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**THE PROTON DECAY CANNOT  
BE SUPPRESSED KINEMATICALLY**

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## 1. Introduction

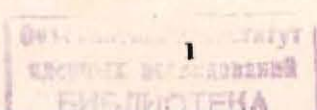
The problem of a consistent relativistic description of decays has attracted attention for a long time (an extensive bibliography can be found in Ref.1). It is connected with some peculiar difficulties. One of them concerns the fact that one cannot regard the commonly used formula

$$P_{\psi}(t) = |(\psi, e^{-iHt}\psi)|^2 \quad (1)$$

as an exact expression of the decay law for a free unstable particle<sup>1,2/</sup>. This is so because (1) relies on the assumption that the unstable particle is characterized by a one-dimensional subspace  $\mathcal{H}_u$  of the state Hilbert space spanned by the vector  $\psi$ . However, the translational invariance together with the fact that the momentum operators have purely continuous spectra require  $\mathcal{H}_u$  to be infinite-dimensional.

Making a physically reasonable choice of this subspace, we have been able in Ref.2 to derive a consistent decay-law formula (cf.(5) below). It gives a non-decay probability  $P_{\psi}(t)$  which depends not only on the mass distribution of the initial state  $\psi$ , but also on its three-momentum spread. Fortunately, this does not complicate the standard treatment of decays, because the deviations of  $P_{\psi}(t)$  from the simple formula related to (1) are very small provided the initial spatial localization of the particle is not extremely sharp. The estimates performed in Ref.2 show that this is true in practically every decay experiment, hence the decay-law dependence on kinematical characteristics of the initial state is negligible within the limit of experimental errors.

There is a possible exception, however. It concerns the proton decay which is sought vigorously at present in order to confirm (or to refute) this prediction of various grand unified theories<sup>3/</sup>. The



recent experimental results give the lower bound of  $10^{32}$  yr <sup>4/</sup> on the proton lifetime which seems to be longer than the prediction of the most popular SU(5) model of grand unification (about  $10^{31}$  yr, however, with large uncertainties <sup>3/</sup>). Before claiming it as inadequate, one must be sure that the decay is not suppressed by an additional effect. There have been some speculations on this point recently. In most of them, the decay slowdown is regarded as a consequence of the short-time non-exponentiality of the decay law <sup>5-9/</sup>; it appears that the effect is negligible unless some highly questionable premises are made <sup>6,10/</sup>. Miglietta and Rimini <sup>11/</sup> predict that the slowdown region is followed by the region of accelerated decay. Again, this region is experimentally hardly accessible. A possible dynamical mechanism for the slowdown of decay was discussed by Goldhaber et al. <sup>12/</sup>.

Another suggestion <sup>13/</sup> is that the decay might be modified by the spreading of the proton wavepacket which is very fast in the time scale given by the theoretically expected lifetime. Such a possibility cannot be excluded a priori, and the present paper is devoted to discussion of this problem. Our treatment is based on the above-mentioned decay-law formula because the latter provides a natural framework in which the effects of the size and shape of the initial wavepacket on the decay law can be studied. Such an analysis should yield a more reliable result than the rough argument employed in Ref.13 which, by the way, leads to the effective proton lifetime as large as  $10^{36}$  yr. <sup>\*</sup>

As we have mentioned, the appropriate conclusions of Ref.2 do not apply to proton. The estimates which worked effectively for the other unstable particles gave a very weak restriction in this case because of the extremely long lifetime <sup>14/</sup>. Since the task of finding finer estimates is difficult, one must approach the problem directly by calculating  $P_{\psi}(t)$ , or rather the decay probability

$$Q_{\psi}(t) = 1 - P_{\psi}(t) \quad (2)$$

This will be done in the following sections for a realistic choice of the mass distribution; in order to simplify the calculations, we neglect spin of the proton and assume the wavepacket to be spherically symmetric.

The results can be summarized briefly as follows. The calculated decay probability is actually less than the "theoretical" one, but its

\* In this connection, see also Ref.9.

decrease is insignificant. Up to higher order terms, we have <sup>\*</sup>

$$Q_{\psi}(t) \approx \Gamma t [1 - s(g)] \quad (3a)$$

where  $s(g)$  is a positive quantity depending on the function  $g$  which characterizes the three-momentum distribution of the initial state (cf.(4) below). A rough estimate yields

$$s(g) \lesssim c(\Delta q)^{-2} \quad (3b)$$

where  $\Delta q$  is the position spread of  $\psi$  and the coefficient  $c$  is of order of  $10^{-28}$  cm<sup>2</sup>. Hence the correction might be substantial only for a proton which is localized initially to a volume of subnuclear size, but in this case the arguments leading to (3) are no longer applicable. Since we are not interested in the protons bound in nuclei <sup>\*\*</sup>, the relations (3) show that the decay under consideration cannot be suppressed by some "kinematical fragmentation" as suggested in Ref.13. This supports the common opinion that there is no other key to the problem than a thorough investigation of the dynamical mechanism that governs the proton decay.

## 2. The decay law

According to Ref.2, the proton wavefunction (in p-representation) at the initial time  $t=0$  is assumed to be of the form

$$\psi(m, \vec{p}) = f(m)g(\vec{p}) \quad (4a)$$

In fact,  $g$  is a two-component function here, but if we neglect spin of the proton, it can be regarded as a scalar one. For technical reasons, one assumes that  $g$  is supported by the ball  $B_{\varepsilon} = \{\vec{p} : |\vec{p}| < \varepsilon\}$  of a radius  $\varepsilon$ . The function  $\psi$  belongs to the Hilbert space  $\mathcal{H}$  which is the carrier of the direct integral of (unitary, irreducible) representations of the Poincaré group referring to the isolated system formed by the proton itself and its decay products <sup>2/</sup>. The mass

\* Since  $\Gamma t \lesssim 10^{-21}$  during the age of our Universe, the linear approximation to the decay law is pretty good. Recall that the role of small-time non-exponentialities is negligible, as pointed out, e.g., in Ref.6.

\*\* Recall that the decay law formula (5) which is the backbone of our considerations has been derived for a free unstable particle. Hence our conclusions apply to real experiments only in the cases when the protons involved can be regarded as free in a reasonable approximation; comments on this points are given in Ref.14. Nuclear effects are considered particularly in Ref.15.

distribution  $|f(\cdot)|^2$  will be specified later in Section 4.

The subspace  $\mathcal{H}_u \subset \mathcal{H}$  which corresponds to the proton alone consists of all functions of the form (4a) with  $g \in L^2(B_\varepsilon)$ . In order to write down the decay law explicitly, one needs the projection  $E_u$  onto  $\mathcal{H}_u$ . Its action can be conveniently expressed using a suitable orthonormal basis in  $\mathcal{H}_u$ ; in this way we obtain<sup>14/</sup>

$$P_\psi(t) = \sum_{kl\mu} \left| \int_{m_0}^{\varepsilon} dm |f(m)|^2 \int_0^{\varepsilon} \frac{p^2 dp}{2(m^2+p^2)^{1/2}} \overline{h_k(p)} \times \right. \\ \left. \times \int_{4\pi} d\Omega_{\vec{p}} Y_{l\mu}(\Omega_{\vec{p}}) g(\vec{p}) \exp\{-it(m^2+p^2)^{1/2}\} \right|^2, \quad (5)$$

where  $Y_{l\mu}$  are the spherical functions,  $p = |\vec{p}|$ , and  $\{h_k\}$  is a complete system of functions with supports in  $[0, \varepsilon]$  which fulfil the orthonormality condition

$$\int_{m_0}^{\varepsilon} dm |f(m)|^2 \int_0^{\varepsilon} \frac{p^2 dp}{2(m^2+p^2)^{1/2}} \overline{h_j(p)} h_k(p) = \delta_{jk}. \quad (6)$$

Suppose now that the initial wavefunction is spherically symmetric, i.e.,

$$\psi(m, p) = f(m)g(p), \quad (4b)$$

where  $g: [0, \varepsilon] \rightarrow \mathbb{C}$ . This is certainly an ad hoc assumption, but it gives us a possibility of evaluating the decay probability from (5). At the same time, our conclusions are not likely to alter qualitatively if the initial state is not rotationally invariant (or if we take the spin into account). For the wavefunction (4b), one can simplify (5) using orthonormality of the spherical functions: it yields

$$P_\psi(t) = 4\pi \sum_k \left| \int_{m_0}^{\varepsilon} dm |f(m)|^2 \int_0^{\varepsilon} \frac{p^2 dp}{2(m^2+p^2)^{1/2}} \overline{h_k(p)} \times \right. \\ \left. \times g(p) \exp\{-it(m^2+p^2)^{1/2}\} \right|^2. \quad (7)$$

Since they belong to  $\mathcal{H}$ , the functions  $(m, p) \mapsto f(m)h_k(p)$ ,  $f(m)g(p)$  are square integrable<sup>12/</sup> with respect to  $dm \otimes p^2 dp / 2(m^2+p^2)^{1/2}$ . The integrations can be then interchanged by Fubini theorem and we can write

$$P_\psi(t) = 4\pi \sum_k \left| \int_0^{\varepsilon} \overline{h_k(p)} g(p) G(p, t) p^2 dp \right|^2, \quad (8)$$

where

$$G(p, t) = \int_{m_0}^{\infty} \frac{|f(m)|^2}{2(m^2+p^2)^{1/2}} \exp\{-it(m^2+p^2)^{1/2}\} dm. \quad (9)$$

The orthonormality condition (6) now reads

$$\int_0^{\varepsilon} \overline{h_j(p)} h_k(p) G(p, 0) p^2 dp = \delta_{jk}. \quad (10)$$

This relation suggests yet another reformulation: the decay law (8) equals

$$P_\psi(t) = 4\pi \sum_k |\langle h_k, s_t \rangle_G|^2, \quad (11a)$$

where  $s_t(p) = g(p)G(p, t)G(p, 0)^{-1}$  (notice that  $G(p, 0) > 0$  in view of (9)) and  $\langle \cdot, \cdot \rangle_G$  is the inner product in  $L^2([0, \varepsilon], G(p, 0) p^2 dp)$ . Since  $\{h_k\}$  is an orthonormal basis in this Hilbert space according to (10), the Parseval equality yields

$$P_\psi(t) = 4\pi \|s_t\|_G^2, \quad (11b)$$

or

$$P_\psi(t) = 4\pi \int_0^{\varepsilon} |g(p)|^2 |G(p, t)|^2 G(p, 0)^{-1} p^2 dp, \quad (12)$$

where  $G(p, t)$  is given by (9). The formula (12) is essential for our following calculations. Since  $P_\psi(0) = 1$  by definition, the function  $g$  has to fulfil the normalization condition

$$4\pi \int_0^{\varepsilon} |g(p)|^2 G(p, 0) p^2 dp = 1. \quad (13)$$

Combining the last two relations with (12), we can express the decay probability

$$Q_\psi(t) = 4\pi \int_0^{\varepsilon} |g(p)|^2 \frac{G(p, 0)^2 - |G(p, t)|^2}{G(p, 0)} p^2 dp. \quad (14)$$

### 3. Simple properties of $E_\psi(t)$

Notice first that the decay law given by (12) is well-defined. As we have mentioned,  $G(p, 0)$  is positive for each  $p \in [0, \varepsilon]$  unless  $f(m) = 0$  a.e. in  $[m_0, \infty)$ , but this is impossible since  $\psi$  is a unit vector in  $\mathcal{H}$ . In fact, the mass distribution is normalized conventionally as

$$\int_{m_0}^{\infty} |f(m)|^2 dm = 1; \quad (15)$$

it fixes the normalization of  $g$  through (13). We have  $|G(p,t)| \leq G(p,0)$  for all  $p,t$ , and therefore  $0 \leq P_{\psi}(t) \leq P_{\psi}(0) = 1$  as it must be. Since the integrated functions in (9), (12) have  $t$ -independent majorants, the dominated-convergence theorem implies the functions  $G(p, \cdot)$  and  $P_{\psi}(\cdot)$  to be continuous. In a similar way, one can prove

$$\lim_{t \rightarrow \infty} P_{\psi}(t) = 0 \quad ; \quad (16)$$

it is enough to change the integration variable to  $\lambda = (m^2 + p^2)^{1/2}$  in (9) and to apply the Riemann-Lebesgue lemma which shows that  $G(p,t) \rightarrow 0$  as  $t \rightarrow \infty$ .

In the case of the proton decay, however, one had to wait too long before the limit (16) might become physically interesting. The following fact is much more important: the dependence of  $P_{\psi}(t)$  on kinematical characteristics (i.e., on the shape of  $g$ ) is suppressed if the three-momentum spread is small enough. In order to see this, consider the scaling transformation  $g \rightarrow g_{\kappa}$ ,  $0 < \kappa \leq 1$ , where

$$g_{\kappa}(p) = \kappa g(p/\kappa) \quad . \quad (17a)$$

The normalization factor here is obtained from (13) to be

$$|c(\kappa)| = \kappa^{-3/2} \left( 4\pi \int_0^{\varepsilon} |g(y)|^2 G(\kappa y, 0) y^2 dy \right)^{-1/2} \quad . \quad (17b)$$

Then the decay law  $P_{\psi, \kappa}(t)$  referring to the rescaled function  $g_{\kappa}$  is given by

$$P_{\psi, \kappa}(t) = \int_0^{\varepsilon} |g(y)|^2 |G(\kappa y, t)|^2 G(\kappa y, 0)^{-1} y^2 dy \times \left( \int_0^{\varepsilon} |g(y)|^2 G(\kappa y, 0) y^2 dy \right)^{-1} \quad . \quad (18)$$

Since the function  $G$  is easily seen to be bounded and  $g \in L^2(0, \varepsilon)$  due to the assumption, one can use the dominated-convergence theorem and perform the limit  $\kappa \rightarrow 0+$  under the integrals. It holds

$$\lim_{\kappa \rightarrow 0+} G(\kappa y, t) = \int_{m_0}^{\infty} \frac{|f(m)|^2}{2m} e^{-imt} dm$$

so the relation (18) yields

$$\lim_{\kappa \rightarrow 0+} P_{\psi, \kappa}(t) = \left| \int_{m_0}^{\infty} \frac{|f(m)|^2}{2m} e^{-imt} dm \right|^2 \left( \int_{m_0}^{\infty} \frac{|f(m)|^2}{2m} dm \right)^{-2} \quad . \quad (19)$$

We see that the limit is independent of  $g$ . Moreover, if the mass distribution  $|f(\cdot)|^2$  has a sharp peak around a value  $M$ , we can approximate the above expression replacing  $(2m)^{-1}$  by  $(2M)^{-1}$ . It gives

$$\lim_{\kappa \rightarrow 0+} P_{\psi, \kappa}(t) \approx \left| \int_{m_0}^{\infty} |f(m)|^2 e^{-imt} dm \right|^2 \quad , \quad (20)$$

i.e., the standard expression related to the formula (1).

The central question is now whether we are near enough to this limit situation in actual experimental arrangements. We will discuss it in the following sections.

#### 4. Choice of $|f(\cdot)|^2$

One has to specify first the mass distribution entering into (12) through (9). In any realistic theory, it should be of a more or less Breit-Wigner shape, with the principal contribution resulting from the pole approximation to solution of the full dynamical problem<sup>1/</sup>. We choose it in the form

$$|f(m)|^2 = \begin{cases} 0 & \dots m < m_0 \\ N \{ (\Gamma/2\pi) [(m-M)^2 + \frac{1}{4}\Gamma^2]^{-1} + \omega(m) \} & \dots m \geq m_0 \end{cases} \quad (21)$$

where  $M = 938.28$  MeV, further  $m_0$  is the threshold mass and  $\Gamma \lesssim 2 \times 10^{-60}$  MeV corresponding to the lifetime  $T \gtrsim 10^{31}$  yr. The function  $\omega$  is supposed to obey the following restrictions (Fig.1)

$$|\omega(m)| \leq \begin{cases} (\Gamma/2\pi) [(m-M)^2 + \frac{1}{4}\Gamma^2]^{-1} & \dots m \geq m_0 + B \\ (\Gamma/2\pi) [(m_0 - M)^2 + \frac{1}{4}\Gamma^2]^{-1} & \dots |m - M| \leq \frac{1}{2}B \end{cases} \quad (22)$$

where we have denoted  $B = 2(M - m_0)$ . This quantity is of the same order as  $M$ , because the lowest open channel in the standard theory<sup>3/</sup> is  $p \rightarrow e^+ \pi^0$  which gives  $m_0 = 135.47$  MeV.

One can use (15) and (21) to find the normalization factor  $N$  in (21), or more exactly, to derive the estimate

$$|N - 1| \leq \frac{4\Gamma}{\pi B} + O(\Gamma^2/B^2) \quad . \quad (23)$$

Since  $\Gamma/B \lesssim 10^{-63}$ , we set  $N = 1$  in the following. With this license, one can write

$$G(p, t) = G_0(p, t) + G_1(p, t) \quad , \quad (24a)$$

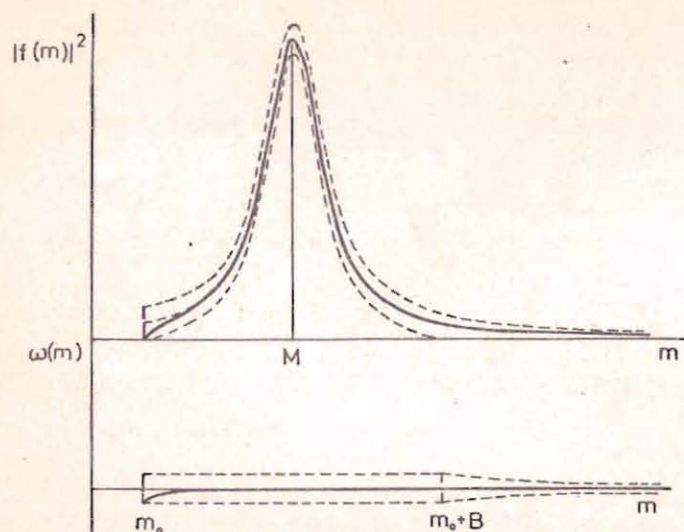


Fig.1. The mass distribution.

where

$$G_0(p, t) = \frac{\Gamma}{2\pi} \int_{m_0}^{\infty} \frac{\exp\{-it(m^2+p^2)^{1/2}\}}{(m-M)^2 + \frac{1}{4}\Gamma^2} \frac{dm}{2(m^2+p^2)^{1/2}}, \quad (24b)$$

$$G_1(p, t) = \int_{m_0}^{\infty} \frac{\omega(m)}{2(m^2+p^2)^{1/2}} \exp\{-it(m^2+p^2)^{1/2}\} dm. \quad (24c)$$

### 5. Evaluation of $G(p, t)$

Let us first estimate the rhs of (24c). The inequalities (22) yield

$$\begin{aligned} |G_1(p, t)| &< \frac{\Gamma}{2\pi} \frac{2}{B^2 + \Gamma^2} \int_{m_0}^{m_0+B} \frac{dm}{m} + \frac{\Gamma}{4\pi} \int_{m_0+B}^{\infty} \frac{dm}{m(m-M)^2} < \\ &\leq \frac{\Gamma}{\pi B^2} \ln\left(1 + \frac{B}{m_0}\right) + \frac{\Gamma}{4\pi} \left[ \frac{1}{M(M-m_0)} + \frac{1}{M^2} \ln\left(1 - \frac{M}{m_0+B}\right) \right] < \\ &< \frac{\Gamma}{\pi m_0 B} + \frac{\Gamma}{2\pi M} \left( \frac{1}{B} - \frac{1}{2M+B} \right) < \frac{\Gamma}{\pi B} \left( \frac{1}{m_0} + \frac{1}{2B} \right). \end{aligned}$$

Since  $2B+m_0 < 4M$ , we obtain

$$|G_1(p, t)| < \frac{2\Gamma M}{\pi m_0 B^2}. \quad (25)$$

The rhs of (24b) can be evaluated by contour integration. The integrated function represents restriction to  $[m_0, \infty)$  of a function which is analytic with exception of two simple poles at  $M \pm \frac{1}{2}\Gamma$  and two cuts on the imaginary axis referring to the factor  $(m^2+p^2)^{1/2}$ . Thus we have (see Fig.2)

$$G_0(p, t) = -2\pi i \frac{\Gamma}{2\pi} \frac{\exp\{-it(m^2+p^2)^{1/2}\}}{2(m^2+p^2)^{1/2}(m-M-\frac{1}{2}\Gamma)} \Big|_{m=M-\frac{1}{2}\Gamma} + G_2(p, t), \quad (26a)$$

$$G_2(p, t) = -\frac{i\Gamma}{2\pi} \int_0^{\infty} \frac{\exp\{-it[(m_0-iz)^2+p^2]^{1/2}\}}{2[(m_0-iz)^2+p^2]^{1/2}} \frac{dz}{(m_0-iz-M)^2 + \frac{1}{4}\Gamma^2}. \quad (26b)$$

Here we have used the fact that the integrated function behaves as  $(\Gamma/4\pi)|m|^{-3}$  for large  $|m|$ , and therefore the integral over  $C_2(R)$  vanishes in the limit  $R \rightarrow \infty$ .

Our intention is to show that  $G(p, t)$  is given essentially by the pole term in (26a). Let us estimate the additional term (26b).

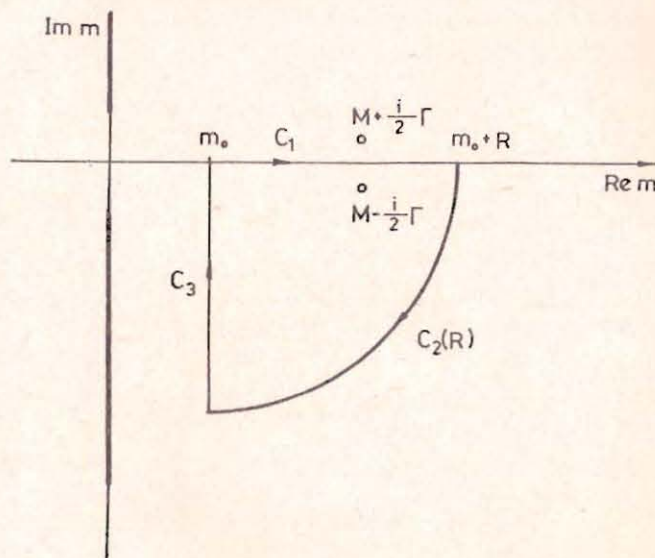


Fig.2. The integration contour for (24b).

We have

$$|(m_0 - iz - M)^2 + \frac{1}{4}\Gamma^2| = (z^2 + \frac{1}{4}B^2)(1 + O(\Gamma^2/B^2)) \quad (27)$$

where the last term can be again neglected. Furthermore,

$$|(m_0 - iz)^2 + p^2|^2 = z^4 + 2(m_0^2 - p^2)z^2 + (m_0^2 + p^2)^2 \equiv F(z^2)$$

We need to estimate  $F(z^2)$  from below. The minimal value  $F(y_0) = 4p^2m_0^2$  is achieved for  $y_0 = p^2 - m_0^2$ , but we are interested in  $F(y)$  for  $y = z^2 \geq 0$  only. If we restrict therefore our attention to those values of  $\varepsilon$  for which the inequality

$$p \leq \varepsilon \leq m_0 \quad (28)$$

holds, then a stronger estimate is possible, namely  $F(y) \geq (m_0^2 + p^2)^2$ , i.e.,

$$|(m_0 - iz)^2 + p^2|^{1/2} \geq m_0 \quad (29)$$

Finally,  $[(m_0 - iz)^2 + p^2]^{1/2}$  lies in the fourth quadrant for  $z \geq 0$  so  $|\exp\{-it[(m_0 - iz)^2 + p^2]^{1/2}\}| \leq 1$  holds for  $t \geq 0$ . Combining this fact with (26) and (29), we get the inequality

$$|G_2(p, t)| \leq \frac{\Gamma}{4\pi m_0} \int_0^\infty \frac{dz}{z^2 + \frac{1}{4}B^2}$$

which yields the following estimate

$$|G_2(p, t)| \leq \frac{\Gamma}{4m_0 B} \quad (30)$$

The estimates (25) and (30) together give

$$|G_1(p, t) + G_2(p, t)| < \frac{\Gamma}{m_0 B} \left( \frac{2M}{B} + \frac{1}{4} \right) = 0.62 \frac{\Gamma}{m_0 B} \quad (31)$$

We shall show a little later that the modulus of the pole term in (26a) is  $\sim (2M)^{-1}$ . Taking then the values of  $M$ ,  $m_0$  and  $\Gamma$  into account, we see that one can neglect the remainder term and write\*

\* The estimates have been performed for the threshold mass  $m_0 \approx 135$  MeV which is given by the standard SU(5) model (cf. Ref. 3, p. 321). If a more complicated Lagrangian is considered, then other decay channels may be open, e.g.,  $p \rightarrow e^+ \nu^c \nu^c$  (Ref. 3, p. 327).

However, the increase of  $M/m_0$  to  $2 \times 10^3$  makes a little difference: the rhs of (32) can be again identified with  $G(p, t)$  for all practical purposes, a possible error being now of order  $\leq 10^{-60}$ . In fact,  $\mathcal{N}_\mu$  should contain also the states referring to the products of secondary decays, etc. This again shifts the value of  $m_0$  to  $m_e$ .

$$G(p, t) = \frac{\exp\{-it[(M - \frac{1}{2}\Gamma)^2 + p^2]^{1/2}\}}{2[(M - \frac{1}{2}\Gamma)^2 + p^2]^{1/2}} \quad (32)$$

## 6. The decay probability

Now one has to insert the calculated  $G(p, t)$  into (14). Before doing that, however, it is useful to find a suitable approximative expression to

$$\frac{G(p, 0)^2 - |G(p, t)|^2}{G(p, 0)} = \frac{1 - \exp\{2t \operatorname{Im}[(M - \frac{1}{2}\Gamma)^2 + p^2]^{1/2}\}}{2[(M - \frac{1}{2}\Gamma)^2 + p^2]^{1/2}} \quad (33)$$

which would make the integration in (14) easier. Recall that the expression (32) itself is approximative. In view of (31), the corresponding error will not change essentially if we replace the denominator in the rhs of (33) by  $2(M^2 + p^2)^{1/2}$ . Furthermore, we shall assume that

$$\varepsilon \ll M \quad (34)$$

In fact, we have already restricted  $\varepsilon$  by (28), where  $m_0$  is at least seven times less than  $M$ , but for a reasonable initial localization of the proton, the inequality (34) is fulfilled even much better. In that case,  $p/M$  is small too and the square root can be expanded in powers of

$$z = \frac{p^2}{M^2} - 1 - \frac{\Gamma}{M} - \frac{\Gamma^2}{4M^2}$$

so that we obtain

$$2t \operatorname{Im}[(M - \frac{1}{2}\Gamma)^2 + p^2]^{1/2} = -\Gamma t \left[ 1 - \frac{p^2}{2M^2} + \frac{3p^4}{8M^4} + \dots \right]$$

neglecting again the terms which contain  $\Gamma/M$ . The numerator in the rhs of (33) is then given by

$$\Gamma t \left[ 1 - \frac{p^2}{2M^2} + \frac{3p^4}{8M^4} + \dots \right] - \frac{1}{2}(\Gamma t)^2 \left[ 1 - \frac{p^2}{M^2} + \frac{p^4}{M^4} + \dots \right] + \dots \quad (35)$$

The second term can be neglected since  $\Gamma t$  is very small\*. If we neglect also the higher powers of  $p^2/M^2$ , then

$$\frac{1}{2}[(M - \frac{1}{2}\Gamma)^2 + p^2]^{-1/2} \approx \frac{1}{2M} \left[ 1 - \frac{p^2}{2M^2} \right]$$

\* Cf. the first footnote on p. 3.

so the relations (33), (35) yield

$$\frac{G(p,0)^2 - |G(p,t)|^2}{G(p,0)} \approx \frac{\Gamma t}{2M} \left[ 1 - \frac{p^2}{M^2} \right], \quad (36)$$

up to higher order terms.

Substituting now from (36) to (14), we get

$$Q_\psi(t) \approx \frac{2\pi\Gamma t}{M} \int_0^\varepsilon |g(p)|^2 \left[ 1 - \frac{p^2}{M^2} \right] p^2 dp. \quad (37)$$

Furthermore, the normalization condition (13) can be in the same way rewritten as

$$\frac{2\pi}{M} \int_0^\varepsilon |g(p)|^2 \left[ 1 - \frac{p^2}{2M^2} \right] p^2 dp \approx 1, \quad (38)$$

again up to higher order terms. Combining the last two relations, we can express finally the decay probability in the form

$$Q_\psi(t) \approx \Gamma t \left\{ 1 - \frac{\pi}{M^3} \int_0^\varepsilon |g(p)|^2 p^4 dp \right\}, \quad (39)$$

i.e., we obtain the formula (3a) with  $s(g)$  given explicitly<sup>\*</sup>.

## 7. Conclusions

The central question concerns now magnitude of the slowdown coefficient  $s(g)$ . Let us first derive an upper bound for it. Suppose that the following inequality holds,

$$s(g) \equiv \frac{\pi}{M^3} \int_0^\varepsilon |g(p)|^2 p^4 dp \leq K, \quad (40a)$$

then one may multiply its rhs by the expression appearing on the lhs of (38), thus obtaining

<sup>\*</sup> The formula (39) can be interpreted as a result of momentum spread and of the relativistic dilatation of lifetime for moving particle. Let  $\tau_0$  be the "theoretical" lifetime at the rest. The lifetime of the moving particle with momentum  $p$  and mass  $M$  is

$$\tau(p) = \tau_0 (1 - v^2)^{-1/2} = \tau_0 (p^2 + M^2)^{1/2} / M.$$

The mean value over the momentum spread

$$\bar{\tau} = 4\pi \int \tau(p) |g(p)|^2 G(p,0) p^2 dp \approx \tau_0 \left\{ 1 + \frac{\pi}{M^3} \int_0^\varepsilon |g(p)|^2 p^4 dp \right\}$$

corresponds just to eq. (39) in our approximation.

$$\pi \int_0^\varepsilon |g(p)|^2 \left[ 2K - (1+K) \frac{p^2}{M^2} \right] p^2 dp \geq 0.$$

This inequality is certainly fulfilled if one requires the square bracket to be non-negative for  $p = \varepsilon$ . It yields a sufficient condition, namely

$$K \geq \frac{\varepsilon^2}{M^2} \left( 2 - \frac{\varepsilon^2}{M^2} \right)^{-1},$$

under which (40a) is valid. In view of the assumption (34), we get the sought bound,

$$s(g) \leq \frac{2}{2M^2} \ll 1. \quad (40b)$$

Of course, the value of  $s(g)$  is determined by the shape of  $g$ , and not only by the three-momentum cut-off  $\varepsilon$ . If, for instance,  $g$  has a sharp peak of a width  $\eta \ll \varepsilon$  and its tail decays rapidly enough in the sense that

$$\int_\eta^\varepsilon |g(p)|^2 p^{2k} dp \ll \int_0^\varepsilon |g(p)|^2 p^{2k} dp, \quad k=1,2,$$

then one can estimate  $s(g)$  rather by  $\eta^2/2M^2$ . On this loose level, therefore, the value of  $s(g)$  is determined by the three-momentum spread  $\Delta p$  of  $\psi$ ,

$$s(g) \approx \frac{(\Delta p)^2}{2M^2 c^2}, \quad (41a)$$

where we have returned to the standard system of units. Continuing this heuristic argument, we express  $p$  from the uncertainty relation<sup>\*\*</sup> obtaining in this way<sup>\*\*</sup>

<sup>\*</sup> Strictly speaking, the uncertainty relation just says that the dispersion  $\Delta p$  is bounded from below by the appropriate multiple of  $(\Delta q)^{-1}$ , and hence it is not immediately applicable here. However, the conventional wisdom which replaces  $\Delta p$  by  $\hbar(\Delta q)^{-1}$  in such situations has some point. As a simple example, consider a spinless particle in one-dimensional non-relativistic quantum mechanics whose wavefunction in the  $p$ -representation is of the form  $\hat{\varphi}(p) = \varepsilon^{-1/2} \chi_{[-\varepsilon/2, \varepsilon/2]}(p)$ . In the  $x$ -representation,  $\varphi(x)$  has a slowly decaying tail so the dispersion  $(\Delta q)_\varphi = \infty$ . Nevertheless,  $\varphi$  forms a distinct peak whose width (measured, e.g., from the distance of neighbouring zeros) is  $\sim \hbar/\varepsilon$ .

<sup>\*\*</sup> In the sense of the previous footnote,  $\Delta q$  here and in (3b) is not necessarily the dispersion but simply a suitable quantity characterizing the size of the proton wavepacket.



$$s(g) \lesssim \frac{3M^2}{8M^2 c^2} (\Delta q)^{-2} \quad (41b)$$

Hence we arrived at the relation (3b); substituting the numerical values we find  $c = 1.6 \times 10^{-26} \text{ cm}^2$ .

It is clear that a more careful analysis is needed if one wants to get some precise statements about the dependence of  $s(g)$  on the shape of the wavefunction. We are convinced, however, that the results of such an effort cannot alter the principal physical conclusion drawn from the above considerations, namely, that the suppression of decay of a free proton due to kinematical effects is unlikely under realistic experimental conditions. We expect also this conclusion to be preserved if one relaxes the technical assumptions made above, i.e., the spherical symmetry and neglect of the spin.

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Диттрих Я., Экснер П.

E2-84-809

Распад протона не может быть подавлен кинематически

При помощи недавно полученной формулы для закона распада релятивистских нестабильных частиц изучается влияние размера и формы волнового пакета на распад свободного протона, который для простоты предполагается сферически симметричным и безспиновым. Оказывается, что вероятность распада уменьшена по сравнению с теоретически ожидаемой, однако в условиях реальных экспериментов этот эффект получается незначительным. Поэтому кинематическое подавление распада протона представляется мало вероятным в противоречии к предсказанию работы Флеминга.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Dittrich J., Exner P.

E2-84-809

The Proton Decay Cannot be Suppressed Kinematically

Using a recently derived decay-law formula for relativistic unstable particles, we examine how the decay of a free proton could be affected by the size and shape of the wavepacket. For simplicity, the proton is assumed to be spherically symmetric and spinless. The decay probability is found to be lessened comparing to the theoretically expected one, however, the effect is insignificant under realistic experimental conditions. Hence the proton decay is not likely to be suppressed kinematically, contrary to the prediction of a paper by Fleming.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1984