb{C}^2$ with domain $\{F^{(\varkappa)}_k : k=0,1,\ldots\}_{lin}$. The representation $\pi_\varkappa$ satisfies the $*$-condition (14) and is determined uniquely up to unitary equivalence.

For the other choice $\psi_\varkappa := \psi^{(-)}_\varkappa, \varkappa>1/2, \varkappa\neq 0$, everything can be repeated step by step. By defining

$$G_{2k} \equiv G^{(\varkappa)}_{2k} := \{0, f_k^{(-\varkappa-1/2)}\}, \quad G_{2k+1} \equiv G^{(\varkappa)}_{2k+1} := \{f_k^{(-\varkappa+1/2)}, 0\}, \tag{16}$$

$$d^{(-)}_k := d_k(-\varkappa),$$

we find that for each $\varkappa<1/2$ the relations

$$\tilde{a} G_k = \epsilon d^{(-)}_k G_{k-1}, \quad \tilde{a}^{\#} G_k = \bar{\epsilon} d^{(-)}_{k+1} G_{k+1} \tag{17}$$

determine an irreducible representation $\rho_\varkappa$ of $B(0,1)$ on $L^2(\mathbb{R}^+)\otimes\mathbb{C}^2$ with domain $\{G_k : k=0,1,\ldots\}_{lin}$. The $*$-condition (14) is satisfied and any other Hilbert-space representation with these properties obtained from $\Omega_\varkappa \restriction \mathcal{D}(\psi^{(-)}_\varkappa)$ is unitarily equivalent to $\rho_\varkappa$.

However, the representations $\rho_\varkappa$ are in fact of no interest as for each $\varkappa<1/2$ the representations $\rho_\varkappa, \pi_{-\varkappa}$ are unitarily equivalent:

$$\rho_\varkappa = U \pi_{-\varkappa} U^{-1}, \quad U := -iI(L^2(\mathbb{R}^+))\otimes\sigma_2 .$$

Proof: By (12,16) we see that $UF^{(-\varkappa)}_k = G^{(\varkappa)}_k$, $k=0,1,\ldots$, i.e., U maps the domains of $\rho_\varkappa$ and $\pi_{-\varkappa}$ onto each other. Further, Eqs.(13) and (17) yield $U\tilde{a}(-\varkappa)F^{(-\varkappa)}_k = \epsilon d_k(-\varkappa)F^{(-\varkappa)}_{k-1} = \epsilon d^{(-)}_k G^{(\varkappa)}_{k-1} = \tilde{a}(\varkappa)G^{(\varkappa)}_k = \tilde{a}(\varkappa)UF^{(-\varkappa)}_k$, $k=0,1,\ldots$; similarly $U\tilde{a}^{\#}(-\varkappa)F^{(-\varkappa)}_k = \tilde{a}^{\#}(\varkappa)UF^{(-\varkappa)}_k$. ∎

The main results of this section can be summarised as follows:

3.1 Theorem: (i) For each $\varkappa>-1/2, \varkappa\neq 0$, the operators $\Omega_\varkappa(z)\restriction\mathcal{D}_\varkappa$, $z\in B(0,1)$, form an irreducible $*$-representation $\pi_\varkappa$ of $B(0,1)$ on $L^2(\mathbb{R}^+)\otimes\mathbb{C}^2$ with domain $\mathcal{D}_\varkappa := \{F^{(\varkappa)}_k : k=0,1,\ldots\}_{lin}$ specified by Eq.(12) and projection $\hat{E} := I(L^2(\mathbb{R}^+))\otimes(\sigma_0+\sigma_3)/2$. In addition, $\pi_\varkappa$ has non-degenerated vacuum $\psi^{(+)}_\varkappa$.

(ii) Any two representations in the family

$$\Pi := \{\pi_\varkappa : \varkappa\in(-\tfrac{1}{2},\infty)\setminus\{0\}\}$$

are non-equivalent.

(iii) The family contains (up to unitary equivalence) all the Hilbert space irreducible $*$-representations of $B(0,1)$ that can be obtained from linear representations $\Omega_\varkappa$, $\varkappa\in\mathbb{R}\setminus\{0\}$, and whose domain contains a vacuum vector.

3.2 Remark: More explicitly, (iii) states the following: Let $\varkappa\in\mathbb{R}\setminus\{0\}$, $\mathcal{D}'$ be a subspace in $C^\infty(\mathbb{R}^+)\otimes\mathbb{C}^2$ having non-trivial intersection with the vacuum subspace $V_\varkappa$ and $\mathcal{H}$ be a Hilbert space such that $\overline{\mathcal{D}'} = \mathcal{H}$ and the operators $\Omega_\varkappa(z)\restriction\mathcal{D}'$, $z\in B(0,1)$, form an irreducible $*$-representation $\pi'_\varkappa$ of $B(0,1)$ on $\mathcal{H}$ with projection $\hat{E}'$. Then there is a unitary map $V: L^2(\mathbb{R}^+)\otimes\mathbb{C}^2 \to \mathcal{H}$ for which

$$\pi'_\varkappa = \begin{cases} V\pi_\varkappa V^{-1} & \text{if } \varkappa>-1/2 \\ V\pi_{-\varkappa}V^{-1} & \text{if } \varkappa\leq -1/2 \end{cases} \tag{18}$$

and $\hat{E}' = V\hat{E}V^{-1}$ or $\hat{E}' = V(\hat{I}-\hat{E})V^{-1}$.

Proof of the Theorem: (i) It remains to verify that $\hat{E}$ fulfils $\hat{E}\mathcal{D}_\varkappa \subsetneqq \mathcal{D}_\varkappa$ and that for each $\phi\in\mathcal{D}_\varkappa$ holds

$$\hat{E}\,\Omega_\varkappa(x)\phi = \Omega_\varkappa(x)\hat{E}\phi \tag{19a}$$

if x is any even element of $B(0,1)$,

$$\hat{E}\,\Omega_\varkappa(y)\phi = \Omega_\varkappa(y)(\hat{I}-\hat{E})\phi \tag{19b}$$

if y is odd (cf.I-Appendix). All these conditions can readily be verified by using Eqs.(12) and (1).

(ii) Let $\pi_\varkappa, \pi_{\varkappa'}\in\Pi$, $\varkappa\neq\varkappa'$; in view of (6) and $[\tilde{n},\tilde{a}^{\#k}] = k\tilde{a}^{\#k}$, the minimal eigenvalue of $\pi_\varkappa(b_{1-1}) \equiv \tilde{n}\restriction\mathcal{D}_\varkappa$ equals $\varkappa+1/2$ and hence $\pi_\varkappa, \pi_{\varkappa'}$ cannot be equivalent.

(iii) By Proposition 2.1, there is non-zero $\psi'_\varkappa$ in $\mathcal{D}'\cap V_\varkappa$ and the alternative (7b) holds for $\Psi'_{\varkappa}$. Then Eq.(18) ensues from the considerations in the beginning of this section. Further Eq.(19a) implies that $\hat{n}' \equiv \pi'_{\varkappa}(b_{1-1})$ commutes with $\hat{E}'$; then, by (18) $\hat{n}'$ has the same spectrum as $\hat{n}$, i.e., a pure-point spectrum with non-degenerate eigenvalues. Hence $\hat{E}'F'_k = p_k F'_k$, $F'_k := VF_k^{(\varkappa)}$ if $\varkappa > -1/2$ or $VF_k^{(-\varkappa)}$ if $\varkappa \leq -1/2$; moreover, $p_k = 0$ or 1 since $\hat{E}'$ is a projection. Finally, (19b) yields for $\hat{a}' \equiv \pi'_{\varkappa}(a_1)$: $\hat{E}'\hat{a}' F'_k = \hat{a}'(\hat{I} - \hat{E}')F'_k$ and, by using (18) and (13c), we find $p_{k-1} = 1 - p_k$, $k=1,2,\ldots$. Thus, one has either $\hat{E}'F'_{2k} = F'_{2k}$, $\hat{E}'F'_{2k+1} = 0$ or $\hat{E}'F'_{2k} = 0$, $\hat{E}'F'_{2k+1} = F'_{2k+1}$. Since

$$\hat{E}F_{2k} = F_{2k}, \quad \hat{E}F_{2k+1} = 0, \tag{20}$$

the first possibility implies $\hat{E}' = V\hat{E}V^{-1}$ and the second $\hat{E}' = V(\hat{I} - \hat{E})V^{-1}$. ∎

4. Essential self-adjointness

The $*$-property (14) means that the operators $\hat{X}_{jk} := \pi_{\varkappa}(x_{jk})$, $\hat{Y}_j := \pi_{\varkappa}(y_j)$ satisfy $\hat{X}_{jk}^{*} = -\hat{X}_{jk}$, $\hat{Y}_j^{*} = -i\hat{Y}_j$, i.e., $i\hat{X}_{jk}$, $\varepsilon\hat{Y}_j$ are symmetric. By using Nelson's analytic-vector theorem /7/, we will now prove that these operators are moreover essentially self-adjoint (e.s.a.).

Hereafter only $\varkappa \in (-\frac{1}{2}, \infty) \setminus \{0\}$ are considered and the notation $\hat{a} \equiv \hat{a}(\varkappa) := \pi_{\varkappa}(a_1)$ is used.

<u>4.1 Lemma</u>: Let $\hat{A}_p$ be a monomial in $\hat{a}$, $\hat{a}^{*}$ of p-th degree, $p=1,2,\ldots$. Then

$$\|\hat{A}_p F_k\|^2 \leq \frac{\Gamma(k+2\varkappa+p+2)}{\Gamma(k+2\varkappa+2)}, \quad k=0,1,\ldots, \tag{21}$$

where $\|.\|$ is the norm on $\mathcal{H} \equiv L^2(\mathbb{R}^+) \otimes \mathbb{C}^2$.

Proof: Since $\varkappa + 1/2 > 0$, Eq.(13b) yields $d_k^2 \leq k+2\varkappa+1$. For $p=1$ the assertion now follows by Eqs.(13a,c) and the proof is finished by induction, if one realizes that $\hat{A}_{p+1}$ equals either $\hat{A}_p\hat{a}^{*}$ or $\hat{A}_p\hat{a}$. ∎

<u>4.2 Proposition</u>: If $\hat{P}_d$ is a homogeneous polynomial in $\hat{a}, \hat{a}^{*}$ of degree d, $d=1,2$, then each F_k, $k=0,1,\ldots$ is an analytic vector of $\hat{P}_d$.

Proof: One has

$$P_d = \sum_{r=1}^{2^d} \alpha_r A_d^{(r)},$$

where $A_d^{(r)}$ are the independent monomials of degree d. Let $M := \max_{1 \leq r \leq 2^d} |\alpha_r|$; then the estimate (21) yields for any F_k, $t > 0$:

$$\sum_{n=0}^{\infty} \|P_d^n F_k\| \frac{t^n}{n!} \leq \sum_{n=0}^{\infty} \sum_{s=1}^{2^{nd}} \frac{(Mt)^n}{n!} \left(\frac{\Gamma(k+2\varkappa+nd+2)}{\Gamma(k+2\varkappa+2)}\right)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2^d Mt)^n}{n!} \left(\frac{\Gamma(k+2\varkappa+nd+2)}{\Gamma(k+2\varkappa+2)}\right)^{\frac{1}{2}}.$$

This series is convergent for any $t \geq 0$ if $d=1$ and for $0 \leq t < (8M)^{-1}$ if $d=2$, whence the assertion. ∎

As $\{F_k: k=0,1,\ldots\}$ is a total set in $\mathcal{H}$, we get by the Nelson theorem:

<u>4.3 Corollary</u>: If $\hat{P}_d = \hat{P}_d^{*}$, $d=1,2$, then $\hat{P}_d$ is e.s.a. ; in particular, this holds true for the operators $i\hat{X}_{jk}$, $\varepsilon\hat{Y}_j$, $j,k = \pm 1$.

If $\hat{B}$ is a biquadratic homogeneous polynomial, then the above proposition implies that the series

$$\sum_{n=0}^{\infty} \|B^n F_k\| \, t^n/(2n)!$$

is convergent for $t < (64M)^{-1}$. Thus $\hat{B}$ has a total set of semi-analytic vectors and by the Nussbaum theorem /7/ $\hat{B}$ is e.s.a. if $\hat{B} \geq 0$. An important example provides the operator

$$\hat{B} \equiv \hat{N} := -(\hat{X}_{11}^2 + \hat{X}_{1-1}^2 + \hat{X}_{-1-1}^2).$$

Essential self-adjointness of $\hat{N}$ implies that the representation $\tau_{\varkappa}$ of $sl(2,\mathbb{R}) \sim sp(2,\mathbb{R})$, which is obtained by restricting $\pi_{\varkappa}$ to the even subalgebra of the unique real form $osp(1,2)$ of $B(0,1)$, is integrable to a unitary representation of the universal covering group of $SL(2,\mathbb{R})$ (see Ref.8). We shall return to this point in the next section.

<u>4.4 Remark</u>: The conclusions concerning integrability of $\tau_{\varkappa}$ and essential self-adjointness of $i\hat{X}_{jk}$ can alternatively be obtained as follows. Introduce a new basis in $sl(2,\mathbb{R})$: $q_1 := x_{1-1}$, $q_2 := (x_{11} - x_{-1-1})/2$, $q_3 := (x_{11} + x_{-1-1})/2 = ib_{1-1}$ (see I-(2.3b)) and set $\hat{Q}_r := \pi_{\varkappa}(q_r)$. The Casimir element of $sl(2,\mathbb{R})$ becomes (cf. Ref.9, Sect. II.2) $2(q_1^2 + q_2^2 - q_3^2)$ and thus $\Delta := -(\hat{Q}_1^2 + \hat{Q}_2^2 + \hat{Q}_2^3)$ commutes with $\hat{Q}_3$. Now $\hat{Q}_3 = i\hat{n}$ and since F_k, $k=0,1,\ldots$ are non-degenerated eigenvectors of $\hat{n}$, they are also eigenvectors (and hence analytic vectors) of Δ. Consequently, Δ is e.s.a., which further implies that any operator $i(a_1\hat{Q}_1 + a_2\hat{Q}_2 + a_3\hat{Q}_3)$, $a_r \in \mathbb{R}$, is e.s.a. /10/.

5. Restriction of $\pi_{\varkappa}$ to the even subalgebra $sl(2,\mathbb{R}) \subset osp(1,2)$

Let $\tau_{\varkappa}$ be the restriction of $\pi_{\varkappa}$ to $sl(2,\mathbb{R})$. According to Eq.(19a), $\tau_{\varkappa}$ is reduced by the projection $\hat{E} \equiv I(L^2(\mathbb{R}^+)) \otimes (\sigma_0 + \sigma_3)/2$:

$$\tau_{\varkappa} = \tau_{\varkappa}^{(+)} \oplus \tau_{\varkappa}^{(-)},$$

the $\tau_{\varkappa}^{(\alpha)}$, $\alpha = \pm 1$ being skew-symmetric representations of $sl(2,\mathbb{R})$ on $L^2(\mathbb{R}^+)$ with domains $\mathcal{D}_{\varkappa}^{(\alpha)} := \{f_k^{(\varkappa - \alpha/2)}: k=0,1,\ldots\}_{lin}$ (cf. Eqs.(12),(20)).

<u>5.1 Proposition</u>: The representations $\tau_{\varkappa}^{(\alpha)}$ are irreducible.

Proof: We have to show that the set $\mathcal{U}^{(\alpha)} := \mathcal{U}(\hat{X}_{11}^{(\alpha)}, \hat{X}_{1-1}^{(\alpha)}, \hat{X}_{-1-1}^{(\alpha)})$, $\hat{X}_{jk}^{(\alpha)} := \hat{X}_{jk} \restriction \mathcal{D}_{\varkappa}^{(\alpha)}$, has no invariant subspaces. By Eqs.I-(2.3) we see that $\hat{X}_{jk}$ are homogeneous quadratic polynomials in $\hat{a}, \hat{a}^{*}$; irreducibility of $\tau_{\varkappa}^{(\alpha)}$ can then be verified with the help of Eqs.(13) by repeating the argument we used for proving absence of invariant subspaces of $\mathcal{U}(\hat{a}, \hat{a}^{*})$ in Sect.3. ∎

<u>5.2 Remark</u>: Let $\mathcal{H}_k^{(\alpha)} \subset L^2(\mathbb{R}^+)$ be the one-dimensional subspace spanned by $f_k^{(\varkappa-\alpha/2)}$. Clearly, each of the domains $\mathcal{D}_{\varkappa}^{(\alpha)}$ can be expressed as the algebraic sum of subspaces $\mathcal{H}_k^{(\alpha)}$

$$\mathcal{D}_k^{(\alpha)} = \sum_{k=0}^{\infty} \mathcal{H}_k^{(\alpha)} .$$

The $f^{(\varkappa-\alpha/2)}$ are eigenvectors of $\hat{n}^{(\alpha)} := \tau_{\varkappa}^{(\alpha)}(b_{1-1})$ corresponding to eigenvalues

$$\lambda_{k,\varkappa}^{(\alpha)} := 2k+\varkappa+(|\alpha|-\alpha+1)/2 .$$

Since the maximal compact subalgebra $u(1) \subset sl(2,\mathbb{R})$ is spanned by $ib_{1-1} = (x_{11}+x_{-1-1})/2$ (see I- Remark 3.2), the restriction $\tau_{\varkappa}^{(\alpha)} \restriction u(1)$ equals direct sum of one-dimensional representations of $u(1)$ on $\mathcal{H}_k^{(\alpha)}$ that are uniquely determined by eigenvalues $\lambda_{k,\varkappa}^{(\alpha)}$. This means that the so-called weight diagram of $\tau_{\varkappa}^{(\alpha)}$ is $\{\lambda_{k,\varkappa}^{(\alpha)} : k=0,1,\dots\}$.

Each of $\tau_{\varkappa}^{(\alpha)}$ is integrable to a representation $\mathcal{T}_{\varkappa}^{(\alpha)}$ of $G = \overline{SL(2,\mathbb{R})}$ as the vectors $f_k^{(\varkappa-\alpha/2)}$, $k=0,1,\dots$, are analytic vectors of

$$\hat{N}^{(\alpha)} := -((\hat{X}_{11}^{(\alpha)})^2 + (\hat{X}_{1-1}^{(\alpha)})^2 + (\hat{X}_{-1-1}^{(\alpha)})^2)$$

and form a total set in $L^2(\mathbb{R}^+)$. Moreover, $\mathcal{T}_{\varkappa}^{(\alpha)}$ is a unitary irreducible representation (UIR) of G on $L^2(\mathbb{R}^+)$ with the following property [11]: let K be the simply connected subgroup of G whose Lie algebra is u(1); then $\mathcal{T}_{\varkappa}^{(\alpha)} \restriction K$ equals direct sum of UIR's of K on $\mathcal{H}_k^{(\alpha)}$, each of them being uniquely determined by the eigenvalue $\lambda_{k,\varkappa}^{(\alpha)}$. In fact, $K \sim \mathbb{R}$ and the UIR of K on $\mathcal{H}_k^{(\alpha)}$ is equivalent to $t \mapsto \exp(it\lambda_{k,}^{(\alpha)} t)$, $t \in \mathbb{R}$.

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Бланк И., Гавличек М. E2-85-160

Неприводимые $*$-представления суперaлгебр Ли $B(0, n)$ с конечнократно вырожденным вакуумом. Результаты для $B(0, 1)$

Общий метод построения неприводимых $*$-представлений суперaлгебр Ли $B(0, n)$, предложенный в первой части работы, применяется к суперaлгебре $B(0,1)$. Получено однопараметрическое семейство Π неэквивалентных неприводимых $*$-представлений с невырожденным вакуумом. Область определения D_π каждого $\pi \in \Pi$ плотна в $L^2(R^+) \otimes C^2$ и для всех элементов $z \in B(0, 1)$, таких, что $z = z^*$, операторы $\pi(z)$ в существенном самосопряжены на области D_π. Сужение π на четную подалгебру $s\ell(2, R)$ вещественной формы $osp(1,2)$ суперaлгебры $B(0,1)$ является прямой суммой двух антисимметрических неприводимых представлений $s\ell(2,R)$ в $L^2(R^+)$, которые интегрируемы в унитарные неприводимые представления универсальной накрывающей группы $SL(2, R)$.

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Blank J., Havlíček M. E2-85-160

Irreducible $*$-Representations of Lie Superalgebras $B(0,n)$ with Finite-Degenerated Vacuum. Results for $B(0,1)$

The general method of constructing irreducible $*$-representations of Lie superalgebras $B(0,n)$ presented in the first part of this study is applied to the superalgebra $B(0,1)$. A one-parameter family Π of non-equivalent irreducible $*$-representations with non-degenerated vacuum is obtained. The domain D_π of each $\pi \in \Pi$ is dense in $L^2(R^+) \otimes C^2$ and for all elements of $B(0,1)$ satisfying $z = z^*$ the operators $\pi(z)$ are essentially self-adjoint on D_π. The restriction of π to the even subalgebra $s\ell(2, R)$ of the real form $osp(1, 1)$ of $B(0,1)$ equals direct sum of two skew-symmetric irreducible representations of $s\ell(2,R)$ on $L^2(R^+)$ that are integrable to unitary irreducible representations of the universal covering group of $SL(2, R)$.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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