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IRREDUCIBLE ∗-REPRESENTATIONS OF LIE SUPERALGEBRAS B(0, n) WITH FINITE-DEGENERATED VACUUM.

General Considerations

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1. Introduction

Recently, we have constructed for the Lie superalgebras $osp(1,2n)$, $n=1,2$, families of infinite-dimensional linear representations $z \mapsto \Omega^{(n)}(z)$. /1-3/ The representations $\Omega^{(1)}$ form a family depending on one parameter that can assume any real value. The family of representations $\Omega^{(2)}$ is labelled by parameters N and $\varkappa$: $N=2,4,\ldots$, and $\varkappa$ takes values in some interval $\mathcal{K}_N \subset \mathbb{R}$. For each $z \in osp(1,2n)$ the operator $\Omega^{(n)}(z)$ is a linear differential operator on $\mathcal{F}_N(M_n)$ – the space of C^∞ vector functions $M_n \ni x \mapsto \phi(x) \in \mathbb{C}^N$, where $M_1 := \mathbb{R}^+$ ($N=2$ for $n=1$) and M_2 is some open subset of $\mathbb{R}^+ \times \mathbb{R}^2$.

Under $osp(1,2n)$ we understand the unique real form of the complex Lie superalgebra LSA $B(0,n)$. This LSA is generated by 2n odd elements y_l, $l=\pm1,\ldots,\pm n$; their symmetric products determine $n(2n+1)$ independent even elements

$$x_{jk} := \frac{1}{2}\langle y_j, y_k\rangle = x_{kj} \tag{1.1a}$$

and the law of multiplication reads

$$\langle x_{jk}, y_l\rangle := -g_{jl}y_k + g_{kl}y_j \,, \quad g_{jl} := \mathrm{sgn}(j)\delta_{j+l} \,. \tag{1.1b}$$

By using the Jacobi identity, one gets

$$\langle x_{jk}, x_{lm}\rangle = g_{jl}x_{km} + g_{jm}x_{kl} + g_{kl}x_{jm} + g_{km}x_{jl} \,. \tag{1.1c}$$

The basis $\{x_{jk}, y_l \;:\; j,k,l=\pm1,\ldots,\pm n\}$ will be called Racah as its even part is identical with the basis of $sp(2n,\mathbb{R})$ considered in Ref.4.

For discussing properties of representations Ω, it is convenient to regard the space $\Lambda(\mathcal{F}_N)$ of linear differential operators on $\mathcal{F}_N(M)$ as an associative $*$-algebra equipped with adjoint operation "#" /5/ and introduce the standard LSA structure on its linear subspace $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ /6/ which is determined via the Racah basis as

$$\mathcal{A}_0 := \{\Omega(x_{jk}) : j,k=\pm1,\ldots,\pm n\}_{lin} \,, \mathcal{A}_1 := \{\Omega(y_l) : l=\pm1,\ldots,\pm n\}_{lin} \,.$$

Basic features of the representations Ω can now be summarized as follows:

(i) Each Ω is a homomorphism of $osp(1,2n)$ on $\mathcal{A}$, the order of the differential operator $\Omega(z)$ being at most one if z is an odd element and at most two if z is even.

(ii) Let $z \mapsto z^*$ be the involution on $B(0,n)$ /7/ defined as the linear extension of

$$x^*_{jk} := -x_{jk}, \quad y^*_l := -iy_l \,. \tag{1.2}$$

Then one has

$$\Omega(z^*) = \Omega(z)^{\#} \quad /5/, \tag{1.3}$$

i.e. Ω is a $*$- homomorphism. In particular, for elements of the even subalgebra $sp(2n,\mathbb{R})$ holds $x^* = -x$, so that $\Omega(x)$ is skew-symmetric:

$$\Omega(x)^{\#} = -\Omega(x) \ , \quad x \in sp(2n,\mathbb{R}) \ .$$

/iii/ All independent Casimir elements of $osp(1,2n)$ (there is just one for $n=1$ and two for $n=2$) are represented by multiples of unity in $\Lambda(\mathcal{F}_N)$ (Schur-irreducibility), the parameters which label Ω being in one-to-one correspondence with these numbers.

Remark: Schureanity does not imply algebraic irreducibility /8/ and the representations Ω are indeed reducible. Finding their irreducible restrictions is one of the basic problems solved in this study.

2. Formulation of the problem

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. A set $\{\hat{a}_j, \hat{a}_j^{\dagger} : j=1,2,..,n\} \subset \mathrm{End}_{\mathcal{H}}\mathcal{D}$ (see Appendix for definitions) is called pB_n-set with domain $\mathcal{D}$ if

$$[\{\hat{a}_j,\hat{a}_k\},\hat{a}_l] = 0, \tag{2.1a}$$

$$[\{\hat{a}_j,\hat{a}_k^{\dagger}\},\hat{a}_l] = -2\,\delta_{k-l}\,\hat{a}_j . \tag{2.1b}$$

The $\hat{a}_j$ ($\hat{a}_j^{\dagger}$) is interpreted as the j-th mode annihilation (creation) operator and

$$\hat{n}_j := \frac{1}{2}\{\hat{a}_j,\hat{a}_j^{\dagger}\} \tag{2.2}$$

as the j-th mode particle-number operator of a para-Bose system with n degrees of freedom /9/.

A simple example of a pB_n-set provide annihilation and creation operators of usual bosons that satisfy the canonical commutation relation (CCR). It is known that for each $n=1,2,..$ such a set for which the operator $\hat{a}_1^{\dagger}\hat{a}_1 + .. + \hat{a}_n^{\dagger}\hat{a}_n$ is e.s.a. is just one up to equivalences /10/. That's why only "non-trivial" solutions of Eqs.(2.1a,b) are of interest, viz. those for which the CCR do not hold. The first example was given for n=1 by Wigner /11/; later on, Green discovered for arbitrary n, including $n=\infty$, an infinite set of solutions labelled by one integer $p=1,2,..$ called "order" and Greenberg with Messiah /12/ selected among them the irreducible ones acting on a Fock space with a unique vacuum.

Our aim is obtaining irreducible pB_n-sets for n=1,2 from the representations $\Omega^{(n)}$. This is possible, at least in principle, because pB_n-sets and representations of $B(0,n)$ are closely related. In order to get a formulation suitable for the purposes of this study, consider another basis $\{b_{jk}, a_l : j,k,l = \pm 1,..,\pm n\}$ in $B(0,n)$ defined via the

following map ζ:

$$a_l \equiv \zeta(y_l) := 2^{-1/2}(y_l - i y_{-l}) \tag{2.3a}$$

$$b_{jk} \equiv \zeta(x_{jk}) := \frac{1}{2}\langle a_j, a_k\rangle = \frac{1}{2}(x_{jk} - x_{-j-k}) - \frac{1}{2}(x_{-jk} + x_{k-j}) \ . \tag{2.3b}$$

Easily can be verified that all the structure constants are identical with those of the Racah basis (see Eqs.(1.1)); thus, ζ is an automorphism of $B(0,n)$. Moreover, one gets from (1.2)

$$a_l^* = a_{-l}, \quad b_{jk}^* = b_{-j-k} \ . \tag{2.3c}$$

By using these facts, one readily gets the sought interrelation of pB_n-sets and $*$-representations of $B(0,n)$:

<u>2.1 Proposition</u>: If π is an irreducible $*$-representation of $B(0,n)$ in terms of operators in $\mathrm{End}_{\mathcal{H}}\mathcal{D}$, then $\{\pi(a_j), \pi(a_j)^{\dagger} : j=1,2,..,n\}$ is an irreducible pB_n-set /13/.

In view of this assertion, the problem of constructing pB_n-sets from representations $\Omega^{(n)}$ can be formulated purely in the language of the representation theory as follows: Given a linear representation Ω on $\mathcal{F}$, find an Ω- invariant subspace $\mathcal{D} \subset \mathcal{F}$ and introduce a scalar product on $\mathcal{D}$ such that the operators

$$\pi(z) := \Omega(z)\restriction\mathcal{D} \ , \quad z \in B(0,n) \tag{2.4a}$$

form an irreducible representation of $B(0,n)$ on $\mathcal{H} := \overline{\mathcal{D}}$ and fulfil

$$\pi(z)^{\dagger} = \Omega(z)^{\#}\restriction\mathcal{D} \ . \tag{2.4b}$$

The required $*$-property means that

$$\hat{N} := \sum_{j=1}^{n}\hat{n}_j = \frac{1}{2}\sum_{j=1}^{n}\{\pi(a_j), \pi(a_j)^{\dagger}\} = \sum_{j=1}^{n}\pi(b_{j-j})$$

has to be a positive operator

$$\hat{N} \geq 0 \ , \tag{2.4c}$$

and, due to its interpretation as particle-number operator, must have non-empty point spectrum. Using the standard considerations based upon the relations $[\hat{N},\hat{a}_j] = -\hat{a}_j$, $j=1,..,n$ (cf.Eq.(2.1b)), one concludes that the sought domain $\mathcal{D}$ must have non-trivial intersection with the vacuum subspace

$$V_{\Omega} := \{\phi \in \mathcal{F} : \tilde{a}_j\phi = 0, \ j=1,..,n\} , \quad \tilde{a}_j \equiv \Omega(a_j) \ . \tag{2.5}$$

The usual requirement of uniqueness of the vacuum will be replaced by a weaker condition

$$1 \leq \dim \mathcal{D} \cap V_{\Omega} < \infty \tag{2.6}$$

since uniqueness, which is essential in the quantum field theory, is a too restrictive condition when systems with finite number of degrees of freedom are concerned. Representations $\pi = \Omega \upharpoonright \mathcal{D}$ which fulfil Eq. (2.6) are said to have finite-degenerated vacuum, FDV-representations. Let us recall in this context that for representations with a unique vacuum ϕ_0 one has [14]

$$\hat{a}_k \hat{a}_1^\dagger \phi_0 = p\, \delta_{k-1} \phi_0 , \quad p > 0,$$

where p is independent of k and is called the order (of parastatistics) [12]. This relation is in general not valid if the uniqueness of vacuum is violated; in particular, the notion of order does not make sense for representations with degenerated vacuum.

3. General features of the construction

We have seen in Sect.2 that the problem consists in finding an Ω-invariant subspace $\mathcal{D} \subset \mathcal{F}$ such that conditions (2.4) and (2.6) are fulfilled. To this purpose we developed a method that can be divided into two steps. In the first one we derive general properties having the form of necessary conditions which follow from the assumption that for a given Ω there exists a domain $\mathcal{D}$ with all the required properties. In this way, the original family of representations Ω is reduced by excluding all those Ω that do not fulfil the necessary conditions. The conditions themselves are obtained by examining the structure of the subspace

$$\mathcal{D}^{(vac)} := \mathcal{D} \cap V_\Omega .$$

The second step is inductive; it deals with the construction of $\mathcal{D}$ starting with a fixed vector ψ in the vacuum subspace V_Ω, this vector being fully specified by the above necessary conditions.

The starting point for analysing the subspace $\mathcal{D}^{(vac)}$ represent the following simple assertions.

3.1 Lemma: (i) The linear envelope of $\{b_{j-k}: j,k=1,..,n\}$ is a subalgebra of $sp(2n,\mathbb{C})$ that is isomorphic to $gl(n,\mathbb{C})$.

(ii) For each $u \in gl(n,\mathbb{C})$ one has

$$\Omega(u)\, V_\Omega \subset V_\Omega . \qquad (3.1)$$

Proof: The first statement is due to the fact that b_{j-k} satisfy the same commutation relations as the elements of the standard basis $\{\varepsilon_{jk}\}$ of $gl(n,\mathbb{C})$ if $\varepsilon_{jk} \leftrightarrow b_{k-j}$. The relation (3.1) follows from $\langle b_{j-k}, a_1 \rangle = -\delta_{k-1} a_j$ and Eq.(2.5).

3.2 Remark: Notice that the real linear envelope of $i(b_{j-k}+b_{k-j})$, $(b_{j-k}-b_{-jk})$ $j,k=1,..,n$, equals $u(n)$ which is isomorphic to $sp(2n,\mathbb{R}) \cap sp(2n)$, i.e., to the maximal compact subalgebra of $sp(2n,\mathbb{R})$.

3.3 Corollary: Suppose that $\Omega \upharpoonright \mathcal{D}$ is a FDV-representation; then $\mathcal{D}^{(vac)}$ is a finite-dimensional subspace invariant under $\Omega(u)$, $u \in gl(n,\mathbb{C})$.

In view of these properties, one can learn much about $\mathcal{D}^{(vac)}$ by applying the theory of finite-dimensional representations of semi-simple Lie algebras [15]. Consider the map ω:

$$sl(n,\mathbb{C}) \ni u \mapsto \omega(u) := \Omega(u) \upharpoonright V_\Omega ;$$

because of 3.1, ω is a representation of $sl(n,\mathbb{C})$ on V_Ω. Now suppose that for a given Ω there is an Ω-invariant domain $\mathcal{D} \subset \mathcal{F}$ such that $\Omega \upharpoonright \mathcal{D}$ fulfils the conditions (2.4b, 2.6). By the corollary,

$$\omega_{vac} := \omega \upharpoonright \mathcal{D}^{(vac)}$$

is a finite-dimensional representation of $sl(n,\mathbb{C})$ and hence

$$\mathcal{D}^{(vac)} = \sum_J{}^{\oplus} V_{\lambda(J)} , \qquad (3.2)$$

each $V_{\lambda(J)}$ being the representation space of an irreducible representation of $sl(n,\mathbb{C})$ with the highest weight (HW) $\lambda(J)$. Consider the particle number operator

$$\tilde{N} := \sum_{j=1}^{n} \Omega(b_{j-j}) ;$$

by Eq.(2.4c), $\tilde{N}_{vac} := \tilde{N} \upharpoonright \mathcal{D}^{(vac)}$ is a positive matrix. Let $\nu \geq 0$ be its eigenvalue, W_ν the subspace of eigenvectors corresponding to ν. As $[\omega_{vac}(u), \tilde{N}_{vac}] = 0$ for each $u \in sl(n,\mathbb{C})$ (Lemma 3.1), the subspace W_ν is invariant under ω_{vac} and thus W_ν equals direct sum of a subsystem of subspaces $V_{\lambda(J)}$ in (3.2). Consequently, to each subspace $V_{\lambda(J)}$ there is an eigenvalue ν of $\tilde{N}_{vac}$ such that $V_{\lambda(J)} \subset W_\nu$; in particular, the corresponding HW-vector $\psi_{\lambda(J)} \in V_{\lambda(J)}$ fulfils

$$\tilde{N}\, \psi_{\lambda(J)} = \nu\, \psi_{\lambda(J)} . \qquad (3.3)$$

By summarizing, we arrive at the following necessary conditions:

3.4 Proposition: Suppose that for a given Ω there is an Ω-invariant domain $\mathcal{D} \subset \mathcal{F}$ such that $\Omega \upharpoonright \mathcal{D}$ is a FDV-representation having the $*$-property (2.4b). Then there exist non-negative integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1}$, a non-negative ν and a non-zero vector $\psi_\lambda \in \mathcal{D}^{(vac)}$ such that Eq.(3.3) and

$$\omega(b_{j-j} - b_{j+1,-j-1})\, \psi_\lambda = \lambda_j\, \psi_\lambda \qquad (3.4a)$$

$$\omega(b_{j+1,-j})\, \psi_\lambda = 0 \qquad (3.4b)$$

are fulfilled for $j=1,2,..,n-1$. Moreover,

$$\omega_\lambda := \omega \upharpoonright \mathcal{U}(\omega(b_{-21}), \omega(b_{-32}), .., \omega(b_{-n,n-1}))\, \psi_\lambda$$ [16]

is an irreducible representation of $sl(n,\mathbb{C})$.

Since $\Omega(u)$ are linear differential operators, the conditions (3.3,4) together with $\psi_\lambda \in V_\Omega$ represent a system of partial differential equations (PDE). By demanding existence of a non-trivial solution, one gets conditions that relate ν and the integers λ_j to parameters labelling Ω (e.g., for n=2 there are two: N and $\varkappa$ - cf. Sect. 1). The first step of our approach consists in finding for given non-negative integers $\lambda_1 \geq \dots \geq \lambda_{n-1}$ the admissible values of these parameters, solving the PDE-systems with these values and verifying irreducibility of ω_λ.

Let us pass to the second step. Suppose we found the function ψ_λ that solves the PDE-system for given $\lambda_1 \geq \dots \geq \lambda_{n-1}$, $\nu \geq 0$ and some fixed representation Ω with admissible parameters. We will find an Ω-invariant domain $\mathcal{D}_\lambda \subset \mathcal{F}$ such that $\psi_\lambda \in \mathcal{D}_\lambda$ and $\pi_\lambda := \Omega \restriction \mathcal{D}_\lambda$ has all the required properties. It is clear that the only candidate is

$$\mathcal{D}_\lambda := \mathcal{U}(\tilde a_1, \tilde a_1^\#, \dots, \tilde a_n, \tilde a_n^\#)\, \psi_\lambda \;. ^{/16/} \tag{3.5}$$

However, one has to verify that the representation π_λ is irreducible and that the FDV-condition $1 \leq \dim \mathcal{D}_\lambda \cap V_\Omega < \infty$ is fulfilled; in addition, a scalar product must be introduced on $\mathcal{D}_\lambda$ such that the $\star$-condition (2.4b) holds.

To this purpose we first try to make the structure of $\mathcal{D}_\lambda$ lucider by finding a basis $\mathcal{E} \subset \mathcal{D}_\lambda$ with some specific properties. Naturally, we demand

$$\psi_\lambda \in \mathcal{E} \,. \tag{3.6a}$$

As a direct consequence of Eqs.(3.3,4) one finds that ψ_λ is a common eigenvector of $\tilde n_j := \Omega(b_{j-j})$, j=1,...,n :

$$\tilde n_j \psi_\lambda = \nu_j \psi_\lambda \;, \tag{3.6b}$$

each eigenvalue ν_j being a simple function of ν and $\lambda_1, \dots, \lambda_{n-1}$. Further, the commutation relation (2.1b) implies for p=1,2,..

$$\tilde n_j \tilde a_k^p = \tilde a_k^p (\tilde n_j - p\,\delta_{j-k}),\quad \tilde n_j \tilde a_k^{\# p} = \tilde a_k^{\# p}(\tilde n_j + p\,\delta_{j-k}). \tag{3.7}$$

A straightforward generalization together with Eq.(3.6b) yield for any monomial $\tilde M \in \mathcal{U}(\tilde a_1, \tilde a_1^\#, \dots, \tilde a_n, \tilde a_n^\#) \equiv \mathcal{U}$

$$\tilde n_j \tilde M \psi_\lambda = (\nu_j + r_j)\tilde M \psi_\lambda ,\quad r_j = 0, \pm 1, \pm 2, \dots,\quad j=1,2,\dots,n.$$

Thus, $\mathcal{D}_\lambda$ is spanned by functions $\phi \in \mathcal{F}$ labelled by integers $r_1, \dots, r_n$, each $\phi \equiv \phi_{r_1 \dots r_n}$ fulfilling

$$\tilde n_j \phi = (\nu_j + r_j)\phi \;,\quad j=1,\dots,n. \tag{3.6c}$$

This is again a system of PDE and we shall see that particular solutions can be found analytically for both the cases n=1,2.

Let $\mathcal{E}$ be a linearly independent set of solutions to (3.6c) for some infinite system of n-tuples $(r_1, \dots, r_n)$ (notice that in view of

Eqs.(3.6a,b) (0,..,0) must belong to the system) such that

$$\mathcal{U}\mathcal{E} \subset \mathcal{E}_{\rm lin} \,. \tag{3.8a}$$

Then one clearly has

$$\mathcal{D}_\lambda \subset (\mathcal{U}\mathcal{E})_{\rm lin} = \mathcal{E}_{\rm lin} \,. \tag{3.8b}$$

Due to the following simple argument one can expect that (3.8a) will hold as soon as $\mathcal{E}$ is sufficiently large. For any $\phi_{r_1 \dots r_n} \in \mathcal{E}$ one gets from (3.6c,3.7)

$$\tilde n_j \tilde a_k \phi = (\nu_j + r_j - \delta_{j-k})\tilde a_k \phi \,.$$

Comparing with Eq.(3.6c) suggests that $\tilde a_k \phi$ should be identified with $\phi_{r_1 \dots r_j - 1 \dots r_n}$ and thus $\tilde a_k \phi_{r_1 \dots r_n} \in \mathcal{E}$ if $\phi_{r_1 \dots r_j - 1 \dots r_n} \in \mathcal{E}$, etc. In fact, in cases we will consider in the following, it is possible to choose $\mathcal{E}$ such that $\tilde a_k \mathcal{E} \subset \mathcal{E} \cup \{0\}$, $\tilde a_k^\# \mathcal{E} \subset \mathcal{E}$, i.e., the action of $\tilde a_k$, $\tilde a_k^\#$ on any $\phi \in \mathcal{E}$ is very simple.

As soon as this action is found, one can verify directly whether $\Omega \restriction \mathcal{E}_{\rm lin}$ is irreducible. If it is so, then from Eq.(3.8b) follows in view of invariance of $\mathcal{D}_\lambda$ under Ω

$$\mathcal{D}_\lambda = \mathcal{E}_{\rm lin}$$

and hence $\mathcal{E}$ is a basis in $\mathcal{D}_\lambda$.

Having such a basis is very helpful in introducing a scalar product on $\mathcal{D}_\lambda$ obeying the $\star$-condition (2.4b). This condition requires the operators $\hat n_j := \tilde n_j \restriction \mathcal{D}_\lambda$ to be symmetric and hence the scalar product must be chosen in such a way that $\mathcal{E}$ becomes an orthogonal set. This can always be done: suppose $\mathcal{E} = \{\phi_r\}_{r=1}^\infty$ (for simplicity the elements of $\mathcal{E}$ are labelled by a single index) and let

$$(\phi_r, \phi_s) := t_r \delta_{r-s},\ r,s=1,2,\dots,\ t_r > 0 \,.$$

The Hilbert space $\mathcal{H} = \overline{\mathcal{D}_\lambda} = \overline{\mathcal{E}_{\rm lin}}$ then consists of all functions

$$\phi = \sum_{r=1}^{\infty} c_r \phi_r \quad \text{such that} \quad \sum_{r=1}^{\infty} t_r |c_r|^2 < \infty \,.$$

However, this choice does not guarantee that (2.4b) is fulfilled. Let us discuss this point in more detail.

First of all, one can replace Eq.(2.4b) by a simpler condition

$$(\tilde a_j \phi, \phi) = (\phi, \tilde a_j^\# \phi) \;,\quad 1 \leq j \leq n,\quad \phi \in \mathcal{D}_\lambda \,, \tag{3.9}$$

since each operator $\Omega(z)$, $z \in B(0,n)$, equals a linear or quadratic function of $\tilde a_j$, $\tilde a_j^\#$. All the operators $\tilde a_j$ can be expressed as (see /5/ and Sect.1)

$$\tilde a_j \equiv \tilde a = f_0 + \sum_{k=1}^{m} f_k p_k \,.$$

Then by the definition of the $\#$-operation /5/, one finds for any $\Phi \in \mathcal{F} \equiv \mathcal{F}_N(M)$, $M \subset \mathbb{R}^m$

$$(\tilde{a}\Phi) \times \Phi - \Phi \times (\tilde{a}^{\#}\Phi) = \sum_{k=1}^{m} p_k \left[\Phi \times (\bar{f}_k \Phi)\right] ,$$

where for $\Phi, \Psi \in \mathcal{F}_N(M)$

$$\Phi \times \Psi := \sum_{r=1}^{N} \varphi_r \overline{\psi_r} ,$$

$\varphi_r, \psi_r \in C^{\infty}(M)$ being the components of Φ and Ψ, respectively. Now the Gauss theorem implies that (3.9) will be fulfilled if $\mathcal{D}_\lambda$ is a subspace in

$$\mathcal{H} = \sum_{r=1}^{N}{}^{\oplus} L^2(M) \tag{3.10a}$$

and if all components of each $\Phi \in \mathcal{D}_\lambda$ vanish on the boundary of M. Of course, this requirement is only sufficient for (3.9) and in case that it were not compatible with the previous conditions imposed upon $\mathcal{D}_\lambda$, one could try another choice of $\mathcal{H}$. However, verifying the condition (3.9) would then probably be difficult.

According to the general definition, the complete specification of each of the sought representations $\Omega \upharpoonright \mathcal{D}$ includes also a projection $\hat{E}$ on the Hilbert space $\mathcal{H}$ that determines the decomposition $\mathrm{End}_{\mathcal{H}}\mathcal{D} = (\mathrm{End}_{\mathcal{H}}\mathcal{D})_0 \oplus (\mathrm{End}_{\mathcal{H}}\mathcal{D})_1$ – cf. Appendix, Eqs.(A.2-4). Assume that

$$\mathcal{H} = \sum_{r=1}^{N}{}^{\oplus} \mathcal{G} , \tag{3.10b}$$

$\mathcal{G}$ being a Hilbert space to which belong all components of each $\Phi \in \mathcal{D}$. Then the projection $\hat{E}$ can be chosen as

$$\hat{E} := I_{\mathcal{G}} \otimes E, \quad E := \sum_{r=1}^{N/2} \varepsilon_{rr} \tag{3.11}$$

This choice is implied by the structure of the operators $\Omega(z)$ /1-3/. The point is that these operators are expressed in terms of two finite subsets $\mathcal{M}_0, \mathcal{M}_1 \subset \mathrm{End}\,\mathbb{C}^N$ and of scalar linear differential operators ξ_α acting on $C^{\infty}(M)$:

$$\Omega(z) = \sum_\alpha \xi_\alpha \otimes T_\alpha ,$$

in such a way that $T_\alpha \in \mathcal{M}_0$ if z is an even element of $B(0,n)$ and $T_\alpha \in \mathcal{M}_1$ if z is odd. In addition, the projection $E \in \mathrm{End}\,\mathbb{C}^N$ satisfies $ET_\alpha = T_\alpha E$ iff $T_\alpha \in \mathcal{M}_0$ and $ET_\alpha = T_\alpha(I-E)$ if $T_\alpha \in \mathcal{M}_1$. It then follows from (3.11) that $\Omega \upharpoonright \mathcal{D}$ will map even and odd elements of $B(0,n)$ in $(\mathrm{End}_{\mathcal{H}}\mathcal{D})_0$ and $(\mathrm{End}_{\mathcal{H}}\mathcal{D})_1$, respectively.

On the other hand, fixing $\hat{E}$ by Eq.(3.11) imposes an additional condition upon the structure of the domain $\mathcal{D}$:

$$\Phi \equiv (\varphi_1, \dots, \varphi_N) \in \mathcal{D} \Rightarrow (\varphi_1, \dots, \varphi_{N/2}, 0, \dots, 0) \in \mathcal{D}$$

(cf.Eq.(A.2)). This will hold, e.g., if

$$\mathcal{D} = \sum_{r=1}^{N}{}^{\oplus} D_r ; \tag{3.12}$$

notice that because of Eq.(3.10b) all D_r must be dense in $\mathcal{G}$.

It appears that for representations $\Omega^{(n)}$, n=1,2, of Refs.1-3 a domain $\mathcal{D}$ can always be found such that (3.12) holds. This is due to the structure of the operators $\tilde{n}_j = \Omega(\Phi_{j-j})$ that allows for decoupling of the system (3.6c) for vector functions $\Phi \equiv (\varphi_1, \dots, \varphi_N)$ into N independent systems of n partial differential equations for individual components φ_r. For getting the basis $\mathcal{E}$, one chooses linearly independent sets $\mathcal{E}_r$ of solutions for the r-th component and puts

$$\mathcal{E} := \bigcup_{r=1}^{N} (0, \dots, \mathcal{E}_r, \dots, 0) .$$

The one has

$$\mathcal{E}_{lin} = \sum_{r=1}^{N}{}^{\oplus} D_r , \quad D_r := (\mathcal{E}_r)_{lin} .$$

In the forthcoming second part of this study the above approach is applied to the linear representations $\Omega^{(1)}$ on $C^{\infty}(\mathbb{R}^+) \otimes \mathbb{C}^2$ of Ref.1. A one-parameter family Π of non-equivalent irreducible representations of $B(0,1)$ on $L^2(\mathbb{R}^+) \otimes \mathbb{C}^2$ with non-degenerated vacuum is obtained. The family Π is complete in the following sense: if $\varkappa \in \mathbb{R} \setminus \{0\}$, $\mathcal{D}'$ is a subspace of $C^{\infty}(\mathbb{R}^+) \otimes \mathbb{C}^2$ having non-trivial intersection with the vacuum subspace $V_{\Omega^{(1)}_\varkappa}$ and $\pi' \equiv \Omega^{(1)}_\varkappa \upharpoonright \mathcal{D}'$ is an irreducible $\ast$-representation, then π' is equivalent to some π in Π.

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<u>Appendix: Hilbert space $\ast$-representations of Lie superalgebras by unbounded operators</u>

It is well-known that Hilbert-space representations of a real LSA whose even subalgebra is non-compact have the following property: if even generators are represented by skew-symmetric operators, then at least one of them must be unbounded /17/. That is how one mostly arrives at $\ast$-representations of LSA by unbounded operators. Corresponding definitions are obtained by generalizing, on the one hand, the definition of finite-dimensional $\ast$-representations of LSA as given, e.g., in Ref. 6 and, on the other hand, that of ∞-dimensional $\star$-representations of Lie algebras [18].

Let $\mathcal{H}$ be an ∞-dimensional separable Hilbert space and D its subspace such that

$$\dim D = \infty \,, \quad \overline{D} = \mathcal{H} \,. \tag{A.1}$$

Consider the set $\mathrm{End}_{\mathcal{H}} D \equiv \mathrm{End}\, D$ of linear operators X on $\mathcal{H}$ satisfying

(i) $D(X) = D$, $\mathrm{Ran}\, X \subset D$,

so that D is a common invariant domain for all $X \in \mathrm{End}\, D$,

(ii) $D(X^+) \supset D$, $\mathrm{Ran}\, X^{\ddagger} \subset D$,

where X^+ is the usual Hilbert-space adjoint of X and

$$X^{\ddagger} := X^+ \upharpoonright D \,.$$

Then End D becomes an associative $\star$-algebra with involution $X \mapsto X^{\ddagger}$.

For a given projection E on $\mathcal{H}$ such that

$$ED \subset D, \quad ED \neq D \tag{A.2}$$

consider the following subsets of End D

$$\begin{aligned} (\mathrm{End}\, D)_0 &:= \{X \in \mathrm{End}\, D : XE\psi = EX\psi ,\ \forall \psi \in D\}, \\ (\mathrm{End}\, D)_1 &:= \{X \in \mathrm{End}\, D : XE\psi = (I-E)X\psi ,\ \forall \psi \in D\}. \end{aligned} \tag{A.3}$$

Then one has

$$\mathrm{End}\, D = (\mathrm{End}\, D)_0 \oplus (\mathrm{End}\, D)_1 \tag{A.4}$$

and End D becomes a LSA if one defines multiplication $X,Y \mapsto \langle X,Y \rangle$ as the bilinear extension of

$$\langle X,Y \rangle := XY - (-1)^{\alpha\beta} YX \,, \quad X \in (\mathrm{End}\, D)_\alpha \,, \ Y \in (\mathrm{End}\, D)_\beta \,, \ \alpha, \beta = 0, 1. \tag{A.5}$$

This LSA, which is completely determined by the associative $\star$-algebra End D and projection E, will be denoted (End D,E) +). The mapping $X \mapsto X^{\ddagger}$ preserves the grading:

$$X \in (\mathrm{End}\, D)_\alpha \Rightarrow X^{\ddagger} \in (\mathrm{End}\, D)_\alpha \,, \quad \alpha = 0, 1 \,. ^{++)}$$

+) Notice that (End D,E) = (End D,I-E) since the conditions (A.2,3) hold for E iff they hold for I-E.

++) Consider, e.g., the case $X \in (\mathrm{End}\, D)_1$; for any $\varphi, \psi \in D$ one has $(EX^{\ddagger}\varphi, \psi) = (\varphi, XE\psi) = (E'\varphi, X\psi)$, $E' := I-E$. Now $E'\varphi \in D \subset D(X^+)$ which implies $(E'\varphi, X\psi) = (X^+E'\varphi, \psi) = (X^{\ddagger}E'\varphi, \psi)$ and as $\overline{D} = \mathcal{H}$, one has $EX^{\ddagger}\varphi = X^{\ddagger}E'\varphi$ for each $\varphi \in D$, i.e., $X^{\ddagger} \in (\mathrm{End}\, D)_1$.

Further this mapping is an involution on End D and thus by (A.5) one sees that (End D,E) is a $\star$-LSA.

Definition: Let a $\star$-LSA $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with multiplication $x, y \mapsto x \cdot y$ and involution $x \mapsto x^{\star}$ be given. Further let D be a subspace in a separable Hilbert space $\mathcal{H}$ and E a projection on $\mathcal{H}$ such that the conditions (A.1,2) are fulfilled. Linear mapping π:

$$\mathcal{A} \ni x \mapsto \pi(x) \in \mathrm{End}\, D$$

is a $\star$-representation of $\mathcal{A}$ on $\mathcal{H}$ with domain D and projection E if:

(i) $\pi(\mathcal{A}_\alpha) \subset (\mathrm{End}\, D)_\alpha$, $\alpha = 0, 1$,

(ii) $\pi(x \cdot y) = \langle \pi(x), \pi(y) \rangle$,

(iii) $\pi(x^{\star}) = (\pi(x))^{\ddagger}$.

References and footnotes

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2. Bednář M., Blank J., Exner P., Havlíček M., JINR E2-82-771, Dubna 1982.
3. Bednář M., Blank J., Exner P., Havlíček M., JINR E2-83-150, Dubna 1983.
4. Racah G., Group Theory and Spectroscopy, CERN Report 61-8, Geneva 1961.
   Notice that also a slightly different basis of osp(1,2n) is called Racah – see Bincer A., J.Math.Phys., 1983, v.24, p.2546.
5. Linear differential operators on $\mathcal{F}_N(M)$, $M \subset \mathbb{R}^m$ are linear maps

$$\phi \mapsto D\phi := \sum_{j_1, \dots, j_m} f_{j_1 \dots j_m} (p_1^{j_1} \dots p_m^{j_m}) [\phi] \,,$$

where $f_{j_1 \dots j_m}$ are C^∞-functions on M with values in the space $\mathrm{End}\, \mathbb{C}^N$ of linear operators on $\mathbb{C}^N$, $p_k^j[\phi] := \partial^j \phi / \partial x_k^j$ and the summation extends over some finite subset of m-tuples of non-negative integers. Clearly these operators form a vector space that will be denoted $\Lambda(\mathcal{F}_N)$. The adjoint of D is defined by

$$D^{\#}\phi := \sum_{j_1, \dots, j_m} (-1)^{j_1 + \dots + j_m} (p_1^{j_1} \dots p_m^{j_m}) [f^+_{j_1 \dots j_m} \phi]$$

(see, e.g., Courant R., Partial Differential Equations, Interscience, New York 1962). By applying the Leibniz formula, one finds that $D^{\#} \in \Lambda(\mathcal{F}_N)$ and similarly can be verified that the set $\Lambda(\mathcal{F}_N)$ is an associative algebra and $D \mapsto D^{\#}$ is an involution on it.

6. Scheunert M., The Theory of Lie Superalgebras, Lect. Notes in Math. Vol. 716, Springer, Berlin 1979.

7. The involution $z \mapsto \vartheta(z) = z^*$ is unique up to equivalence transformations $\vartheta \mapsto \varphi \vartheta \varphi^{-1}$ generated by automorphisms φ of B(0,n)– see Parker M., J.Math.Phys., 1980, v.21, p.689.

8. Representation π (possibly ∞-dimensional) with domain $\mathcal{D}$ is algebraically irreducible if $\{0\}$ and $\mathcal{D}$ are the only π-invariant subspaces of $\mathcal{D}$. Throughout this paper irreducibility always means algebraic irreducibility.

9. Notice that due to Eq.(2.1b) the operators $\hat{n}_j$ have the basic property of usual particle-number operators, viz $[\hat{n}_j, \hat{a}_j] = -\hat{a}_j$, $[\hat{n}_j, \hat{a}_j^{*}] = \hat{a}_j^{*}$. One further has $[\hat{n}_j, \hat{n}_k] = 0$ if $j \neq k$, although different modes need not commute: $[\hat{a}_j, \hat{a}_k] \neq 0$, $[\hat{a}_j^{*}, \hat{a}_k^{*}] \neq 0$ for $j \neq k$.

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11. Wigner E., Phys.Rev., 1950, v.77, p.711.

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Greenberg O.W., Messiah A.M.L., Phys.Rev., 1965, v.138, p.B1155.

13. Conversely, to each irreducible pB_n-set $\{\hat{a}_j, \hat{a}_j^{*} : j=1,2,\dots,n\}$ there is an irreducible $*$-representation π of B(0,n):
$\pi(y_j) := 2^{-1/2}(\hat{a}_j + i\hat{a}_j^{*})$, $\pi(y_{-j}) := 2^{-1/2}(\hat{a}_j^{*} + i\hat{a}_j)$, $j=1,\dots,n$
$\pi(x_{jk}) := 1/2\,\{\pi(y_j), \pi(y_k)\}$, $j,k = \pm 1, \dots, \pm n$,
and by Eq.(2.3a) one has $\pi(a_j) = \hat{a}_j$. This interrelation of B(0,n) and pB_n-sets was formulated algebraically by Ganchev A., Palev T., J.Math.Phys., 1980, v.21, p.797.

14. If the number of degrees of freedom is finite, then p may assume any positive value (see the second part of this study), whereas for infinite systems p is always integer.

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16. If $T_1, \dots, T_n$ are linear operators on a Hilbert space $\mathcal{H}$ with a common invariant domain D, then $\mathcal{U}(T_1, \dots, T_n)$ denotes the subalgebra of $\mathrm{End}_{\mathcal{H}} D$ generated by $T_1, \dots, T_n$.

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Неприводимые $*$-представления супералгебр Ли B(0, n).
Общие соображения

Изучается задача получения неприводимых $*$-представлений π супералгебр Ли B(0, n), n = 1,2 на основе недавно построенного семейства линейных представлений дифференциальными операторами в пространстве $\mathfrak{F}_N^{(n)}$ C^{∞}-функций со значениями в C^N. Применяется эквивалентная формулировка при помощи опреаторов рождения-уничтожения пара-Бозе системы с n степенями свободы. Показано, что пространство $\mathfrak{D}$ любого представления π является подпространством в $\mathfrak{F}_N^{(n)}$, содержащим вакуумные векторы. Получены необходимые условия и показано, что всякое представление имеет циклический вектор, который полностью определяется некоторым $\nu \geq 0$ и старшим весом алгебры sl(n, C). Строится базис в подпространстве $\mathfrak{D}$, который состоит из общих собственных векторов операторов числа частиц соответствующих отдельным степеням свободы рассматриваемой пара-Бозе системы.

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Blank J., Halvíček M. E2-85-112

Irreducible $*$-Representations of Lie Superalgebras B(0, n) with Finite-Degenerated Vacuum. General Considerations

We study the problem of getting irreducible $*$-representations π of Lie superalgebras B(0, n), n = 1,2, starting with a recently constructed family of linear representations in terms of differential operators on the space $\mathfrak{F}_N^{(n)}$ of C^N-valued C^{∞}-functions. Equivalent formulation via creation-annihilation operators of a para-Bose system with n degrees of freedom is used and the domian $\mathfrak{D}$ of any π is shown to be a subset of $\mathfrak{F}_N^{(n)}$ containing a non-zero vacuum subspace. By assuming its dimension finite, we derive necessary conditions for existence of π and show that each representation has a cyclic vector that is fully specified by a real $\nu \geq 0$ and a highest weight of sl(n, C). A basis in the domain $\mathfrak{D}$ is constructed; this basis consists of common eigenvectors of particle-number operators corresponding to individual degrees of freedom of the para-Bose system under consideration.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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