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EXACT CALCULATIONS OF THE LOWEST-ORDER ELECTROMAGNETIC CORRECTIONS FOR THE PROCESSES $e^+e^- \to \mu^+\mu^- (\tau^+\tau^-)$

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1. INTRODUCTION

The process of e^+e^--annihilation into two fermions, observable at collider facilities

$$e^+ + e^- \to f^+ + f^- \qquad (1)$$

is of permanent interest for several reasons. Concerning radiative corrections, in case of leptoproduction one notices the possibility of observing effects independent of strong interactions which allows rather clean tests of the electroweak theory. For quark production corrections from (or to) quantum chromodynamic processes are important, too. In any case, the first necessity in the analysis of process (1) is, of course, the calculation of all QED radiative corrections which may be determined with reasonable effort. The knowledge on QED radiative corrections has been developing rapidly during recent years (see, e.g., the review /1/), mainly motivated by the demand of data analyzers, but partly this is due to accumulated experiences and methodical advances including new facilities of analytic calculations on computers which drastically reduced the effort necessary to get definite results (see /2/ and literature cited therein).

The total cross section and several spectra for (1) and the related Bremsstrahlung process

$$e^+ + e^- \to f^+ + f^- + \gamma \qquad (2)$$

including hard photons have been got to order a^3 in QED in /3,4/ in the ultrarelativistic approximation, $(m/E)^2$, $(\mu/E)^2 < 1$, where E is the beam energy in the c.m.s.; and $m(\mu)$, the mass of initial (final) fermions *. In /6/, the influence of the mass of final fermions on (1,2) has been studied.

Here, we go one step further and determine the total cross section of processes (1,2) to order a^3 in QED for arbitrary masses both in the initial and final states. Since the exact virtual corrections are well known to this order /7/, we have

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**Recently, formulae have been derived for the e.m. corrections to annihilation of e^+e^--pairs into a photon from the initial state only but taking into account leading log's to all orders in a /5/.*

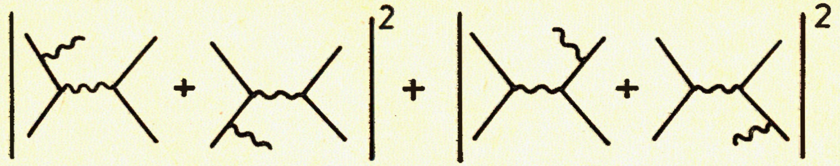

Fig.1. Bremsstrahlung contribution to the energetic fermion spectrum and to the total cross section of $e^+e^- \to f^+f^-\gamma$.

to do the analytic fourfold integration of Bremsstrahlung diagrams over the whole phase space being complicated by the occurrence of two different masses. This became possible by a dedicated choice of variables and integrations and by the extensive use of the system of analytic calculations SCHOONSCHIP /8/. Since the total cross section is a C-even quantity and initial and final state radiation have different C-parity, their interference does not contribute and initial and final state radiation may be calculated separately as is shown in Fig.1. Despite cumbersome calculations, we got rather compact expressions for the total cross section and, on the way, for the energetic fermion spectrum which may be of interest at high energies (compared to the masses) too.

The article is organized as follows. In Sec.2 we formulate the problem to be solved and determine the energetic fermion spectrum. Finally, in Sec.3 the total cross section is presented.

2. THE ENERGETIC FERMION SPECTRUM

We start with the following matrix element for the Bremsstrahlung process (2):

$$M_\beta = Q_i \bar{u}(-k_2, m)\left[\frac{1}{z}(2k_{2\beta} - \gamma_\beta \hat{p})\gamma_\mu - \frac{1}{z}\gamma_\mu(2k_{1\beta} - \hat{p}\gamma_\beta)\right]u(k_1, m) \times$$
$$\times \frac{1}{M^2}\bar{u}(p_1, \mu)\gamma_\mu u(-p_2, \mu) + Q_f \bar{u}(-k_2, m)\gamma_\mu u(k_1, m)\frac{1}{S} \times \qquad (3)$$
$$\times \bar{u}(p_1, \mu)\left[\frac{1}{v}(2p_{1\beta} + \gamma_\beta \hat{p})\gamma_\mu - \frac{1}{v}\gamma_\mu(2p_{2\beta} + \hat{p}\gamma_\beta)\right]u(-p_2, \mu).$$

The Q_i (Q_f) are initial (final) particle charges. The denominators of propagators in (3) are:

$$S = -(k_1 + k_2)^2, \quad M^2 = -(p_1 + p_2)^2, \quad z = -2k_1 p,$$

$$\bar{z} = -2k_2 p = S - M^2 - z, \quad v = -2p_1 p = S - X, \quad \bar{v} = -2p_2 p = X - M^2 .$$

Here and in the following we will use:

$$X = -2p_2 (k_1 + k_2), \quad t = -2k_1 p_2, \quad r = -(p_1 + p)^2 = S - X + \mu^2 .$$

The X may be interpreted in terms of the c.m.s. energy of f^+, $X = 2\sqrt{S}\, p_{20}$. The t is connected with the c.m.s. scattering angle of f^+, $\theta = (\vec{k}_1 \vec{p}_2)_{cms}$:

$$t = \frac{X}{2} - \frac{1}{2S} (\lambda_S \lambda_X)^{1/2} \cos\theta ,$$

where, with

$$\lambda(x, y, z) = (x - y - z)^2 - 4yz,$$

$$\lambda_S \equiv \lambda(S, m^2, m^2) = S^2 - 4m^2 S = 4S |\vec{k}_{1,2}|^2_{cms} ,$$

$$\lambda_X \equiv \lambda(S, r, \mu^2) = X^2 - 4\mu^2 S = 4S |\vec{p}_2|^2_{cms} .$$

Finally, r is the invariant mass of $(f^- \gamma)$. The process (2) depends on five invariants, so the integration over the invariant phase space is four-dimensional. The following choice allows us to perform the analytic integrations without approximations:

$$\sigma_{tot} = \frac{\alpha^3}{\pi^2 \sqrt{\lambda_S}} Q_i^2 Q_f^2 \int d\Gamma \sum_{spins} |M|^2 ,$$

$$\Gamma = \frac{\pi^2}{4S} \int_{2\mu\sqrt{S}}^{S} \sqrt{\lambda_X}\, dX \frac{1}{2} \int_{-1}^{+1} d\cos\theta \cdot \frac{v}{4\pi r} \int_{-1}^{+1} d\cos\theta_R \int_0^{2\pi} d\phi_R . \tag{4}$$

The integrations over dX and $d\cos\theta$ reflect the $d^3 p_2$ and the remaining two integrations result from the choice of the so-called R-system, the rest system of $(f^- \gamma)$: $\vec{p}_1 + \vec{p} = 0$. The θ_R, ϕ_R are the photon angles in this system.

Having properly defined the problem, we will say only a few words on the more or less straightforward but tedious calculations. They are completely contained in a computer program written in SCHOONSCHIP /8/, which run at a CDC-6500 approximately 130 seconds to do the organization of all necessary integrations (including the treatment of infrared - (IR) divergence to be mentioned later)*. Our input information to the program consisted of the matrix elements (3) and three tables of basic integrals over (i) $d\Omega_R = d\cos\theta_R\, d\phi_R$, (ii) $d\cos\theta$, (iii) dX, which have been calculated analytically by hand and are published separately /10/. They contain 29, 15, 11 integrals, respectively, and may be useful in similar calculations, too. Besides several checks on the basic integrals used, we also proved explicitly that after the integral over $d\cos\theta$ the interference of initial and final Bremsstrahlung vanishes as it should be and as is explained in the Introduction and shown in Fig.1.

The resulting energetic fermion spectrum is:

$$\frac{1}{\sigma_0} \frac{d\sigma}{dx} = \frac{\alpha}{\pi} \frac{1}{\beta} [Q_i^2 S_I(x) + Q_f^2 (1 + 2\rho) S_F(x)], \quad \sigma_0 = \frac{4\pi\alpha^2}{3S} Q_i^2 Q_f^2 , \tag{5}$$

$$\begin{aligned} S_I(x) = {} & \frac{2x\beta_x}{1-x} (1 + 2\rho)(1 + 2\rho_f)[(1 - 2\rho) \frac{L(\beta)}{\beta} - 1] + x\beta_x [(7 + 30\rho) \frac{L(\beta)}{\beta} - \\ & - 10 - 26\rho(1 + 2\rho \frac{L(\beta)}{\beta})] - 2x^2 \beta_x [(4 + 21\rho) \frac{L(\beta)}{\beta} - 7 - 17\rho(1 + 2\rho \frac{L(\beta)}{\beta})] + \\ & + 3L(\beta_x)[(1 + 2\rho_f + 2\rho(1 - 2\rho)(1 + 4\rho_f)) \frac{L(\beta)}{\beta} - (1 + 2\rho)(1 + 4\rho_f)] - \\ & - 3x(1 - x) L(\beta_x)[(1 + 4\rho)(\frac{L(\beta)}{\beta} - 1) - 8\rho^2 \frac{L(\beta)}{\beta}] - 3L(x, \beta_x)[\frac{L(\beta)}{\beta} - 1], \end{aligned} \tag{6}$$

$$\begin{aligned} S_F(x) = {} & L(x, \beta_x)[\frac{2(1 - 4\rho_f^2)}{1 - x} - 1 - x - 4\rho_f] - \frac{x\beta_x}{4}[1 + \frac{8(1 + 2\rho_f)}{1 - x} - \\ & - \frac{1}{1 - x + \rho_f} + \frac{\rho_f (1 - \rho_f)}{(1 - x + \rho_f)^2}]. \end{aligned} \tag{7}$$

The following dimensionless variables and functions are introduced:

$$x = \frac{X}{S} = \frac{p_{20}}{E} \epsilon [\frac{\mu}{E}, 1], \quad \rho = \frac{m^2}{S}, \quad \rho_f = \frac{\mu^2}{S},$$

$$\beta = (1 - 4\rho)^{1/2} = \frac{\lambda_S^{1/2}}{S}, \quad \beta_x = (1 - 4\rho_f / x^2)^{1/2} = \frac{\lambda_X^{1/2}}{X},$$

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* *The computer output containing all the subsequent steps is, in principle, available at Dubna, see also /9/.*

$$\beta_f = \beta_x \,|_{x=1} = (1 - 4\rho_f)^{1/2} ,$$

$$L(\beta) = \ln\frac{1+\beta}{1-\beta}, \quad L(x, \beta_x) = \frac{1}{2}[L(\beta_x) + \ln\frac{2 - x(1-\beta_x)}{2 - x(1+\beta_x)}],$$

where x and β_x are the c.m.s. energy (in units of E) and velocity of one of the final fermions, β_f its maximal c.m.s. velocity, and β is the velocity of annihilating fermions.

As is evident also from the corresponding Feynman diagrams, final state radiation is - up to a factor - dependent on and only on the final mass μ, whereas initial state radiation has a mixed dependence on both the masses. As an example, in Fig.2 we show the energetic fermion spectrum for $p\bar{p} \to e^+e^-\gamma$ at E = =100 GeV. For not too large values of x initial state radiation is enhanced compared to final state radiation. This may be traced back to the emission of hard photons from the initial fermion which leads to a smaller photon propagator denominator

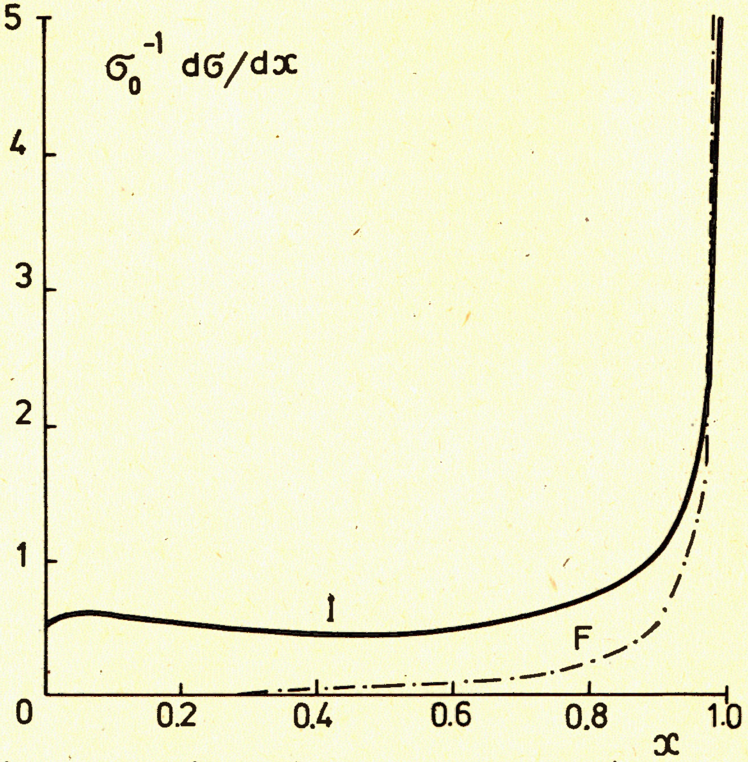

Fig.2. Energetic fermion spectrum for $p\bar{p} \to e^+e^-\gamma$ at E = 100 GeV. (Solid curve - initial state radiation, dashed curve - final state radiation).

M^2. For sufficiently small final fermion mass this produces even a weak local maximum at small values of x (in Fig.2 at x = 0.06). The usual IR-divergence resulting from soft photon emission may be seen at $x \simeq 1$. There, radiation from the lighter electrons dominates that of heavier protons *.

In the case of small masses, $(m/E)^2$, $(\mu/E)^2 \ll 1$ the spectrum may be approximated as follows:

$$\frac{1}{\sigma_0}\frac{d\sigma}{dx} = \frac{\alpha}{\pi}[Q_i^2 S_I(x) + Q_f^2 S_F(x)],$$

$$S_I(x) = (L-1)[\frac{2x\beta_x}{1-x} + 3(1-x+x^2)L(\beta_x) - 3L(x,\beta_x)] + \qquad (8)$$

$$+ x\beta_x[(7-8x)L + 2(7x-5)],$$

$$S_F(x) = \frac{1+x^2}{1-x}L(x,\beta_x) - \frac{x\beta_x}{4}[1 + \frac{8}{1-x} - \frac{1-x}{(1-x+\rho_f)^2}], \qquad (9)$$

where $L = L(\beta)|_{\beta \simeq 1} = \ln\frac{1}{\rho}$.

3. THE TOTAL CROSS SECTION

The remaining integration over x would be trivial without the IR divergence from terms proportional to $(1-x)^{-1}$ in (6), (7). The IR problem forces us to take into account also the vertex corrections to process (1) for the calculation of the total cross sections. Since there is no contribution from box diagrams to order α^3 , the σ_{tot} reads:

$$\sigma_{tot} = \sigma_0 \frac{\beta_f}{4\beta}(3-\beta^2)(3-\beta_f^2)[1 + \delta_I + \delta_F + \delta_{VP}], \qquad (10)$$

where δ_{VP} is the contribution from vacuum polarization /7,12/ to (1) not to be considered here, and

$$\delta_I = \delta_{IB} + Q_i^2\delta_V(m,\beta), \quad \delta_F = \delta_{FB} + Q_f^2\delta_V(\mu,\beta_f).$$

** Of course, rather similar conclusions may be drawn from other spectra too, as has been done in /11/ for the energetic photon spectrum.*

The $\delta_{IB}(\delta_{FB})$ being the integrated over x initial (final) Bremsstrahlung contribution (6), (7), and

$$\delta_V(m,\beta) = \frac{\alpha}{\pi}\{(-P_{IR} + \ln\frac{\eta}{m})[\frac{1+\beta^2}{\beta}L(\beta) - 2] - 2 + \tag{11}$$

$$+ \frac{4-\beta^2}{3-\beta^2}\beta L(\beta) + \frac{1+\beta^2}{2\beta}[\Phi(\frac{2\beta}{\beta-1}) - \Phi(\frac{2\beta}{\beta+1}) + \pi^2]\} .$$

Here $P_{IR} = (n-4)^{-1} + \gamma/2 - \ln 2\sqrt{\pi}$ contains the infrared pole (n - the regularized space-time dimension, γ -the Euler constant), and η is some parameter with dimension of mass,

$$\Phi(x) = -\int_0^1 \frac{dt}{t}\ln(1-xt) .$$

The Bremsstrahlung IR singularity has also been isolated using dimensional regularization following the method developed in/13/. Some technical points are described in/10/ including a list of necessary integrals. Here we only remark that most of the corresponding calculations have been done within the SCHOONSCHIP programme already mentioned. We end with the following expressions for δ_I, δ_F:

$$\delta_I = \frac{\alpha}{\pi}Q_i^2\{[\frac{1+\beta^2}{\beta}L(\beta) - 2](\ln\frac{\beta_f^2}{\rho_f} + \frac{1}{2}\ln\frac{1}{\rho}) -$$

$$- \frac{4L(\beta_f)[\beta L(\beta) - 1]}{\beta_f(3-\beta^2)(3-\beta_f^2)} + \frac{1+\beta^2}{\beta}[\Phi(\frac{2\beta}{(\beta-1)}) - \Phi(\frac{2\beta}{(\beta+1)}) + \frac{\pi^2}{2}] -$$

$$- \frac{L(\beta)}{3\beta(3-\beta^2)}[1 + \frac{(1-\beta^2)}{2(3-\beta_f^2)}\cdot(24 + 9\beta^2 - 2\beta_f^2 - 5\beta^2\beta_f^2)] +$$

$$+ \frac{2}{3}[1 + 2(\frac{1-\beta^2}{3-\beta^2})(\frac{3-2\beta_f^2}{3-\beta_f^2})]\} , \tag{12}$$

$$\delta_F = \frac{\alpha}{\pi}Q_f^2\{\frac{1+\beta_f^2}{\beta_f}[\ln b_f\cdot\ln((1-b_f)(1-b_f^2)) + 4\Phi(b_f) + 2\Phi(-b_f)] -$$

$$- 2\ln\frac{\beta_f^2}{\rho_f^{3/2}} - \frac{33 + 22\beta_f^2 - 7\beta_f^4}{8\beta_f(3-\beta_f^2)}\ln b_f + \frac{3}{4}(\frac{5-3\beta_f^2}{3-\beta_f^2})\} , \tag{13}$$

where

$$b_f = \frac{1-\beta_f}{1+\beta_f} .$$

The δ_F being only dependent on the final mass μ has been given already in /6/ which article has been devoted to the study of final mass effects. It may also be taken from/14/, where the imaginary part of the two-loop vacuum polarization has been calculated. The δ_I is a new result which in the limit of small initial mass m coincides with the approximated result given in/6/.

Deviation between the exact and the approximated expressions arises not too far away from the threshold. In Fig.3 both values are shown for $p\bar{p} \to f^+f^-(\gamma)$ ($f = e, \mu, \tau, \ldots$) at E = 2 GeV. For comparison, we included the δ_F too. Whereas δ_F takes for small final masses μ the constant value 0.2%, δ_I is non-negligible both for

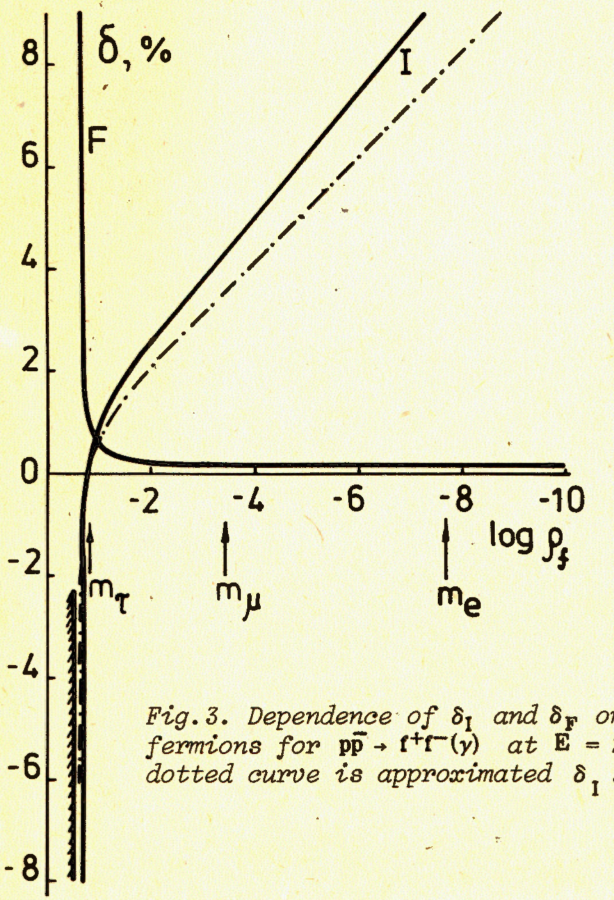

Fig. 3. Dependence of δ_I and δ_F on the mass of final fermions for $p\bar{p} \to f^+f^-(\gamma)$ at E = 2 GeV; dashed-dotted curve is approximated δ_I.

small and large μ showing some deviation between the exact and approximated results over the full range of final masses.

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Точное вычисление электромагнитных поправок низшего порядка к процессам $e^+e^- \to \mu^+\mu^-(\tau^+\tau^-)$

В рамках квантовой электродинамики получены точные выражения для энергетического спектра фермионов и полного сечения процесса $e^+e^- \to f^+f^-(\gamma)$ в порядке α^3 без пренебрежения массами частиц.

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