объединенный
институт
ядерных
исследований
дубна

E2-85-128

A.Galperin,[1] E.Ivanov, V.Ogievetsky, E.Sokatchev[2]

HARMONIC SUPERGRAPHS. FEYNMAN RULES AND EXAMPLES

Submitted to "Classical and Quantum Gravity"

---

[1] Institute of Nuclear Physics, Tashkent, USSR

[2] Institute of Nuclear Research and Nuclear Energy, Sofia, Bulgaria

1985

This is the second part of our exposition of the quantization scheme for N=2 matter and SYM theories in harmonic superspace. In "Green functions"[1] we gave the Green functions for those theories. Here we spell out the Feynman rules and show a number of examples of their application. Section I is devoted to the matter hypermultiplets. In particular, the most general self-couplings of the hypermultiplets are given, including those with broken rigid SU(2) invariance. Also, the duality relation between the hypermultiplet and the N=2 linear multiplet is discussed [x]. Section II deals with N=2 SYM. We explain how to expand the action in terms of the prepotentials in order to find the infinite set of vertices in this non-polynomial theory. The vertices are rather simple, with no derivatives at all. Section III contains many examples of supergraph calculations. They illustrate some important points like how to deal with harmonic distributions and why they do not lead to new divergences, what is the meaning of the nonlocality in the harmonic coordinates and how it disappears in the final results. Here we also present a very simple proof of the finiteness of a class of nonlinear N=4 σ-models in two dimensions. The Appendix contains a discussion of the duality between the two forms of the hypermultiplet.

All the references to formulas from Part I[1] are given by a couple of Roman and Arabic numerals, e.g., (III.15). The formulas in Part II are marked by Arabic numerals only, e.g., (15).

I. Hypermultiplets. General couplings and Feynman rules

The kinetic part of the action for q^+ and ω hypermultiplets and the corresponding Green functions were considered in [1]. Here we shall start with a discussion of their interactions.

Consider first a hypermultiplet q^+_s in a certain representation of a gauge group (index s). It may interact with the SYM superfield V_a^{++} and with itself, e.g.,

$$S = \int d\zeta^{(-4)} du \left\{ \overset{*}{q}{}^{+r} \left[D^{++} \delta^s_r + i g V_a^{++} (t^a)^s_r \right] q^+_s + \frac{\lambda}{4} \overset{*}{q}{}^{+r} \overset{*}{q}{}^{+s} q^+_r q^+_s \right\} \qquad (1)$$

---

[x] Sec.I contains our answers to some comments on harmonic superspace recently made in [2].

Here t^a are the generators of the representation of the YM group. It is remarkable that this representation may be complex. This possibility leads to an extension of the class of finite models, as discussed in /3/. If the representation is real, one can also add self-couplings of the type /4/

$$(q^+)^4 + h.c. \,, \quad \overset{*}{q}{}^+ (q^+)^3 + h.c.$$

The above self-couplings of q^+ are the most general ones if the rigid SU(2) invariance of the theory is to be preserved. If one abandons this invariance (but never the U(1) invariance!), a much larger class of self-interactions is allowed, e.g.,

$$S = \int dz^{(-4)} du \,\big[\overset{*}{q}{}^+ D^{++} q^+ \cdot f(\gamma\, q^+ \overset{*}{q}{}^+ u_1^- u_2^-) + g(\varkappa\, q^+ \overset{*}{q}{}^+ u_1^- u_2^-) \cdot \lambda\, (\overset{*}{q}{}^+)^2 (q^+)^2 \big] , \quad (2)$$

etc. Here $u^-_{1,2}$ are components of the SU(2) isospinor u^-_i (SU(2) is broken), γ and $\varkappa$ are coupling constants of dimension -2, f and g are arbitrary functions.

Next we consider the ω hypermultiplet. It is most natural to assign ω to a real representation of the YM group. (ω is a real analytic superfield), e.g., to the adjoint representation. Then the coupling of ω to the N=2 SYM has the form

$$S = \int dz^{(-4)} du\, \frac{1}{2}\, \mathrm{tr}\, [\omega (D^{++} + i g V^{++})^2 \omega] . \quad (3)$$

In contrast to the q^+ hypermultiplet, putting ω in a complex representation would mean doubling the number of physical fields.

Concerning the self-interactions of ω, in /4/ we have shown that they are possible only for several (not just one) ω's. Once again, this is so under the condition of preservation of the rigid SU(2) invariance. If the latter is broken, even a single ω can self-interact, e.g.,

$$S = \int dz^{(-4)} du\, [\omega (D^{++})^2 \omega + f(\varkappa\omega) (u_1^+ u_2^+)^2] . \quad (4)$$

In the above context we would like to comment on a claim recently made in /2/. The claim was that q^+ and ω have very restricted

self-interactions, whereas the linear N=2 multiplet /5/ allows more general ones. First of all, the class of self-couplings of the linear multiplet considered in /2/ corresponds to broken SU(2) invariance, and we saw above that in such a case the self-interactions of both q^+ and ω are most general. Second, the N=2 linear multiplet is nothing but the dual to the ω hypermultiplet. Indeed, the N=2 linear multiplet can be described by a real analytic superfield L^{++} constrained by the equation

$$D^{++} L^{++} = 0 . \quad (5)$$

It is the N=2 generalization of the N=1 linear multiplet L ($D^2 L = \bar{D}^2 L = 0$) and of the N=0 notoph L_m ($\partial^m L_m = 0$). The kinetic action for L^{++} is

$$S = \int dz^{(-4)} du\, L^{++} L^{++}$$

and the Green function is $\frac{1}{\Box} \Pi^{(2,2)}$ (see (IV.26)). It should be stressed that if the SU(2) invariance is preserved L^{++} cannot self-interact at all. The ω multiplet which is dual to L^{++} is introduced as a Lagrange multiplier for the constraint (5):

$$S = \int dz^{(-4)} du\, \big(L^{++} L^{++} + \omega D^{++} L^{++} + f^{(4)}(L^{++}, u^{\pm}) \big) \quad (6)$$

Varying (6) with respect to L^{++} gives an expression of L^{++} in terms of ω and putting it back in (6) produces the action for ω. So, the self-interactions of L^{++} are equivalent to the self-interactions of ω. This situation is analogous to the case of N=1 (N=0) where the linear multiplet (notoph) is dual to chiral one (scalar field). At the same time the N=2 linear multiplet L^{++} cannot be coupled to the N=2 SYM (in contrast with hypermultiplets). So the coupling of L^{++} are of less generality than those of ω and q^+.

A final comment on ref. /2/ concerns the truncation of the off-shell ω hypermultiplet to the relaxed hypermultiplet of ref. /6/. In /4/ we showed that the latter is obtained by constraining ω,

$$(D^{++})^3 \omega = 0 .$$

In /2/ a "solution" of this constraint was proposed,

$$\omega = (D^+)^4 \, [(\bar{D}^-)^2 \Psi(z) + (D^-)^2 \bar{\Psi}(z)] \quad (7)$$

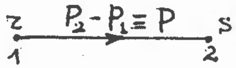
which was claimed to trivialize the action for ω. In fact, eq. (7) does not make sense since the l.h.s. has U(I) charge 0, and the r.h.s. +2. Then it is obvious that inserting (7) into the action for ω makes the harmonic integral vanish (recall (II.9)). This emphasizes once again how important is to keep the balance of U(I) charges in any harmonic calculation.

Before proceeding to the Feynman rules we recall here the action for the Faddeev-Popov ghosts /1/ because it resembles (3):

$$S_{FP} = \int dz^{(-4)} du \; i\, tz \left[F D^{++} (D^{++} + igV^{++}) P \right]. \tag{8}$$

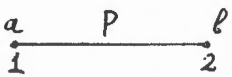
Now we give a list of Feynman rules for the N=2 matter theories. The rules will be formulated in momentum space, i.e., after Fourier transforming the x-dependence of the analytic superfields $\Phi(\zeta(x,\theta,u),u)$.

The q^+ propagator $\langle \overset{*}{q}{}^{+z}(p_1,\theta_1,u_1)\, q^+_s(p_2,\theta_2,u_2)\rangle$ is (see (III.10)



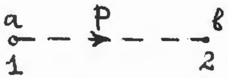
$$\frac{i}{p^2}\, \frac{(D_1^+)^4 (D_2^+)^4}{(u_1^+ u_2^+)^3}\, \delta^8(\theta_1 - \theta_2)\, \delta^z_s . \tag{9}$$

The ω propagator $\langle \omega_a(1)\, \omega_b(2) \rangle$ is (see (III.14))



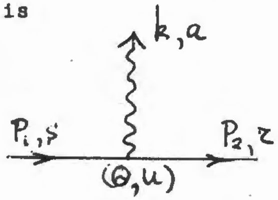
$$\frac{i}{p^2}\, (D_1^+)^4 (D_2^+)^4\, \delta^8(\theta_1-\theta_2)\, \frac{u_1^- u_2^-}{(u_1^+ u_2^+)^3}\, \delta_{ab}. \tag{10}$$

The ghost propagator $\langle F_a(1) P_b(2) \rangle$ has a similar form



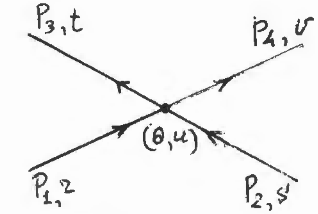
$$-\frac{1}{p^2}\, (D_1^+)^4 (D_2^+)^4\, \delta^8(\theta_1-\theta_2)\, \frac{u_1^- u_2^-}{(u_1^+ u_2^+)^3}\, \delta_{ab}. \tag{11}$$

The vertices can be read off from (1), (3) and (8) (we consider only the case with unbroken SU(2) invariance). The vertex $\overset{*}{q} V q$ is



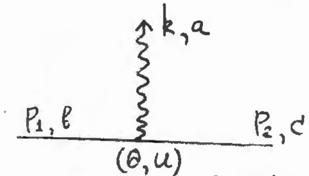
$$-g\,(t^a)_z^s\, (2\pi)^4\, \delta(p_1 - p_2 - k). \tag{12}$$

The vertex $(\overset{*}{q})^2 (q)^2$ is



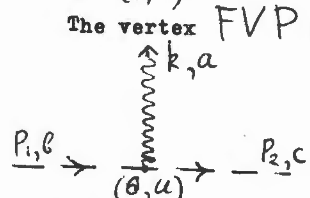
$$i\lambda\, \delta_t^{(z} \delta_v^{s)}\, (2\pi)^4\, \delta(p_1 + p_2 - p_3 - p_4). \tag{13}$$

The vertex $\omega V \omega$ is



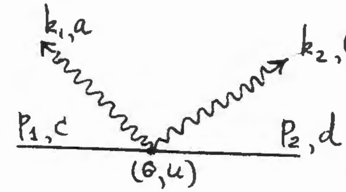
$$-ig\, f^{abc}\left(D^{++}_{(b)} - D^{++}_{(c)}\right)(2\pi)^4\, \delta(p_1 - p_2 - k). \tag{14}$$

The vertex FVP is



$$g\, f^{abc}\, D^{++}_{(b)}\, (2\pi)^4\, \delta(p_1 - p_2 - k). \tag{15}$$

In (14) and (15) the subscripts (b) and (c) mean that the harmonic derivatives act on the corresponding lines. Finally, the vertex $\omega V V \omega$ is



$$ig^2\left(f^{ade} f^{ebc} + f^{ace} f^{ebd}\right) \cdot (2\pi)^4\, \delta(p_1 - p_2 - k_1 - k_2). \tag{16}$$

At each vertex one integrates over all the internal momenta (with the measure $(2\pi)^{-4} d^4p$). Besides, an integration $\int d^4\theta^+ du$ (remaining from $\int dz^{(-4)} du$ in the momentum representation) is also implied. Inspecting the propagators (9)-(11) one can see that at each analytic vertex there are factors $(D^+)^4$ coming from the propagators which can always be used to restore the full Grassmann measure $d^8\theta$ at the vertex. This important feature of the N=2 supergraph technique will be illustrated in a number of examples in Sec. III.

II. Feynman rules for N=2 SYM

The N=2 SYM action is nonpolynomial in the prepotential V^{++}, so there are infinitely many vertices of self-interaction of V^{++}. To find them one should expand the action in terms of V^{++}:

$$S=\sum_{n=0}^{\infty}\frac{g^n}{n!}\int d^{12}z\,du_1\dots du_n\left[\frac{\delta^n S}{\delta V^{++}_{a_1}(z,u_1)\dots\delta V^{++}_{a_n}(z,u_n)}\right]_{V^{++}=0}\cdot \quad (17)$$

$$\cdot V^{++}_{a_1}(z,u_1)\dots V^{++}_{a_n}(z,u_n).$$

The structure of this expansion is suggested by the first variation of S (IV.15) and the expression (IV.8) of e^{iv} in terms of V^{++} (note that in (17) and elsewhere in Part II we indicate explicitly the gauge coupling constant g).

The problem now is how to evaluate the functional derivatives of S or, equivalently, its variations $\delta_n\dots\delta_1 S$. We shall first compute the second variation and then show that the rest can easily be derived from it.

The second-order variation is obtained from (IV.15) with the help of (IV.11) (see also (IV.10)) and (II.17), (II.15):

$$\delta_2\delta_1 S=-\frac{i}{g}\,\mathrm{tr}\int d^{12}z\,du\,\delta_1V^{++}\delta_2(e^{iv}D^{--}e^{-iv})=$$

$$=\frac{i}{g}\,\mathrm{tr}\int d^{12}z\,du\,(\delta_1V^{++})_\tau D^{--}(e^{-iv}\delta_2 e^{iv})= \quad (18)$$

$$=\mathrm{tr}\int d^{12}z\,du_1du_2\,\frac{(\delta_1V^{++})_\tau(z,u_1)(\delta_2V^{++})_\tau(z,u_2)}{(u_1^+u_2^+)^2}=$$

$$=\int d^{12}z\,du_1du_2\,\frac{(\delta_1V_a^{++})_\tau(\delta_2V_a^{++})_\tau}{(u_1^+u_2^+)^2},$$

where $(\delta V^{++})_\tau=(\delta V^{++}_a)_\tau T^a$. Using this expression as a starting point one may compute the higher-order variations by just varying $(\delta V^{++})_\tau$ in (18). From (IV.10), (IV.11) one finds

$$\delta_2(\delta_1V^{++})_\tau=\delta_2(e^{-iv}\delta_1V^{++}e^{iv})=$$

$$=-ig\int du_2\,\frac{u_1^+u_2^-}{u_1^+u_2^+}\left[(\delta_1V^{++})_\tau,(\delta_2V^{++})_\tau\right]$$

or, taking into account (IV.1),

$$\delta_2(\delta_1V_a^{++})_\tau=g f_{abc}\int du_2\,\frac{u_1^+u_2^-}{u_1^+u_2^+}(\delta_1V_b^{++})_\tau(\delta_2V_c^{++})_\tau. \quad (19)$$

So, the third-order variation is

$$\delta_3\delta_2\delta_1 S=2g f_{abc}\int d^{12}z\,du_1du_2du_3\,(\delta_1V_b^{++})_\tau\cdot$$

$$(\delta_2V_a^{++})_\tau(\delta_3V_c^{++})_\tau\cdot\frac{u_1^+u_3^-}{(u_1^+u_3^+)(u_1^+u_2^+)^2}\cdot\frac{u_2^+u_3^+}{u_2^+u_3^+}= \quad (20)$$

$$=g f_{abc}\int d^{12}z\,du_1du_2du_3\,\frac{(\delta_1V_a^{++})_\tau(\delta_2V_b^{++})_\tau(\delta_3V_c^{++})_\tau}{(u_1^+u_2^+)(u_1^+u_3^+)(u_2^+u_3^+)}.$$

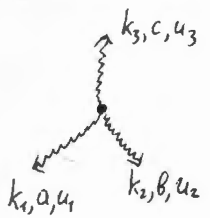
In the derivation the symmetry of the integrand in (18) and the antisymmetry of fabc are used.

The process of variation of S shown above goes on straightforwardly. The higher-order variations will contain a product of V^{++}'s, of the structure constants fabc and of pairs of harmonics arranged according to the symmetry. The singularities of various factors of the denominator will not coincide. It is remarkable that there are no derivatives in the vertices obtained (unlike the case N=1) . The harmonic nonlocalities will be shown to disappear in supergraph calculations (see Sec. III).

Now we are ready to formulate the Feynman rules for N=2 SYM. We prefer to work in the Fermi-Feynman gauge $\alpha=-1$ (IV.28). The propagator is (recall III.7))

$$\frac{i}{k^2}(D_1^+)^4\delta^8(\theta_1-\theta_2)\delta^{(-2,2)}(u_1,u_2)\delta_{ab}. \quad (21)$$

The three-particle vertex is



$$\frac{i g f^{abc}}{(u_1^+u_2^+)(u_1^+u_3^+)(u_2^+u_3^+)}(2\pi)^4\delta(k_1+k_2+k_3) \qquad (22).$$

Its symmetry is evident. The usual momentum integration is implied. Note that the configuration space integral at the vertex is $\int d^8\theta\, du_1 du_2 du_3$, so the Grassmann measure is already complete, in contrast to the matter and matter-gauge vertices (Sec.I). The n-particle gauge field vertices can be found as explained above.

III. Examples of supergraph calculations

Above we derived a set of Feynman rules for N=2 matter and SYM and now we may apply them to manifestly supersymmetric supergraph computations. We hope to convince the reader that it is indeed very easy to handle the quantum harmonic superfields. We shall demonstrate that no divergences related to the singularities of the harmonic distributions arise. The harmonic nonlocality will be shown to disappear when the external legs of a graph are put on-shell or their superisospins are fixed. We shall also confirm earlier conjectures /7/ that all the quantum corrections to the effective action can be written as integrals with the full N=2 Grassmann measure $d^8\theta$ (this fact is important when discussing the ultraviolet behaviour). A simple proof of the finiteness of a class of two-dimensional N=4 supersymmetric σ-models will be given.

The first example is the one-loop correction to the 4-point function for a self-interacting q^+ hypermultiplet (Fig.1). The corresponding analytic expression is

$$\Gamma=\lambda^2\int\frac{d^4p_1\ldots d^4p_4\, d^4k}{(2\pi)^{16}}\, d^4\theta_1^+ d^4\theta_2^+ du_1 du_2\, \delta(p_1+p_2-p_3-p_4)\cdot$$
$$\cdot\, q^+(p_1,\theta_1,u_1)\, q^+(p_2,\theta_1,u_1)\, \overset{*}{q}{}^+(p_3,\theta_2,u_2)\, \overset{*}{q}{}^+(p_4,\theta_2,u_2)\cdot \qquad (23)$$
$$\cdot\frac{(D_1^+)^4(D_2^+)^4}{(u_1^+u_2^+)^3}\,\delta^8(\theta_1-\theta_2)\cdot\frac{(D_1^+)^4(D_2^+)^4}{(u_1^+u_2^+)^3}\,\delta^8(\theta_1-\theta_2)\,\frac{1}{k^2(p_1+p_2-k)^2}$$

The general rule for handling such expressions is first to do all the θ integrations but one using the Grassmann δ-functions from the propagators. For this purpose one has to restore the full measures $d^8\theta_1 d^8\theta_2$. This can be achieved by taking $(D_1^+)^4(D_2^+)^4$ off one of the propagators and using (III.5) (note that the other propagator and the external superfields are analytic, so D_1^+, D_2^+ do not act on them). Then one can apply the identity

$$\delta^8(\theta_1-\theta_2)(D_1^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2)\equiv(u_1^+u_2^+)^4\delta^8(\theta_1-\theta_2) \qquad (24)$$

(that follows from (III.2) and the algebra of $D^i_{\alpha(i)}$) and do the integral. The result is

$$\Gamma=\lambda^2\int\frac{d^4p_1\ldots d^4p_4\, d^4k}{(2\pi)^{16}}\, d^8\theta\, du_1 du_2\, \delta(p_1+p_2-p_3-p_4)\cdot \qquad (25)$$
$$\cdot\,\frac{q^+(p_1,\theta,u_1)\, q^+(p_2,\theta,u_1)\, \overset{*}{q}{}^+(p_3,\theta,u_2)\, \overset{*}{q}{}^+(p_4,\theta,u_2)}{(u_1^+u_2^+)^2\, k^2(p_1+p_2-k)^2}.$$

In (25) we observe an important phenomenon. Although in the initial expression (23) there seemed to be a product of two singular harmonic distributions, in the process of doing the D-algebra one of them cancelled out. The distribution remaining in (25) does not lead to new, harmonic divergences. This can be most easily demonstrated if the external lines are put on-shell, i.e., $D^{++}q^+=0$. In this case (see (II.3) and (II.5))

$$q^+(u_1)\,q^+(u_1)\equiv\frac{1}{2}D_1^{++}D_1^{--}\left(q^+(u_1)\,q^+(u_1)\right)$$

and the u_2 integral can be computed (see (II.18)):

$$\frac{1}{2}\int du_1 du_2\, D_1^{++}D_1^{--}(q_1^+q_1^+)\cdot(\overset{*}{q}{}_2^+\overset{*}{q}{}_2^+)\frac{1}{(u_1^+u_2^+)^2}=$$
$$=-\frac{1}{2}\int du_1 du_2\, D_1^{--}(q_1^+q_1^+)\cdot(\overset{*}{q}{}_2^+\overset{*}{q}{}_2^+)\, D_1^{--}\delta^{(2,-2)}(u_1,u_2)= \qquad (26)$$
$$=\frac{1}{2}\int du\,(\overset{*}{q}{}^+\overset{*}{q}{}^+)(D^{--})^2(q^+q^+).$$

One sees that the harmonic nonlocality present in (25) has disappeared and there are no harmonic divergences x). The momentum integral diverges logarithmically. Its divergent part is local in x-space (and thus in superspace):

$$\Gamma_\infty = C_\infty \cdot \lambda^2 \int d^{12}z\, du (\overset{*}{q}{}^+)^2 (D^{--})^2 (q^+)^2 =$$

$$= C_\infty \lambda^2 \int d\zeta^{(-4)} du (\overset{*}{q}{}^+)^2 (D^+)^4 (D^{--})^2 (q^+)^2 =$$

$$= -2 C_\infty \lambda^2 \int d\zeta^{(-4)} du (\overset{*}{q}{}^+)^2 \Box (q^+)^2$$

(see (III.12)). Obviously, Γ_∞ differs from the initial action (1) and the theory is nonrenormalizable (which is not surprising since the coupling constant λ has dimension M^{-2}).

It is remarkable that in three-dimensional space-time (d=3) the graph in Fig.1 is convergent. Moreover, in d=2 we may easily prove xx) that the theory of the self-interacting q^+ hypermultiplet is finite off-shell (the same applies to the ω hypermultiplet, as well as to the more general couplings (2), (4)). Indeed, in d=2 $[\lambda] = M^0$, $[q^+(z,u)] = M^0$, so $[q^+(p,\theta,u)] = M^{-2}$. The n-particle contribution to the effective action has the generic form

$$\Gamma_n = \int d^8\theta\, du (d^2p)^{n-1} [q(p,\theta,u)]^n I(p).$$

The fact that the θ integral has the full measure $d^8\theta$ follows from the Feynman rules, as explained above. We see that the momentum integral I(p) has dimension M^{-2} and hence is convergent. Note that on-shell the q^+ theory is equivalent to some class of nonlinear supersymmetric σ-models /4/ (N=2 in d=4, N=4 in d=2). The finiteness of some hyper-Kahler N=4 models in d=2 has been proved in /8,9/ by completely different means.

---

x) It should be pointed out that one may go even further in (26) and compute the remaining u integral. Indeed, the equation of motion $D^{++}q^+=0$ means $q^+(p,\theta,u) = u^{+i} q_i(p,\theta)$, so (26) amounts to $\overset{*}{q}{}^i \overset{*}{q}{}^j q_i q_j$ and the harmonic dependence disappears.

xx) The proof of this nonrenormalization theorem goes along the lines presented in /7/.

So, we have seen that on-shell the graph in Fig.1 gives a contribution local in harmonic space. This is so because the infinite set of auxiliary fields are eliminated by the equation of motion. Off-shell they contribute to the effective action and the latter remains nonlocal in u-space. This nonlocality is similar to the nonlocality of the effective action in x-space. Nevertheless, it can be removed if one fixes the superisospins of the external superfields (just as one can fix the momenta of the external legs), e.g.,

$$q^+_{(n)}(p,\theta,u) = (u^+)^{(n+1}(u^-)^{n)} q_{(i_1 \ldots i_{2n+1})}(p,\theta).$$

Then one may write down

$$q^+_{(n)} = D^{++}[(u^+)^{(n}(u^-)^{n+1)}] q_{(i_1 \ldots i_{2n+1})} \cdot \frac{1}{n+1} .$$

So, $q^+_{(n)}(u_1) q^+_{(m)}(u_1)$ can be presented as a total harmonic derivative D^{++}. Integrating by parts and using (II.18) one obtains a harmonic δ-function and may do one of the u integrals. The remaining expression is local in u-space because the harmonics in the numerator combine to form an SU(2) singlet (otherwise, the u integral would vanish) and cancel out the denominator. No harmonic divergences occur. Of course, the remaining u integral may also be computed.

As an exercise the reader can compute the graph in Fig.2 and show that on-shell the ultraviolet divergent part vanishes. Another exercise is the graph in Fig.3 that describes the one-loop self-energy correction for q^+ coupled to N=2 SYM. In the SYM propagator (21) there are too few spinor derivatives (at least 8 are required in a loop, see (27)) and the contribution of this graph vanishes.

Our second example is the one-loop correction to the V^{++} self-energy in N=2 SYM. The relevant graphs are shown in Figs.4 and 5. The Yang-Mills contribution (Fig.4) is given by

$$\Gamma_2^{YM} = g^2 \int \frac{d^4k\, d^4p}{(2\pi)^8 p^2 (k-p)^2} d^8\theta\, d^8\eta\, du_1 du_2 du_3 dw_1 dw_2 dw_3 \cdot$$

$$\frac{V^{++}_a(k,\theta,u_1) V^{++}_a(k,\eta,w_1)}{(u_1^+u_2^+)(u_1^+u_3^+)(u_2^+u_3^+)(w_1^+w_2^+)(w_1^+w_3^+)(w_2^+w_3^+)} \cdot$$

$$\cdot (D^+_\theta(u_2))^4 \delta^8(\theta-\eta) \delta^{(-2,2)}(u_2,w_2) (D^+_\eta(u_3))^4 \delta^8(\theta-\eta) \delta^{(-2,2)}(u_3,w_3).$$

Unlike the previous example (23) the Grassmann measures are already complete. So, we may start doing the D-algebra by taking the derivatives $D^+_\theta(u_2)$ off the first δ-function and integrating by parts. The result vanishes unless all of the $D^+_\theta(u_2)$ hit the second δ-function. Indeed, one may easily see that

$$\delta^8(\theta-\eta)(D)^m\delta^8(\theta-\eta)=0\,,\quad if\quad m<8. \tag{27}$$

According to (24) we get a factor of $(u_2^+u_3^+)^4$. Next we do the w_2 and w_3 integrals with the help of the harmonic δ-functions. The result is

$$\Gamma_2^{YM}=g^2\int\frac{d^4k\,d^4p}{(2\pi)^8p^2(k-p)^2}d^8\theta\,du_1du_2du_3dw_1\,\cdot$$

$$\cdot\frac{(u_2^+u_3^+)^2\,V_a^{++}(k,\theta,u_1)\,V_a^{++}(k,\theta,w_1)}{(u_1^+u_2^+)(u_1^+u_3^+)(w_1^+u_2^+)(w_1^+u_3^+)}.$$

Note the absence of coinciding harmonic singularities. To do the u_3 and w_1 integrals, we write down

$$(u_2^+u_3^+)^2=D_2^{++}D_3^{++}[(u_2^-u_3^-)(u_2^+u_3^+)],$$

then integrate by parts and use (II.18). At the end we get (replacing w_1 by u_2)

$$\Gamma_2^{YM}=-2g^2\int\frac{d^4k\,d^4p\,d^8\theta\,du_1du_2}{(2\pi)^8p^2(k-p)^2}\,\frac{u_1^-u_2^-}{u_1^+u_2^+}\,V_a^{++}(1)\,V_a^{++}(2). \tag{28}$$

The ghost contribution shown in Fig.5 is (a factor of -2 is due to the statistics and number of ghosts)

$$\Gamma_2^{gh}=2g^2\int\frac{d^4k\,d^4p}{(2\pi)^8p^2(k-p)^2}\,d^4\theta_1^+d^4\theta_2^+du_1du_2\,V_a^{++}(1)\,V_a^{++}(2)\,\cdot$$

$$\cdot(D_1^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2)\,D_1^{++}\left(\frac{u_1^-u_2^-}{(u_1^+u_2^+)^3}\right)\cdot$$

$$\cdot(D_1^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2)\,D_2^{++}\left(\frac{u_1^-u_2^-}{(u_1^+u_2^+)^3}\right).$$

The first step is to restore the measures $d^8\theta_1$, $d^8\theta_2$ and do the θ_2 integration. As a result, we get $(u_1^+u_2^+)^4$ which commutes with D_1^{++}, D_2^{++} and cancels out one of the denominators completely and the other partially. Thus, the multiplication of coinciding singularities is once again avoided. The final result is

$$\Gamma_2^{gh}=2g^2\int\frac{d^4k\,d^4p}{(2\pi)^8p^2(k-p)^2}\,d^8\theta\,du_1du_2\,\frac{(u_1^+u_2^-)(u_1^-u_2^+)}{(u_1^+u_2^+)^2}\,V_a^{++}(1)\,V_a^{++}(2). \tag{29}$$

Putting together (28) and (29) we arrive at the total one-loop self-energy contribution

$$\Gamma_2=-2g^2\int\frac{d^4k\,d^4p}{(2\pi)^8p^2(k-p)^2}\,d^8\theta\,du_1du_2\,\frac{V_a^{++}(1)\,V_a^{++}(2)}{(u_1^+u_2^+)^2} \tag{30}$$

(recall (II.15)). The logarithmically divergent part of Γ_2 has the form of the linearized action (see (IV.16)), as is to be expected.

The next example is the computation of the one-loop contribution of hypermultiplet matter to the V^{++} self-energy (Fig.6). For the hypermultiplet in the adjoint representation (any other real representation is admissible) we find

$$\Gamma_2^{\omega}=2g^2\int\frac{d^4k\,d^4p}{(2\pi)^8p^2(k-p)^2}\,d^4\theta_1^+d^4\theta_2^+du_1du_2\,V_a^{++}(1)\,V_a^{++}(2)\,\cdot$$

$$\cdot(D_1^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2)\cdot(D_1^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2)\,\cdot$$

$$\cdot\left[\frac{u_1^-u_2^-}{(u_1^+u_2^+)^3}\,D_1^{++}D_2^{++}\left(\frac{u_1^-u_2^-}{(u_1^+u_2^+)^3}\right)-D_1^{++}\left(\frac{u_1^-u_2^-}{(u_1^+u_2^+)^3}\right)D_2^{++}\left(\frac{u_1^-u_2^-}{(u_1^+u_2^+)^3}\right)\right].$$

The two terms originate from the two possible positions of the harmonic derivatives at the vertices (recall (14)). The computation goes along the same lines as in the case of the ghost contribution. The result just cancels out Γ_2 (30), so the total SYM and matter one--loop contribution to YM self-energy is zero /7/ . In particular, this means the absence of ultraviolet divergences at this level in this particular combination of N=2 SYM and hypermultiplet matter.

Of course, above one can recognize the well-known N=4 SYM theory written down in terms of N=2 superfields. To see this explicitly, one may check that the classical action $S = S^{N=2}_{SYM} + S^{N=2}_{\omega}$ is invariant under the following N=4 supersymmetry transformations

$$\delta V^{++} = \varepsilon^{i\alpha} u_i^+ \theta_\alpha^+ \cdot \omega + (\overset{*}{\ })$$

$$\delta \omega = \frac{1}{2} (D^+)^4 (\varepsilon^{i\alpha} u_i^- \theta_\alpha^- e^{iv} D^{--} e^{-iv}) + (\overset{*}{\ }).$$

Another manifestation of the N=4 supersymmetry is the fact that the graph in Fig.3 with an ω matter line vanishes, as can easily be seen. It is just the superpartner of the vanishing contribution to the V^{++} self-energy considered above.

The above cancellation of N=2 SYM and matter contributions to the V^{++} self-energy can be extended to the case of q^+ matter in some (in general, complex) representation R. The contribution of the graph in Fig.6 with a q^+ loop is

$$\Gamma_2^{\,q} = 2g^2 T(R) \int \frac{d^4k\, d^4p}{(2\pi)^8 p^2 (k-p)^2}\, d^8\theta\, du_1 du_2 \frac{V_a^{++}(1) V_a^{++}(2)}{(u_1^+ u_2^+)^2} \qquad (31)$$

where $tr\,(T_R^a T_R^b) \equiv \delta^{ab} T(R)$ (T(adjoint) = 1). Comparing (30) with (31) we see that cancellation occurs provided T(R)=1, or, if there are n hypermultiplets q_i in representation R_i, provided $\sum_{i=1}^{n} T(R_i) = 1$ /3,10/.

At the end we shall show an example of a graph giving a nonvanishing finite contribution (Fig.7). The evaluation is straightforward. First we restore the measures $d^8\theta$ at all the vertices. The D-algebra is trivial because only 8 D's are available. The result of the θ integration and of half of the u integrations is

$$\Gamma_4 \sim g^4 \int \frac{d^4k\, d^4p_1 \dots d^4p_4\, \delta(p_1+p_2-p_3-p_4)}{k^2 (k-p_1)^2 (k-p_1-p_2)^2 (k-p_4)^2}\, d^8\theta\, du_1 du_2 \frac{1}{(u_1^+ u_2^+)^2}$$

$$\cdot \overset{*}{q}{}^{+r}(p_1, \theta, u_1)\, \overset{*}{q}{}^{+s}(p_2, \theta, u_1)\, q_s^+(p_3, \theta, u_2)\, q_r^+(p_4, \theta, u_2).$$

On-shell the u_2 integral can be computed as explained in (26), and one finds

$$\Gamma_4 \sim g^4 \int d^8\theta\, du\, \overset{*}{q}{}^{+r}(1)\, \overset{*}{q}{}^{+s}(2) (D^{--})^2 [q_s^+(3)\, q_r^+(4)]$$

times the momentum integral which is convergent.

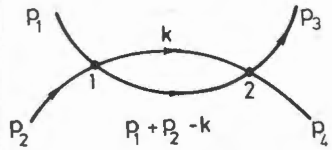


Fig.1. One-loop correction to the 4-point function for a self-interacting q^+ hypermultiplet.

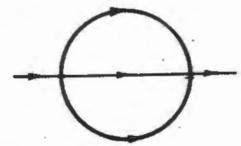


Fig.2. An example of two-loop supergraph.

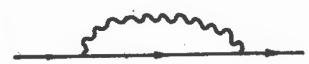


Fig.3. One-loop self-energy correction for q^+ coupled to N=2 SYM.

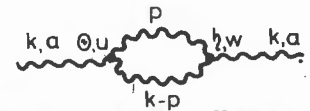


Fig.4. The Yang-Mills contribution to the one-loop V^{++} self-energy.

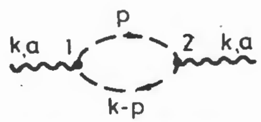


Fig.5. The ghost contribution to the one-loop V^{++} self-energy.

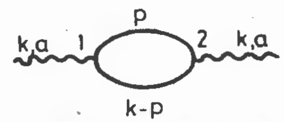


Fig.6. The hypermultiplet contribution to the one-loop V^{++} self-energy.

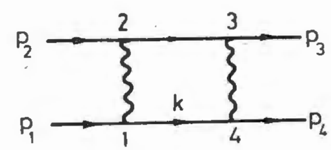


Fig.7. A supergraph giving nonvanishing finite correction.

IV. Conclusion

In the present paper (Parts I and II) a manifestly N=2 supersymmetric supergraph technique has been developed for the first time. This has become possible owing to the constructive application of the recently introduced concept of harmonic superspace. This superspace is an adequate framework for describing both the N=2 matter and gauge multiplets. A complete set of Feynmann rules has been presented. The examples given above show that handling these rules is not more difficult than in the case N=1. The crucial advantage is the preservation of manifest N=2 supersymmetry at each step of the calculations.

Previous experience, e.g., with the quantized Kaluza-Klein theories, seemed to indicate that introducing additional bosonic coordinates should lead to new divergences. This is not the case with the harmonic coordinates, as we have show n in a number of examples above. Apparently, the reason is that the harmonic coordinates give rise to infinite towers of auxiliary or (and) gauge degrees of freedom which do not propagate. On the contrary, in the Kaluza-Klein context the additional coordinates imply infinite sets of new propagating modes, i.e., the physical content of the theory is much larger than required.

The actions for the various N=2 multiplets under consideration are given as integrals over either the analytic (matter ghosts) or chiral (SYM) subspaces of harmonic superspace. Nevertheless, the investigation of the Feynman rules shows that the quantum corrections can always be written as integrals with the full Grassmann measure $d^8\theta$. The integrand is constructed of analytic superfields depending on the same coordinate θ but on different harmonic coordinates (as well as different momenta). This harmonic nonlocality is natural, it resembles the nonlocality in x-space. Moreover, if one fixes the superisospins of the external lines (e.g., by putting them on-shell) all the harmonic integrals can be computed by simple algebraic manipulations and the dependence on the harmonic coordinates disappears.

The fact that the effective action is an integral of the type $\int d^8\theta$ yields a significant improvement in the ultraviolet behaviour [7]. Indeed, in N=2 theories with a dimensionless coupling constant the maximal divergences are logarithmic, and graphs with external matter lines are even superficially convergent. This situation will further improve in the N=3 SYM theory [11] where the dimensionality of the full measure $d^{12}\theta$ automatically implies the ultraviolet finiteness. A generalization of the quantization technique developed here to the case N=3 will be reported elsewhere.

Staying within the framework of the N=2 theories one can investigate the finiteness of N=2 SYM coupled to matter by employing the powerful background field method [7,12]. In our approach the splitting of the gauge superfield into quantum and background parts is as simple as in the case N=0 due to the linear transformation law of the prepotential. The generalization of this method to the case N=2 will soon be reported. It will help to study the mechanisms for soft breaking of supersymmetry and rigid SU(2) -symmetry.

One of the most intriguing problems ahead is the quantization of N=2 Einstein supergravity. In [4] we found the relevant prepotentials and their gauge group. The action for this theory is given as an integral over the chiral N=2 superspace just as the N=2 SYM action. Above we have seen that the latter can be rewritten as an analytic integral if the integrand is expanded in terms of the analytic prepotentials by means of subsequent variations of the action. We believe that the same method will help to develop suitable perturbation expansion technique for N=2 supergravity too.

Acknowledgements. Authors are indebted to S.Kalitzin and B.Zupnik for comments and collaboration at early stage of the work and to R.E.Kallosh, E.Nissimov, S.Pacheva, D.V.Shirkov, O.V.Tarasov for useful discussions.

Appendix: On the duality between q^+ and ω hypermultiplets

Here we shall show that any self-interaction of q^+ hypermultiplets of the type (2) admits a dual form in terms of ω hypermultiplets. First we shall find the duality transformation for a pair of q^+-multiplets and then apply it for a single q^+-multiplet.

Consider two hypermultiplets q^+_i, i= 1,2 in the fundamental representation of an extra (gauge) group $SU_g(2)$ (not to be confused with the proper SU(2) symmetry of the hypermultiplet). One can write down the following invariant Lagrangian

$$\mathcal{L}^{(4)} = \overset{*}{q}{}^{+i} D^{++} q^+_i + \lambda (\overset{*}{q}{}^{+i} q^+_i)^2 . \qquad (A.1)$$

Using the fact that $u_i^\pm$ form a complete set in the two-dimensional doublet space one may decompose $q_i^+, \overset{*}{q}^{+i}$ as follows

$$q_i^+ = u_i^+\omega + u_i^- f^{(2)}, \quad \overset{*}{q}^{+i} = u^{+i}\overset{*}{\omega} + u^{-i}\overset{*}{f}^{(2)}. \qquad (A.2)$$

Here $\omega, f^{(2)}$ are analytic (but not necessarily real) superfields. Substituting (A.2) into (A.I) one obtains (up to total harmonic derivatives)

$$\mathcal{L}^{(4)} = -f^{(2)}D^{++}\overset{*}{\omega} - \overset{*}{f}^{(2)}D^{++}\omega - \overset{*}{f}^{(2)}f^{(2)} + \lambda\left[\overset{*}{\omega}^2(f^{(2)})^2 + \omega^2(\overset{*}{f}^{(2)})^2 - 2\omega\overset{*}{\omega}f^{(2)}\overset{*}{f}^{(2)}\right]. \qquad (A.3)$$

Clearly, $f^{(2)}$ does not propagate and can be eliminated with the help of the equations of motion. After that the Lagrangian (A.3) is expressed only in terms of $\omega, \overset{*}{\omega}$,

$$\mathcal{L}^{(4)} = (1+4\lambda\omega\overset{*}{\omega})^{-1}\left[(1+2\lambda\omega\overset{*}{\omega})D^{++}\omega D^{++}\overset{*}{\omega} + \lambda\omega^2(D^{++}\overset{*}{\omega})^2 + \lambda\overset{*}{\omega}^2(D^{++}\omega)^2\right]. \qquad (A.4)$$

This Lagrangian has an automorphism group $SU_A(2)$ which is the diagonal subgroup in the direct product $SU_g(2) \times SU(2)$. Transformations from the coset $SU_g(2) \times SU(2) / SU_A(2)$ are realized implicitly in (A.4).

Now we may apply the above trick to the case of a single hypermultiplet. To this end we rewrite q^+ and $\overset{*}{q}^+$ as a "doublet" q_i^+ subject to the reality condition

$$\overset{*}{q}^{+i} = \varepsilon^{ij}q_j^+; \quad q^+ = q_1^+ + iq_2^+, \quad \overset{*}{q}^+ = q_2^+ + iq_1^+. \qquad (A.5)$$

The kinetic term is invariant under the extra SU(2) realized on the indices i,j:

$$\mathcal{L}_K^{(4)} = \overset{*}{q}^+D^{++}q^+ = \overset{*}{q}^{+i}D^{++}q_i^+$$

(again, this equality holds up to total harmonic derivatives).

However, this invariance is not preserved in the interaction, which is not essential.

The doublet (A.5) can be decomposed as in (A.2). The reality condition (A.5) means

$$\overset{*}{\omega} = \omega, \quad \overset{*}{f}^{(2)} = f^{(2)}.$$

Now one can insert (A.5) into the general self-interaction Lagrangian (2) for q^+. Once again $f^{(2)}$ obeys an algebraic equation of motion

$$f^{(2)} = -\frac{1}{2}D^{++}\omega + F^{(2)}(\omega, f^{(2)}),$$

solving which one arrives at the ω -representation of the q^+ Lagrangian (2).

At the quantum level the q^+-ω duality manifests itself as a relation between the ω and q^+ propagators. For instance, for the case (A.2) one has

$$\langle\overset{*}{\omega}(1)\,\omega(2)\rangle \doteq -u^-_{1i}\,u_2^{-j}\langle\overset{*}{q}^{+i}(1)\,q_j^+(2)\rangle \sim \frac{i}{p^2}(D_1^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2)\frac{u_1^-u_2^-}{(u_1^+u_2^+)^3}$$

which coincides with (10).

References

1. A.Galperin, E.Ivanov, V.Ogievetsky, E.Sokatchev, JINR preprint E2-85-29, Dubna (January, 1985).
2. W.Siegel, Univ.of California prepr. UCB-PTH-84/25, Berkeley, (September, 1984).
3. J.P.Derendinger, S.Ferrara, A.Masiero, Phys.Lett. 143B (1984) 133.
4. A.Galperin, E.Ivanov, S.Kalitzin, V.Ogievetsky, E.Sokatchev. Class. Quantum Grav. 1 (1984) 469.
5. A.Karlhede, U.Lindstrom, M.Roček, Stony Brook preprint, ITP-SB-84-54 (June, 1984).

6. P.S.Howe, K.S.Stelle, P.K.Townsend, Nucl. Phys. B214 (1983) 519.
7. S.J.Gates, M.T.Grisaru, M.Roček, W.Siegel, Superspace (Benjamin/Cummings, Reading (1983)).
8. A.Morozov, A.Perelomov, Preprint ITEP-131, Moscow (1984).
9. Ya.Kogan, A.Morozov, A.Perelomov, Pisma ZhETP, 40 (1984) 38.
10. P.S.Howe, K.S.Stelle, P.C.West, Phys.Lett. 124B (1983) 55.
11. A.Galperin, E.Ivanov, S.Kalitzin, V.Ogievetsky, E.Sokatchev. JINR preprint E2-84-441, Dubna (June, 1984).
12. M.T.Grisaru, D.Zanon, Phys.Lett. 142B (1984) 359.

Received by Publishing Department
on January 29, 1985.

Гальперин А. и др. E2-85-128

Гармонические суперграфики. Правила Фейнмана и примеры

Эта статья завершает описание процедуры квантования в гармоническом суперпространстве. Выведены правила Фейнмана для теорий $N=2$ материи и Янга-Миллса и даны различные примеры вычисления гармонических суперграфиков. Вычисления выглядят не намного сложнее, чем в $N=1$ случае. Интегрирование по гармоническим переменным не ведет к каким-либо трудностям. Нелокальности по этим переменным исчезают на массовой оболочке. Важно, что квантовые поправки всегда записываются как интегралы по полному гармоническому суперпространству, несмотря на то, что исходное действие было интегралом по аналитическому подпространству. В качестве побочного результата мы получаем очень простое доказательство конечности широкого класса $N=4$, $d=2$ нелинейных σ моделей. Мы рассматриваем самые общие взаимодействия гипермультиплетов, включая те, которые нарушают $SU(2)$. Установлены соотношения дуальности между $N=2$ линейным мультиплетом и обоими типами гипермультиплетов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Galperin A. et al. E2-85-128

Harmonic Supergraphs. Feynman Rules and Examples

This paper completes a description of the quantization procedure in the harmonic superspace approach. The Feynman rules for $N=2$ matter and Yang-Mills theories are derived and the various examples of harmonic supergraph calculations are given. Calculations appear to be not more difficult than those in the $N=1$ case. The integration over harmonic variables does not lead to any troubles, a non-locality in these disappears on-shell. The important property is that the quantum corrections are always written as integrals over the full harmonic superspace even though the initial action is an integral over the analytic subspace. As a by-product our results imply a very simple proof of finiteness of a wide class of the $N=4$, $d=2$ nonlinear σ-models. We consider the most general self-couplings of hypermultiplets including those with broken $SU(2)$. The duality relations among the $N=2$ linear multiplet and both kinds of hypermultiplet are established.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985