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BĂCKLUND TRANSFORMATIONS FOR THE SUPER-LIOUVILLE EQUATIONS

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1. Introduction

Recently, we have constructed new integrable systems with extended supersymmetry (SUSY) in two dimensions, N=2 and N=4 superextensions of the Liouville equation (LE) [1-3]. These systems have been written in a manifestly covariant superfield form and have proved to exhibit a number of remarkable properties. In particular, they possess local internal symmetries U(1)xU(1) (in the N=2 case) and SU(2)xSU(2) (in the N=4 case). By analogy with the N=0 and N=1 LE's [4] one may expect that the N=2 and N=4 LE's are closely related to the realistic theories in higher dimensions: superstrings, supergauge theories, etc., and may thus have nontrivial physical applications. The bosonic sector of the N=4 LE contains, in addition to the ordinary (N=0) LE, the equations of nonlinear σ-model of the Wess-Zumino type [5].

In the present paper we continue studying of the group-theoretical structure of the N=2 and N=4 LE's. A common feature of the integrable systems is the presence of the Bäcklund transformations (BT) relating to each other their different solutions. Here we derive the BT's for the N=2,4 LE's and discuss the group-theoretical meaning of these within the general approach [1-3] based on the nonlinear realizations of infinite-dimensional symmetries. In Sect. 2, the general techniques are illustrated by the familiar examples of the N=0 and N=1 LE's. In Sect. 3 the BT's for the N=4 case are constructed and then the reduction to the N=2 case is performed. Besides the BT's acting among the solutions of a given super-LE, we find the BT's to the solutions of the related free equations.

2. Bäcklund transformations as the right gauge shifts on coset spaces: the N=0 and N=1 cases

The standard BT is defined as a family (τ-parametric in general) of transformations converting a solution of a given nonlinear equation to a solution of the same or other equation. These transformations do not affect the space-time coordinates and operate on the fields and their derivatives. The BT's for N=0 and N=1 LE's are well known [6,7]. Here we reobtain them within a new method which admits a straightforward generalization to the N=2 and N=4 cases.

The basic idea of constructing BT's for the N=0 LE and its superextensions is to consider most general gauge transformations which do not shift the coordinates and preserve the zero curvature representation

$$d\Omega = \Omega \wedge \Omega \tag{1}$$

for the corresponding super-LE. We begin with the BT relating to each other solutions of the N=0 LE. We have shown in [1] that this equation results from the covariant reduction of an infinite-dimensional coset space G/H to its fully geodesic subspace SL(2,R)/H. Here G is the conformal group in two dimensions with the algebra

$$\begin{cases} i[L^n_\pm, L^m_\pm] = (n-m)L^{n+m}_\pm \\ i[L^n_+, L^m_-] = 0 \quad , \quad (n,m=-1,0,1,\dots), \end{cases} \tag{2}$$

$H \propto \{U = L^0_+ - L^0_-\}$ is the corresponding Lorentz group, and the group SL(2,R) has generators $R_+ = L^{-1}_+ + m^2 L^1_-$, $R_- = L^{-1}_- + m^2 L^1_+$, U ($[m^2] = L^{-2}$). The covariant reduction means that in the decomposition

$$g^{-1}dg = \sum_{n=-1}^{+\infty} \omega^\pm_n L^n_\pm \tag{3}$$

with

$$g \equiv G/H = e^{ix^\pm L^{-1}_\pm} e^{iz^\pm_1(x) L^1_\pm} \dots e^{iu(x)(L^0_+ + L^0_-)} \tag{4}$$

one nullifies all the 1-forms $\omega^\pm_k$ except for those lying in the algebra sl(2,R). By that procedure, the Goldstone fields $\{z^\pm_n(x)\}$ are expressed in terms of the dilaton $u(x)$. The surviving 1-form Ω^{Red} in (3) satisfies the zero-curvature condition (1) on the algebra sl(2,R) which is equivalent to the LE for the field $u(x)$:

$$u_{+-} = m^2 e^{-2u} \tag{5}$$

Explicitly, the 1-form Ω^{Red} is as follows

$$\Omega^{Red} = e^{-u}\left(\eta\, dx^+ R_+ + \frac{1}{\eta} dx^- R_-\right) + (u_- dx^- - u_+ dx^+)\,U. \tag{6}$$

Note that Ω^{Red} involves a "spectral" parameter η which is introduced, according to the prescription of refs. [1], via a constant right shift by the stability subgroup H :

$$\Omega(\eta) = g^{-1}(\eta)\, d g(\eta)\,; \quad g(\eta) = g e^{i\alpha U}, \quad \eta = e^{\alpha}. \tag{7}$$

Since the 1-form Ω^{Red} (6) is completely defined by the coset elements g (4), the transformations of Ω are induced by those of these elements. The structure of g implies that the transformations having no effect on the coordinates $x^\pm$ are generated by certain right shifts on the coset space (4). With account of the relation between $z^\pm_1(x)$ and $u(x)$ [1]

$$z^\pm_1(x) = \partial_\pm u(x)$$

most general gauge transformations realized on $u(x)$ and $\partial_\pm u(x)$ are the following ones

$$g \to \tilde g = g\, e^{imb^\pm(x) L^1_\pm}\, e^{ia(x)(L^0_+ + L^0_-)}, \tag{8}$$

$a, b^\pm$ being certain functions of $x^\pm$, arbitrary for the moment. Now, let us require that the transformed 1-form $\tilde\Omega$

$$\tilde\Omega = \tilde g^{-1} d\tilde g \tag{9}$$

obeys the same zero-curvature condition (1) on the algebra $sl(2,R)$ as the initial 1-form Ω^{Red} does. In other words, we require:

$$\tilde\Omega \in sl(2,R). \tag{10}$$

Computing $\tilde\Omega$ explicitly, one gets two systems of the equations on parameters $a(x)$ and $b^\pm(x)$:

$$\begin{aligned} &a) \quad m\eta e^{-u} e^{-2a} = \partial_+ b^- + m\eta e^{-u} - b^- \partial_+ u \\ &b) \quad \partial_- b^- + b^- \partial_- u + \frac{m}{\eta}(b^-)^2 e^{-u} = 0 \\ &c) \quad \partial_- a = \frac{m}{\eta} e^{-u} b^- \end{aligned} \tag{11}$$

and

$$\begin{aligned} &a) \quad \frac{m}{\eta} e^{-u} e^{-2a} = \partial_- b^+ + \frac{m}{\eta} e^{-u} - b^+ \partial_- u \\ &b) \quad \partial_+ b^+ + b^+ \partial_+ u + m\eta (b^+)^2 e^{-u} = 0 \\ &c) \quad \partial_+ a = m\eta e^{-u} b^+. \end{aligned} \tag{12}$$

It is demonstrated in the Appendix that the systems (11) and (12) are equivalent so we may consider only the first one.

The second and third equations of (11) imply

$$\partial_- a = -\partial_-(u + \ln b^-) \tag{13}$$

whence

$$a = -(u + \ln b^-) + \varphi(x^+).$$

Without loss of generality, one may put $\varphi(x^+) = 0$ (see the Appendix). Therefore,

$$a = -u - \ln b^-. \tag{14}$$

Having in mind that the structure of the coset element (4) and of the right shift (8) suggests the following relation between the transformed and initial fields

$$\tilde u = u + a \tag{15}$$

we immediately get from eq. (14).

$$b^- = e^{-\tilde u}. \tag{16}$$

Substituting this expression into the first two equations of the system (11) we arrive at the BT for the N=0 LE (5)

$$\begin{cases} \tilde u_+ + u_+ = 2\eta m\, sh(\tilde u - u) \\ \tilde u_- - u_- = \frac{m}{\eta} \exp\{-(\tilde u + u)\}. \end{cases} \tag{17}$$

It is a simple exercise to check that the integrability conditions of the system (17) are just the LE's for the fields $u(x)$ and $\tilde u(x)$.

To obtain the BT transforming a solution of eq. (5) to that of the free massless equation it suffices to demand that the transformed 1-form $\tilde\Omega$ possesses the zero-curvature representation on the algebra of the two-dimensional Poincaré group $\mathcal{P} \propto \{L^{-1}_\pm, U\}$.

In this case, we have:

$$\begin{cases} \tilde{u}_+ + u_+ = \eta m \exp(\tilde{u}-u) \\ \tilde{u}_- - u_- = \frac{m}{\eta}\exp\{-(\tilde{u}+u)\}. \end{cases} \qquad (18)$$

The field $\tilde{u}(x)$ satisfies the free equation

$$\tilde{u}_{+-} = 0 \qquad (19)$$

which is the compatibility condition for eqs. (18). So, eqs. (18) define BT's from the solutions of the N=0 LE to those of eq. (19).

The geometric significance of the constructed BT's is thus as follows. These transformations relate to each other different geodesic hypersurfaces of the coset space G/H (a pseudosphere SL(2,R)/H to the other pseudosphere or pseudoplane $\mathcal{P}/H$) and have a uniform representation by the right gauge shifts (8) which act in the whole coset space G/H and are constrained by the requirement of preserving the zero-curvature condition (1). The difference between the BT's (17) and (18) is traced to the difference in geometries of the hypersurfaces to which one goes (pseudosphere or pseudoplane).

Construction of BT's for the N=1 LE proceeds in a complete parallel with the case of N=0. G is now the N=1 superextension of the conformal group based on superalgebra $\mathbb{K}^+(1|1) \oplus \mathbb{K}^-(1|1)$ /2/:

$$\begin{cases} i[L^n_\pm, L^m_\pm] = (n-m)L^{n+m}_\pm \\ i[L^n_\pm, G^r_\pm] = (\frac{n}{2}-r)G^{r+n}_\pm \\ \{G^r_\pm, G^s_\pm\} = -2L^{r+s}_\pm, \qquad (n,m=-1,0,1,\ldots;\ r,s=-\frac12,\frac12,\frac32,\ldots). \end{cases} \qquad (20)$$

The stability subgroup H includes, as before, only the Lorentz rotations with the generator U while the zero-curvature representation is written now on the superalgebra $osp(1|2) = \{R_+, R_-, U, Q_+ = G_+^{-\frac12} + m G_-^{\frac12}, Q = G_-^{-\frac12} - m G_+^{\frac12}\}$ /2/. An element of the relevant coset space G/H, with the Goldstone superfields covariantly expressed in terms of the superdilaton $u(x,\theta)$ has the form

$$g = e^{ix^\pm L^{-1}_\pm} e^{\theta^\pm G^{-\frac12}_\pm} e^{i\partial_\pm u L^1_\pm} e^{\mathcal{D}_\pm u G^{\frac12}_\pm} \ldots e^{iu(L^0_+ + L^0_-)}, \qquad (21)$$

where $\mathcal{D}^\pm$ are the flat N=1 covariant spinor derivatives

$$\mathcal{D}^\pm = i\theta^\pm \frac{\partial}{\partial x^\pm} + \frac{\partial}{\partial \theta^\pm}. \qquad (22)$$

We will be interested in the right gauge transformations of (21) which do not affect the superspace coordinate $\{x^\pm, \theta^\pm\}$ and are realizable on u, $\partial_\pm u$ and $\mathcal{D}_\pm u$

$$g \to \tilde{g} = g e^{imb^\pm L^1_\pm} e^{\zeta^\pm G^{\frac12}_\pm} e^{ia(L^0_+ + L^0_-)}. \qquad (23)$$

In the element $\tilde{g}$, the parameters $b^\pm, \zeta^\pm, a$ are general N=1 superfields, $b^\pm, a$ being bosonic and $\zeta^\pm$ fermionic. The latter shift $\mathcal{D}_\pm u$.

Requiring the 1-form

$$\tilde{\Omega} = \tilde{g}^{-1} d\tilde{g}$$

to lie in superalgebra $osp(1|2)$ and using the explicit expression for the 1-form $\Omega = g^{-1}dg$ /2/ we obtain, as in the N=0 case, two systems of equations on the superfields $b^\pm, \zeta^\pm, a$. Again, it is sufficient to consider only one of these systems (see the Appendix):

$$\begin{cases} i\mathcal{D}_+ a = -\eta e^{-\frac{u}{2}}\zeta^+ \\ \mathcal{D}_+\zeta^+ - \eta e^{-\frac{u}{2}} b^+ - \frac12 \zeta^+ \mathcal{D}_+ u = 0 \\ \frac{m}{\eta} e^{-\frac12(a+u)} = e^{\frac{a}{2}}[\mathcal{D}_-\zeta^+ - \frac{m}{\eta}e^{-\frac{u}{2}} + \frac12\zeta^+\mathcal{D}_- u] \\ i\mathcal{D}_+ b^+ + ib^+\mathcal{D}_+ u - 2\eta e^{-\frac{u}{2}}\zeta^+ b^+ + \zeta^+\mathcal{D}_+\zeta^+ = 0 \\ i\mathcal{D}_- b^+ + ib^+\mathcal{D}_- u - \frac{2m}{\eta}e^{-\frac{u}{2}}\zeta^+ + \zeta^+\mathcal{D}_-\zeta^+ = 0. \end{cases} \qquad (24)$$

It follows from eqs. (21), (23) that the transformed superfield $\tilde{u}$ is related to u as

$$\tilde{u} = u + a. \qquad (25)$$

Solving eqs. (24) with respect to b^+ we get the BT's for the N=1 LE /7/:

$$\begin{cases} \mathcal{D}_+\tilde{u} - \mathcal{D}_+ u = i\xi^+\eta e^{-\frac12(\tilde{u}+u)} \\ \mathcal{D}_-\tilde{u} - \mathcal{D}_- u = \frac{2i}{\eta}\xi^+ \operatorname{ch}\{\frac12(\tilde{u}-u)\} \\ \mathcal{D}_+\xi^+ = m\eta\exp\{-\frac12(\tilde{u}+u)\} \\ \mathcal{D}_-\xi^+ = \frac{2m}{\eta}\operatorname{sh}\{\frac12(\tilde{u}-u)\} \end{cases} \qquad (26)$$

with $\xi^+ = e^{\frac{\tilde{u}}{2}}\zeta^+$.

It is easy to see that the consistency condition for (26) is the N=1 LE for the superfield $\tilde{u}$:

$$\mathcal{D}_+\mathcal{D}_-\tilde{u} = ime^{-\tilde{u}}. \qquad (27)$$

Analogously to the case of N=0 LE, one may impose also different conditions on the transformed 1-form $\tilde{\Omega}$, e.g., demand it to belong to the flat N=1 Poincaré superalgebra /2/. In this case, instead of (26) we have

$$\begin{cases} \mathcal{D}_+\tilde{u} - \mathcal{D}_+ u = i\eta\xi^+ e^{-\frac12(\tilde{u}+u)} \\ \mathcal{D}_-\tilde{u} + \mathcal{D}_- u = \frac{i}{\eta}\xi^+ e^{\frac12(\tilde{u}-u)} \\ \mathcal{D}_+\xi^+ = m\eta\exp\{-\frac12(\tilde{u}+u)\} \\ \mathcal{D}_-\xi^+ = \frac{m}{\eta}\exp\{\frac12(\tilde{u}-u)\} \end{cases} \qquad (28)$$

and the superfield $\tilde{u}$ obeys now the free equation

$$\mathcal{D}_+\mathcal{D}_-\tilde{u} = 0. \qquad (29)$$

It is worth noting that the spinor superfields $\xi^\pm$ have a clear geometric meaning in the present approach: they are parameters of the right gauge transformation (23) preserving the reduction conditions.

Before going to the construction of BT's for the N=2 and N=4 LE's let us point out relevancy just of the superfield form of BT's for supersymmetric integrable equations. To get the component BT's, one should repeatedly act on eqs. (26), (27) by the covariant derivatives (22) and use the anticommutation relation

$$\{\mathcal{D}_\pm, \mathcal{D}_\pm\} = 2i\partial_\pm . \tag{30}$$

However, even with the auxiliary fields eliminated, the component BT's for the N=1 LE involve ten rather cumbersome equations and this number increases when going to the N=2 and N=4 LE's. Therefore, in what follows we restrict ourselves to considering only the superfield BT's.

3. Bäcklund transformations for N=4 and N=2 super-Liouville equations

The superfield N=2 and N=4 LE's have been constructed for the first time in our papers /3/ by the method analogous to that one working in the N=0 and N=1 cases. We have started with the coset spaces G/H where G is the N=2 or N=4 superextension of the conformal group in two dimensions and H is one of its subgroups. The covariant reduction of G/H to its proper geodesic subsupermanifold naturally results in the equations for essential superfields of the theory which generalize the N=1 LE. An important distinction of the N=2 and N=4 LE's from the N=1 LE is the presence of the internal symmetry groups in the relevant stability subgroups H (U(1) and SU(2) in the N=2 and N=4 cases, respectively). Such an enlargement of the stability subgroup means, according to the concept of refs. /1-3/, a possibility to introduce two and four spectral parameters in the N=2 and N=4 cases. Except for the increase in the number of spectral parameters, the construction of BT's for the N=2 and N=4 LE's goes along the same line as for their N=0 and N=1 prototypes. Since the N=2 LE follows from the N=4 one by a dimensional reduction in Grassmann coordinates /3/, we shall build first BT's for the N=4 LE and then descend to the N=2 case.

The N=4 LE emerges as one of the constraints of the covariant reduction of the coset space G^A/H to its subspace SU(1,1|2)/H where G^A is the N=4 superextension of the conformal group in two dimensions with the superalgebra $\mathcal{G}^A = \mathbb{K}^A_+(1|2) \oplus \mathbb{K}^A_-(1|2)$ /3/:

$$\begin{aligned}
&i[L^n_\pm, L^m_\pm] = (n-m)L^{n+m}_\pm, \quad \{G^r_{\alpha\pm}, G^s_{\beta\pm}\} = \{\bar{G}^{r\alpha}_\pm, \bar{G}^{s\beta}_\pm\} = 0, \\
&\{G^r_{\alpha\pm}, \bar{G}^{s\beta}_\pm\} = -2\delta^\beta_\alpha L^{r+s}_\pm + 2(r-s)(\sigma^k)^\beta_\alpha T^{r+s}_{k\pm}, \\
&i[L^n_\pm, G^r_{\alpha\pm}] = \left(\tfrac{n}{2} - r\right) G^{r+n}_{\alpha\pm}, \quad i[L^n_\pm, T^p_{k\pm}] = -p\,T^{p+n}_{k\pm}, \\
&i[T^p_{k\pm}, G^r_{\alpha\pm}] = -\tfrac{1}{2}(\sigma_k)^\beta_\alpha G^{r+p}_{\beta\pm}, \quad [T^p_{k\pm}, T^l_{j\pm}] = \varepsilon_{kji} T^{p+l}_{i\pm}
\end{aligned} \tag{31}$$

$(n,m = -1,0,1,\ldots;\ r,s = -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \ldots;\ p,l = 0,1,2,\ldots;\ \alpha,\beta = 1,2;\ i,j,k = 1,2,3)$.

the stability subgroup H=SO(1,1)xSU(2) is generated by the set $H \propto \{U = L^0_+ - L^0_-,\ T_i = T^0_{i+} + T^0_{i-}\}$ and the zero-curvature representation superalgebra su(1,1|2) is spanned on the generators $\{R_+, R_-, U, T_i, Q_{\alpha+} = G^{-\frac{1}{2}}_{\alpha+} + m G^{\frac{1}{2}}_{\alpha-}, \bar{Q}^\alpha_+, Q_{\alpha-} = G^{-\frac{1}{2}}_{\alpha-} - m G^{\frac{1}{2}}_{\alpha+}, \bar{Q}^\alpha_-\}$. An infinite array of the Goldstone superfield parameters in the coset element

$$\begin{aligned}
g \equiv G^A/H = {} & e^{ix^\pm L^{-1}_\pm}\, e^{\theta^\pm G^{-\frac{1}{2}}_\pm + \bar{\theta}^\pm \bar{G}^{-\frac{1}{2}}_\pm}\, e^{iz_1^\pm L^1_\pm}\, e^{\xi^\pm_{\frac{1}{2}} G^{\frac{1}{2}}_\pm + \bar{\xi}^\pm_{\frac{1}{2}} \bar{G}^{\frac{1}{2}}_\pm} \cdot \\
& \cdot e^{v_1^{\pm k} T^1_{k\pm}} \ldots\, e^{iu(L^0_+ + L^0_-)}\, e^{\varphi^k (T^0_{k+} - T^0_{k-})}
\end{aligned} \tag{32}$$

is expressed in terms of four essential superfields $\{u, \varphi^k\}$ which turn out to be subjected to the system of equations

$$\begin{cases}
\mathcal{D}^{(\alpha}_- q^{\beta)}_\gamma = 0, \quad \bar{\mathcal{D}}_{+(\alpha} q^\beta_{\gamma)} = 0 \\
\mathcal{D}^\alpha_- (q\, \mathcal{D}^\gamma_+ q^{-1})^\beta_\gamma = 0 \\
\mathcal{D}^\alpha_+ (q^{-1} \bar{\mathcal{D}}_{-\gamma} q)^\gamma_\beta + 4im\, \bar{q}_\beta{}^\alpha = 0
\end{cases} \tag{33}$$

$$q^\beta_\alpha \equiv (e^{-u - i\varphi\cdot\sigma})^\beta_\alpha; \quad \bar{q}^\beta_\alpha \equiv (e^{-u + i\varphi\cdot\sigma})^\beta_\alpha = -\varepsilon_{\alpha\gamma}\varepsilon^{\beta\rho} q^\gamma_\rho \tag{34}$$

$$\begin{gathered}
\mathcal{D}^\alpha_\pm = i\theta^{\alpha\pm}\frac{\partial}{\partial x^\pm} + \frac{\partial}{\partial \bar{\theta}^\pm_\alpha}, \quad \bar{\mathcal{D}}_{\pm\alpha} = i\bar{\theta}^\pm_\alpha \frac{\partial}{\partial x^\pm} + \frac{\partial}{\partial \theta^{\alpha\pm}} \\
\{\mathcal{D}^\alpha_\pm, \bar{\mathcal{D}}_{\pm\beta}\} = 2i\delta^\alpha_\beta \partial_\pm .
\end{gathered} \tag{35}$$

The system (33) is the N=4 superextension of the LE.

In the general Cartan 1-form

$$\Omega = g^{-1} dg$$

extending first over the whole infinite dimensional superalgebra $\mathcal{G}^A$ (31) there remains after the covariant reduction only the component with values in superalgebra su(1,1|2). It obeys the zero-curvature condition on that superalgebra. The reduced 1-form Ω^{Red} has been explicitly given in our papers /3/.

Let us consider right gauge transformations of the coset element (32) which affect the superfield q^β_α and its first derivatives but not the coordinates $\{x^\pm, \theta^\pm_\alpha, \bar{\theta}^{\pm\alpha}\}$. The most general form of such transformations directly generalizing that of the N=0 and N=1 cases is as follows

$$g \to \tilde{g} = g\, e^{imb^\pm L^1_\pm}\, e^{\xi^\pm G^{\frac{1}{2}}_\pm + \bar{\xi}^\pm \bar{G}^{\frac{1}{2}}_\pm}\, e^{ia(L^0_+ + L^0_-)}\, e^{\psi^k (T^0_{k+} - T^0_{k-})} . \tag{36}$$

Computing the transformed form $\tilde{\Omega}$

$$\tilde{\Omega} = \tilde{g}^{-1} d\tilde{g}$$

explicitly and requiring it to have a zero-curvature representation on su(1,1|2), i.e.,

$$\tilde{\Omega} \in su(1,1|2), \tag{37}$$

we derive the following system of equations for the parameters of the right shift (36) (we again restrict ourselves to one of the arising systems, see the Appendix):

$$\begin{aligned}
&(q' \mathcal{D}^\beta_+ q'^{-1})^\gamma_\alpha - (q \mathcal{D}^\beta_+ q^{-1})^\gamma_\alpha = 2i\delta^\beta_\alpha (\xi\bar{q})^\gamma \\
&im\,\mathcal{D}^\alpha_+ b + \tfrac{i}{4}(q\mathcal{D}^\beta_+ q^{-1})^\alpha_\beta m b - (\xi\sigma_k\bar{\xi})(\tilde{q}\mathcal{D}^\alpha_+ \tilde{q}^{-1}\sigma^k) + 2i(\xi\bar{\xi})(\xi\bar{q})^\alpha - \\
&\qquad - 2b(\xi\bar{q})^\alpha m + \xi\mathcal{D}^\alpha_+ \bar{\xi} + \mathcal{D}^\alpha_+ \xi\bar{\xi} = 0
\end{aligned}$$

$$im\mathcal{D}_-^\alpha b-\frac{i}{4}(\bar q\,\mathcal{D}_-^\beta\bar q^{-1})_\beta^\alpha mb-(\zeta\sigma_\kappa\bar\zeta)(\tilde q^{-1}\mathcal{D}_-^\alpha\tilde q\,\sigma^\kappa)-2m(\zeta\tilde q)^\alpha+$$
$$+\zeta\mathcal{D}_-^\alpha\bar\zeta+\mathcal{D}_-^\alpha\zeta\bar\zeta=0$$
$$\mathcal{D}_+^\alpha\zeta^\beta+2i(\zeta\bar{\tilde q})^\alpha\zeta^\beta+\frac{1}{2}(\zeta\sigma_\kappa)^\beta(\tilde q\,\mathcal{D}_+^\alpha\tilde q^{-1}\sigma^\kappa)-\frac{1}{8}\zeta^\beta(q\,\mathcal{D}_+^\gamma q^{-1})_\gamma^\alpha=0$$
$$\mathcal{D}_+^\alpha\bar\zeta_\beta-mb(\bar{\tilde q})_\beta^\alpha+i(\zeta\bar\zeta)(\bar{\tilde q})_\beta^{\ \alpha}-\frac{1}{2}(\sigma^\kappa\mathcal{D}_+^\alpha\bar{\tilde q}^{-1}\bar{\tilde q})(\sigma_\kappa\bar\zeta)_\beta-$$
$$-\frac{1}{8}(\mathcal{D}_+^\gamma q\,q^{-1})_\gamma^\alpha\,\bar\zeta_\beta=0$$
$$\mathcal{D}_-^\alpha\bar\zeta_\beta-m(\tilde q)_\beta^\alpha-\frac{1}{2}(\mathcal{D}_-^\alpha\bar{\tilde q}\,\bar{\tilde q}^{-1}\sigma^\kappa)(\sigma^\kappa\bar\zeta)_\beta+ \tag{38}$$
$$+\frac{1}{8}(\mathcal{D}_-^\gamma\bar q\,\bar q^{-1})_\gamma^\alpha\,\bar\zeta_\beta^+=-m\,(q_0\tilde q)_\beta^{\ \alpha}$$
$$\mathcal{D}_-^\alpha\zeta^\beta+\frac{1}{2}(\zeta\sigma_\kappa)^\beta(\tilde q^{-1}\mathcal{D}_-^\alpha\tilde q\,\sigma^\kappa)+\frac{1}{8}\zeta^\beta(\bar q\,\mathcal{D}_-^\gamma\bar q^{-1})_\gamma^{\ \alpha}=0.$$

Here

$$\begin{cases}\tilde q_\alpha^{\ \beta}=\left(e^{-\frac{u}{2}-i\frac{\varphi\cdot\sigma}{2}}\right)_\alpha^{\ \beta}\\ (q_0)_\alpha^\beta=\left(e^{-a-i\psi\cdot\sigma}\right)_\alpha^{\ \beta}\end{cases} \tag{39}$$

and the transformed quaternionic superfield $(q')_\alpha^{\ \beta}$ is given by:

$$(q')_\alpha^{\ \beta}\equiv\left(e^{-u'-i\varphi'\cdot\sigma}\right)_\alpha^{\ \beta}=(\tilde q\,q_0\tilde q)_\alpha^{\ \beta}. \tag{40}$$

Just as in the previously considered cases of the N=0 and N=1 BT's, the superfield b can be expressed in terms of the remaining quantities:

$$b=e^{-u'}-\frac{e^{u'}}{2m^2}(\zeta\bar\zeta)^2. \tag{41}$$

Substituting this expression into the system (38) we obtain the BT for the N=4 LE:

$$\mathcal{D}_+^\alpha u'-\mathcal{D}_+^\alpha u=i(\zeta\bar{\tilde q})^\alpha$$
$$\mathcal{D}_-^\alpha u'+\mathcal{D}_-^\alpha u=i(\zeta\tilde q)^\alpha\left[e^{u'}-\frac{ie^{2u'}}{m}(\zeta\bar\zeta)\right]+i(\zeta q_0\tilde q)^\alpha\left[e^{u'}+\frac{ie^{2u'}}{m}(\zeta\bar\zeta)\right]$$
$$\mathcal{D}_+^\alpha(\bar{\tilde q}\bar\zeta)_\beta-m\left(e^{-u'}-\frac{e^{u'}}{2m^2}(\zeta\bar\zeta)^2\right)(\bar q)_\beta^\alpha+i(\zeta\bar\zeta)(\bar q)_\beta^\alpha+ \tag{42}$$
$$+\mathcal{D}_+^\alpha u\,(\bar{\tilde q}\bar\zeta)_\beta=0$$
$$\mathcal{D}_+^\alpha(\zeta\tilde q)^\beta+2i(\zeta\bar{\tilde q})^\alpha(\zeta\tilde q)^\beta-(\zeta\tilde q)^\beta\mathcal{D}_+^\alpha u=0$$
$$\mathcal{D}_-^\alpha(\tilde q\bar\zeta)_\beta=m(q)_\beta^\alpha-m(q')_\beta^\alpha$$
$$\mathcal{D}_-^\alpha(\zeta\bar{\tilde q})^\beta=0.$$

It may be checked that the consistency condition of the system (42) is just the system (33) for $(q')_\alpha^{\ \beta}$. The dependence on spectral parameters is implicitly contained in $(q)_\alpha^{\ \beta}(\lambda)$:

$$(q)_\alpha^{\ \beta}(\lambda)=(q)_\alpha^{\ \gamma}(\lambda)_\gamma^{\ \beta} \tag{43}$$
$$(\lambda)_\gamma^{\ \beta}=\left(e^{-(\lambda_0+i\lambda\cdot\sigma)}\right)_\gamma^{\ \beta}. \tag{44}$$

BT from solutions of the N=4 LE to those of the corresponding mass-

less equation ($m=0$ in eq. (33))[x)] are given basically by the same system (42); minor differences appear only in the second and fifth equations:

$$\mathcal{D}_-^\alpha u'+\mathcal{D}_-^\alpha u=i(\zeta\tilde q)^\alpha\left[e^{u'}-\frac{ie^{2u'}}{m}(\zeta\bar\zeta)\right] \tag{45}$$
$$\mathcal{D}_-^\alpha(\tilde q\,\bar\zeta)_\beta=m(q)_\beta^{\ \alpha}.$$

Thus, following the general method of constructing BT's described in Sect. 2, we have constructed BT's for the N=4 LE. Now, it is easy to find the BT for the N=2 LE /2/ :

$$\bar{\mathcal{D}}^-\mathcal{D}^+v=-2im\,e^{-v^+}. \tag{46}$$

Indeed, taking into account that the N=2 LE follows from the N=4 LE when the values of indices are restricted as /3/

$$\alpha,\beta=1\quad;\ i=j=k=3 \tag{47}$$

and introducing the superfields v and v^+ by

$$(q)_1^{\ 1}\equiv\left(e^{-(u+i\varphi^3\sigma^3)}\right)_1^{\ 1}=e^{-(u+i\varphi^3)}=e^{-v};\ v^+=u-i\varphi^3 \tag{48}$$

we arrive at the following N=2 BT's

$$\mathcal{D}^+\tilde v-\mathcal{D}^+v=2i\eta\,\xi\,e^{-\frac{1}{2}(\tilde v^++v^+)}$$
$$\mathcal{D}^-\tilde v^++\mathcal{D}^-v^+=\frac{4i}{\eta}\,\xi\,ch\left(\frac{\tilde v-v}{2}\right)$$
$$\mathcal{D}^+\xi=0\quad,\quad\mathcal{D}^-\xi=0 \tag{49}$$
$$\mathcal{D}^+\bar\xi=m\bar\eta\,e^{-\frac{1}{2}(\tilde v^++v^+)}$$
$$\bar{\mathcal{D}}^-\bar\xi=\frac{2m}{\eta}sh\left(\frac{\tilde v-v}{2}\right)\quad;\quad\xi=e^{\frac{\tilde v^+}{2}}\zeta.$$

The integrability conditions of the system (49) are just the equations (46) for v and $\tilde v$. The same reduction (47), (48), being applied to the BT from the N=4 LE to its $m=0$ relict, yields the following system:

$$\mathcal{D}^+\tilde v-\mathcal{D}^+v=2i\eta\,\xi\,e^{-\frac{1}{2}(\tilde v^++v^+)}$$
$$\mathcal{D}^-\tilde v^++\mathcal{D}^-v^+=\frac{2i}{\eta}\,\xi\,e^{\frac{1}{2}(\tilde v-v)}$$
$$\mathcal{D}^+\xi=0,\quad\mathcal{D}^-\xi=0 \tag{50}$$
$$\mathcal{D}^+\bar\xi=m\bar\eta\,e^{-\frac{1}{2}(\tilde v^++v^+)}$$
$$\bar{\mathcal{D}}^-\bar\xi=\frac{m}{\eta}\,e^{\frac{1}{2}(\tilde v-v)}.$$

These equations define the BT's from the N=2 LE to the free equation since the integrability condition for (50) is

$$\bar{\mathcal{D}}^-\mathcal{D}^+\tilde v=0. \tag{51}$$

Note that the spectral parameter in N=2 BT's is complex (H=SO(1,1)xSO(2)) and, because H is now Abelian, the dependence of

x) In fact, even in the limit $m=0$ the N=4 LE describes interacting fields because of the presence of the Wess-Zumino nonlinear σ-model in its bosonic sector.

the BT's (49), (50) on this parameter is much simpler than in the N=4 case. In principle, one may get rid of the spectral parameter at all, putting it equal to unity by the proper right gauge transformation of the 1-form Ω^{Red} (this property is exclusively inherent in the LE and its superextensions, for other integrable systems it is, in general, not the case). However, it is convenient to explicitly keep η (and $\lambda_\alpha{}^\beta$ in the N=4 case) in order to have a manifest H-covariance.

To close this Section, we discuss the status of the additional spinor superfields which appear in the BT's for super-LE's. In our approach these superfields are parameters of right gauge transformations of the coset elements (23), (36). The need for them is obvious already from the fact that the spinor objects cannot be formed from $q_\alpha{}^\beta$, v otherwise as by applying the spinor derivatives. So, if we want to have BT's of the first order in spinor derivatives, we are forced to include additional spinor superfields in the right-hand sides of such BT's. Of course, these objects can be eliminated from BT's but in this case one immediately earns the equations of the second order in spinor derivatives which are nothing but the original super-LE's. Thus, the structure of BT's given by eqs. (49), (50) is unique.

4. In the present paper we have described the uniform method of obtaining BT's for the ordinary and super-LE's and have constructed with its help, the BT's for the N=2 and N=4 super-LE's. It is well known [8,9] that, having an explicit form of BT's, one may easily find infinite series of the conservation laws inherent in the integrable systems, expose the relevant hidden symmetries (different from those with respect to the original (super) group G), etc. Besides, it has been shown recently [10] that in the case of N=1 LE the BT's to the free $(m=0)$ equation provide a straightforward tool to obtain the general superfield solution of the N=1 LE (this solution has been found earlier in our paper [2] by a different method). One may expect that the BT's for the N=4 LE presented here can also be used to produce the general solution of the N=4 LE which is unknown as yet. We intend to consider all these questions in more detail elsewhere.

APPENDIX

We demonstrate here the equivalency of the systems (11) and (12). It follows from eqs.(11b,c), (12b,c) that

$$\begin{cases} \partial_+ \ln \tilde{b}^+ + \partial_+ (2u + a) = 0 \\ \partial_- \ln \tilde{b}^- + \partial_- (2u + a) = 0, \end{cases} \tag{A1}$$

where $\tilde{b}^+ = e^{-u} b^+$, $\tilde{b}^- = e^{-u} b^-$. (A2)

The system (A1) has the general solution

$$\begin{aligned} \tilde{b}^+ &= \exp\{-2u - a + \Psi(x^-)\} \\ \tilde{b}^- &= \exp\{-2u - a + \varphi(x^+)\} \end{aligned} \tag{A3}$$

with Ψ and φ being arbitrary functions of x^-, x^+. Requiring the systems (11) and (12) to be compatible to each other constrains the functions φ and Ψ as follows

$$(-2u_- + \Psi_-) e^{\Psi} = (-2u_+ + \varphi_+) e^{\varphi} \tag{A4}$$

or, bearing in mind (11c) and (12c):

$$(-2\tilde{u}_- + \Psi_-) e^{\Psi} = (-2\tilde{u}_+ + \varphi_+) e^{\varphi}. \tag{A5}$$

Now, inserting (A3) into (11), (12) we arrive at the two systems of BT's:

$$\begin{cases} e^{\Psi}(\tilde{u}_- + u_- - \Psi_-) = \mathrm{sh}(\tilde{u} - u) \\ e^{\Psi}(\tilde{u}_+ - u_+) = \exp\{-(\tilde{u} + u)\} \end{cases} \tag{A6}$$

$$\begin{cases} e^{\varphi}(\tilde{u}_+ + u_+ - \varphi_+) = \mathrm{sh}(\tilde{u} - u) \\ e^{\varphi}(\tilde{u}_- - u_-) = \exp\{-(\tilde{u} + u)\}. \end{cases} \tag{A7}$$

Since the conditions (A4), (A5) imply

$$\begin{aligned} e^{\Psi}(\tilde{u}_- + u_- - \Psi_-) &= e^{\varphi}(\tilde{u}_+ + u_+ - \varphi_+) \\ e^{\Psi}(\tilde{u}_+ - u_+) &= e^{\varphi}(\tilde{u}_- - u_-) \end{aligned}$$

the systems (A6), (A7) and, hence, (11) and (12) are equivalent, provided the constraints (A4) and (A5) hold. So, it remains to show that eq. (A4) (and (A5)) is actually solvable with respect to φ, Ψ. The easiest way to be convinced about that is to substitute for u into eq. (A4) the general solution of the LE:

$$e^{-2u} = -\frac{g_- f_+}{(g+f)^2}.$$

Then it immediately follows

$$\varphi = -\ln f_+, \quad \Psi = -\ln g_-$$

that completes the proof of equivalency of the systems (11) and (12).

Note that the BT (A6) can be cast in the standard from (17) by the following conformal transformation

$$\begin{cases} \tilde{u}_1 = \tilde{u} - \frac{\Psi}{2}, \quad u_1 = u - \frac{\Psi}{2} \\ \frac{\partial \tilde{x}^-}{\partial x^-} = e^{-\Psi(x^-)}. \end{cases} \tag{A8}$$

Thus the presence of functions φ, Ψ in BT's (A6), (A7) reflects the freedom of the latter with respect to conformal group.

In the case of superextended LE's, the arising two systems of the superfield BT's also contain originally a number of superfunctions connected with the freedom under the relevant superconformal group. Fixing properly this freedom (constraints of the type (A4)), it is again possible to show the equivalency of two systems of BT's. With the restriction to one of these systems only, one may always find a superconformal transformation which puts these BT's into the standard form (26), (42), (49).

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Преобразования Бэклунда для суперрасширений уравнения Лиувилля

Описана общая техника построения преобразований Бэклунда для уравнения Лиувилля и его суперрасширений в рамках метода, основанного на нелинейных реализациях бесконечномерных симметрий. Найдены суперполевые преобразования Бэклунда для N=2 и N=4 суперсимметричных уравнений Лиувилля и преобразования Бэклунда к решениям соответствующих свободных уравнений. Выяснен геометрический смысл этих преобразований.

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Bäcklund Transformations for the Super-Liouville Equations

We describe a procedure of obtaining the Bäcklund transformations for the Liouville equation and its superextensions within the nonlinear realization method. We treat first the familiar N=0 and N=1 cases and then apply general techniques to derive the Bäcklund transformations for the newly constructed N=2 and N=4 super-Liouville equations as well as those to the solutions of the related free equations. All these transformations are shown to have a common geometric meaning as the constrained right gauge shifts on the corresponding infinite-dimensional coset spaces.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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