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D.Kirschbaum

THE ALGEBRA OF THE CLASSICAL HAMILTONIAN MECHANICS AS THE CLOSURE OF TWO FINITE-DIMENSIONAL ALGEBRAS

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I. Introduction

In the structural analysis of physical systems geometrization and mappings are concepts of basic importance. The structures of observables and the properties of states of the considered system are mapped on (expressed by) the elements of some geometrical space and permit us then a classification of physical objects according to their transformation and invariance properties or to the specific time evolution. A lot of information about the structure of physical objects results from the interpretation of definite varieties of mappings (e.g., symmetry groups) or by comparison of expectation values of the observables in different reference frames. The study of the interplay between geometrical structures and transformation properties for the elements of any physical systems is a powerful tool for the analysis of consisting theories and for the development of new theories.

The problem is that for the familiar physical theories the set of allowed continuous transformation is in general an infinite-dimensional one. This concerns for instance classical mechanics, the quantum mechanics and the relativistic field theory. It is therefore desirable to determine some finite-dimensional subgroups in the set of all admissible transformations with definite transformation properties of the physical objects in such a way, that the result of any continuous transformation of these objects may be derived from the transformation behaviour under the subgroups only. Consider for example the theory of general relativity. The group of continuous coordinate transformations has the form

$$x_\mu \longrightarrow x'_\mu = f_\mu(x_\nu), \tag{I.1}$$

$f_\mu(x)$ is some continuous-differentiable function.

It is well-known /1/, that any transformation (I.1) may be derived by repeated use of mappings resulting from elements of two of their finite-dimensional subgroups, namely of the special linear group Sl (4, R) and of the conformal group C_{15}. The algebra of generators for the transformations (I. 1) may be given by

$$ {}^{n}L_{\nu}^{n_0\, n_1\, n_2\, n_3} = -i\cdot x_0^{n_0}\cdot x_1^{n_1}\cdot x_2^{n_2}\cdot x_3^{n_3}\cdot \partial_\nu \tag{I.2}$$

$\left(n = n_0 + n_1 + n_2 + n_3\,,\ \partial_\nu = \frac{\partial}{\partial x_\nu}\right)$.

The Ogievetsky-theorem for this algebra of generators for the group of general covariant coordinate transformations (I. 1) reads: Any generator (I.2) can be expressed as linear combination of repeated commutators of generators of the special linear (the affine) group and of those of the conformal group. The algebra of generators (I. 2) is the closure of these two finite-dimensional Lie-algebras /1,2/ . The use of this theorem permits us to define statements for the construction of relativistic invariant theories /3/ .

We consider in this paper the Hamiltonian formulation of classical mechanics based on the 2 n canonical variables (qi, pi), i = 1, ... n . The observables of this physical system are all functions f(qi,pi). They form with the four operations-addition, multiplication with real numbers (considering real observables), - associative commutative multiplication of functions and Poisson bracket operation between functions, the Heisenberg algebra $\mathcal{a}_H$ of observable quantities of this mechanical system having n degrees of freedom /4/. The diffeomorphisms of this algebra $\mathcal{a}_H$ plays a basic role in the qualitative theory of Hamiltonian systemsas they characterize possible time evolutions, symmetry transformations, invariance conditions and so on for this closed physical system. The structure of this diffeomorphism group (and of the algebra of its generators) and the determination of subgroups is important in many problems of classical mechanics. The group of inner continuous automorphism is called the canonical transformation group of the Heisenberg algebra $\mathcal{a}_H$. The elements of $\mathcal{a}_H$ generate by means of algebraic operations the one-parameter maps called canonical transformations. The algebra of observables and the algebra of the canonical group together constitute the Hamilton algebra /5/ as structural basis of the classical mechanics for the considered system.

Any one-parameter canonical transformation group of $\mathcal{a}_H$ may be generated by a vector field L_f, which is 1-1 determined by an element $f \in \mathcal{a}_H$ using the Poisson bracket operation as followes /6/

$$L_f = \{\cdot, f\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \cdot \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \cdot \frac{\partial}{\partial p_i} \right). \tag{I.3}$$

The kernel of this homomorphism $\phi : \mathcal{a}_H \to \{L_f\}$ consists of the constants $R \in \mathcal{a}_H$. In contrares to the vector fields (I.2) the L_f (I.3) consist in the general case of 2 n terms in analogy to the expressions (I.2).

We prove in this paper, that any generator (I.3) of canonical transformations in the Hamilton algebra may be given as linear combination of repeated Poisson bracket operations of generators from

two Lie-subalgebras of $\{L_f\}$ - the Lie-algebra of the linear canonical transformation group Sp (2n, R) and a Lie algebra of some "conformal canonical group" of dimension n(n+2). The algebra $\{L_f\}$ of generators (I.3) is the closure of these two Lie-algebras. This result has in its structure some similarity to the Ogievetsky-theorem, but the physical systems, the geometrical structures correlated with them and transformation groups are quite different in there physical and mathematical content.

The structure of the Lie-algebra of Sp (2n, R) is well known from literature /7/, the algebra of the "conformal canonical group" (we write the notation c.c.g.) has some common properties with the Lie-algebra of the group SU(n, 1) discussed as spectrum generating algebra for the n-dimensional harmonic oscillator.

The paper is organized as follows: In chapter II we briefly repeat some concepts of the classical Hamilton algebra to fix our notation. We characterize in chapter III the two Lie-algebras of the groups Sp (2n, R) and c.c.g. as subalgebras of the algebra $\{L_f\}$ of canonical transformations generated by the vector fields (I.3). The properties of the group Sp (2n, R) and its Lie-algebra are collected from the literature and are written in a suitable form for our further considerations. It seems that the group c.c.g. and its Lie-algebra have not been used so far as a transformation group in the phase space. We point at some common properties of this Lie-algebra with those of the group SU (n, 1) /8/ .

In chapter IV we prove our theorem and give some conclusions. Some usual relations in phase space are given in Appendix A, and Poisson bracket expressions of generators of the two finite-dimensional Lie-algebras in Appendix B. If not mentioned otherwise, we denote in the formulas the summation by repeated indices. Lie-groups are denoted by capital symbols; the Lie-algebras, by the corresponding small letters.

II. The Hamilton algebra of the classical mechanics

We consider the classical mechanics of a closed physical system of pointlike massive particles moving according to the Newtonian law. In the standard Hamiltonian formulation /4/ the 2n dynamical variables $(\hat{q}, \hat{p}) = (\hat{q}_1, \dots \hat{q}_n, \hat{p}_1, \dots \hat{p}_n)$ with Poisson bracket

$$\{\hat{q}_i, \hat{q}_j\} = 0, \quad \{\hat{p}_i, \hat{p}_j\} = 0, \quad \{\hat{q}_i, \hat{p}_j\} = -\{\hat{p}_j, \hat{q}_i\} = \delta_{ij} \tag{II.1}$$

form a generating system for all observables of this system. Any observable of the Heisenberg algebra $\mathfrak{a}_H$ may be generated from this basis observables (II.1) as result of operations: addition, multiplication with reals $\tau \in R$ and an associative, commutative product [5].

The complete algebraic structure of the dynamical observables is given by the following set of operations between any two elements $\hat{f}, \hat{g} \in \mathfrak{a}_H, (\tau \in R)$

$$\hat{f} + \hat{g}, \quad \tau \cdot \hat{f}, \quad \hat{f} \cdot \hat{g}, \quad \{\hat{f}, \hat{g}\} = -\{\hat{g}, \hat{f}\}. \tag{II.2}$$

The skew-symmetric bilinear Poisson bracket operation $\{\cdot,\cdot\}$ satisfies the requirements for a Lie product including the Jacobi condition implying that $\mathfrak{a}_H$ has the algebraic structure of a Lie ring.

The evolution law of the basic observables (II.1) and for any other observables $\hat{f} \in \mathfrak{a}_H$ is given in the canonical form as

$$\dot{\hat{p}}_k = \{\hat{p}_k, \hat{H}\}, \quad \dot{\hat{q}}_k = \{\hat{q}_k, \hat{H}\}, \quad \dot{\hat{f}} = \{\hat{f}, \hat{H}\}, \tag{II.3}$$

$\hat{H} \in \mathfrak{a}_H$ is some observable characterizing the special dynamical system. In fact, the Poisson bracket operation $\{\cdot,\cdot\}$ in (II.2) is the algebraical form of the evolution law (II.3) for this closed physical system of classical particles [4].

Any continuous transformation of the canonical variables (II.1)

$$\hat{Q} = \hat{Q}(\hat{q}, \hat{p}, \tau), \quad \hat{P} = \hat{P}(\hat{q}, \hat{p}, \tau) \tag{II.4}$$

(τ - real parameter) is called canonical iff it preserves the form of the evolution equations (II.3). This means that from (II.3) and (II.4) results

$$\dot{\hat{Q}} = \{\hat{Q}, \hat{H}(\hat{Q}, \hat{P})\}, \quad \dot{\hat{P}} = \{\hat{P}, \hat{H}(\hat{Q}, \hat{P})\}. \tag{II.5}$$

The transformations (II.4) induce a transformation on the whole algebra $\mathfrak{a}_H$, for any element $\hat{f} \in \mathfrak{a}_H$ results $\hat{f} \longrightarrow \hat{F}$ using $\hat{f}(\hat{q}, \hat{p}) = \hat{F}(\hat{Q}, \hat{P})$.

The finite transformations (II.4) may be expanded for an infinitesimal parameter ε as [9]

$$\hat{Q} = \hat{q} + \varepsilon \cdot \{\hat{q}, \hat{F}\}, \quad \hat{P} = \hat{p} + \varepsilon \cdot \{\hat{p}, \hat{F}\}, \tag{II.6a}$$

for any element $\hat{f} \longrightarrow \hat{F}$ of $\mathfrak{a}_H$ follows

$$\hat{f}(\hat{Q}, \hat{P}) = \hat{f}(\hat{q}, \hat{p}) + \varepsilon \cdot \{\hat{f}(\hat{q}, \hat{p}), \hat{F}(\hat{q}, \hat{p})\}. \tag{II.6b}$$

The function $\hat{F}$ is called generating element of the transformation (II.4). The evolution law (II.3) is clearly a special case of (II.6).

From (II.6) follows, that the relation $\phi : \mathfrak{a}_H \longrightarrow \{L_f\}$ between the generators $L_{\hat{F}}$ and the elements $\hat{F}$

$$\hat{F} \longrightarrow L_{\hat{F}} = \{\cdot, \hat{F}\}, \quad L_{\hat{F}}(\hat{f}) = \{\hat{f}, \hat{F}\}, \tag{II.7}$$

is a linear map of the elements of the algebra $\mathfrak{a}_H$ into the algebra of generators of its continuous transformations. From the Jacobi identity for the Poisson brackets in $\mathfrak{a}_H$ follows for (II.7) $\{\hat{f}, \hat{g}\} \longrightarrow [L_{\hat{f}}, L_{\hat{g}}]$, i.e., the map (II.7) is an homeomorphism of the related two algebras. The kernel of (II.7) consists of the elements (the real numbers) of $\mathfrak{a}_H$.

For the algebra $\{L_{\hat{f}}\}$ of linear operators results from the properties of the Poisson bracket operation in (II.2).

The Poisson bracket operation of (II.2) between any two observables defines on the phase space a symplectic binary product as

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \cdot \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \cdot \frac{\partial g}{\partial q_i} \right), \tag{II.10}$$

especially the canonical evolution equations (II.3) become the well-known expression $\dot{p}_k = -\frac{\partial H}{\partial q_k}$ and $\dot{q}_k = \frac{\partial H}{\partial p_k}$, where H is called the Hamilton function of the considered dynamical system.

Equations (II.7), (II.10) define analytical expressions for the generating vector fields of canonical transformations as follows

$$L_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \cdot \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \cdot \frac{\partial}{\partial p_i} \right). \tag{II.11}$$

These analytical expressions (II.10) and (II.11) are very useful for practical calculations of the Poisson bracket operation between the elements of $\mathfrak{a}_H$.

A basis set in the algebra $\mathfrak{a}_H$ are the monomials in (q,p), using the expansion

$$f(q,p)=\sum_{(i_\nu),(k_\mu)} a^{i_1\cdots i_n}_{k_1\cdots k_n}\cdot q_1^{i_1}\cdots q_n^{i_n}\cdot p_1^{k_1}\cdots p_n^{k_n}\,, \tag{II.12}$$

from this expression follows the corresponding basis for the generators of canonical transformations as

$$L^{(i)}_{Q}{}^{(k)}_{\cdot P}(f)=L^{i_1\cdots i_n}_{k_1\cdots k_n}(f)=\{f, q_1^{i_1}\cdots q_n^{i_n}\cdot p_1^{k_1}\cdots p_n^{k_n}\}. \tag{II.13}$$

Finite canonical transformations (II.4) generated from monomials (II.13) have a simple explicit expression /9/ . The set of all polynomials of degree l ($l=\sum_{\nu=1}^{n} i_\nu+k_\nu$) constitute a vector space $\mathfrak{a}^{l}_H$ of dimension $(2n+l-1)!/(2n-1)!\cdot l!$ in $\mathfrak{a}_H$. Finally we note, that the algebra (and the set of generators (II.11) too) is -2 graded. For $f^{l_1}\in\mathfrak{a}^{l_1}_H,\ f^{l_2}\in\mathfrak{a}^{l_2}_H$ follows

$$\{f^{l_1},f^{l_2}\}\in\mathfrak{a}_H^{l_1+l_2-2}. \tag{II.14}$$

For further use we note the following transformation in the basic variables (II.9), (II.1) of phase space corresponding to the calculus of creation and annihilation operators in quantum mechanics /7,10/

$$z_i=\frac{1}{\sqrt{2}}\cdot(q_i+i\cdot p_i)\quad ,\quad z_i^*=\frac{1}{\sqrt{2}}\cdot(q_i-i\cdot p_i). \tag{II.15}$$

(II.15) maps the real 2n-dimensional phase space into a n dimensional complex space Cn . The functions $f(q,p)\in\mathfrak{a}_H$ are mapped onto the function space about Cn with reality condition

$$f(z,z^*)=\left(f(z,z^*)\right)^*=f^*(z^*,z)\,. \tag{II.16}$$

The algebraical operations (II.1), (II.10) become the expressions

$$\{z_i,z_j^*\}=-i\cdot\delta_{ij}\,,\quad \{f,g\}=-i\cdot\sum_{k=1}^{n}\left(\frac{\partial f}{\partial z_k}\cdot\frac{\partial g}{\partial z_k^*}-\frac{\partial f}{\partial z_k^*}\cdot\frac{\partial g}{\partial z_k}\right). \tag{II.17}$$

The transformation (II.15) is an isomorphism between two realizations of the phase space and the algebraical structure of the Hamiltonian system for the particle classical mechanics. Some more details concerning isomorphic realizations of the phase space are given in Appendix A.

We characterize in the next chapter the canonical transformations of the Lie-groups Sp(2n,R) and c.c.g. in $\mathfrak{a}_H$ and characterize the Lie-algebras of their generators in the set (II.11) using the transformations (II.15), (II.17).

III. Two finite-dimensional Lie algebras of canonical transformations

Using (II.15)-(II.17), we express those observables $f(q,p)$ as $f(z,z^*)$, which are at most quadratic in the variables. They have the form

$$f(z,z^*)=f_i^{(1)}\cdot z_i+f_i^{(2)}\cdot z_i^*+f_{jk}^{(1)}\cdot z_j\cdot z_k+f_{jk}^{(2)}\cdot z_j\cdot z_k^*+f_{jk}^{(3)}\cdot z_j^*\cdot z_k^*\,, \tag{III.1}$$

($f_i^{(\nu)}$, $f_{jk}^{(\nu)}$ - real constants).
They form a Lie-subalgebra. This result follows immediately from (II.14).

However the functions with $f_i^{(\nu)}=0$ generate the real symplectic group Sp (2n,R) of dimension n·(2n+1) /7/.

Let as consider the symplectic group. This noncompact simple Lie-group is generated by the monomial basis /7/

$E_{ij}=-z_i\cdot z_j$	dimension $\binom{n+1}{2}$	(III.2a)
$E^{ij}=z_i^*\cdot z_j^*$	dimension $\binom{n+1}{2}$	(III.2b)
$E_i{}^j=z_i\cdot z_j^*$	dimension n^2 .	(III.2c)

The monomials (III.2) satisfy the symmetry relations $E_{ij}=E_{ji}$, $E^{ij}=E^{ji}$, $(E_i{}^j)=(E_i{}^j)^+$, $(E_{ij})^+=-E^{ij}$.

From (II.17) follows, that the generators (III.2a) and (III.2b) form separately two Abelian subalgebras, the generators (III.2c) satisfy the commutation relations

$$\{E_i{}^j,E_k{}^l\}=i\cdot(\delta_k{}^j\cdot E_i{}^l-\delta_i{}^l\cdot E_k{}^j). \tag{III.3}$$

The set of generators (III.2c) is isomorphic to the Lie-algebra u(n), they constitute the maximal compact subalgebra in the Lie-algebra $sp(2n,R)$ of generators (III.2) for the symplectic group. Note, that $E_0^0 = z_i \cdot z_i^*$ generates the $u(1)$ subalgebra of $u(n)$, the remaining generators (III.2c) form the (n^2-1) dimensional Lie-algebra $su(n)$. E_0^0 is in the basis (II.15) the expression of the Hamiltonian for the classical harmonic oscillator. The remaining commutators of generators (III.2) and some suitable details about the resulting symplectic group transformations are given in the Appendix B.

An essential property of the symplectic group $Sp(2n,R)$ of canonical transformations in the Heisenberg algebra $\mathcal{A}_H$ is the fact, that these group elements act as matrix transformations in any vector space $\mathcal{A}_H^l \subset \mathcal{A}_H$ (i.e., the set of functions defined on phase space with fixed degree l in (q,p)). The translations T_{2n} in phase space (and in $\mathcal{A}_H$) are generated by the 2n variables (z_i, z_i^x). The Lie-algebra (III.1) is therefore the semidirect sum $sp(2n,R) \ni T_{2n}$. The elements (III.1) form the Lie algebra of the inhomogeneous symplectic group of canonical transformations in $\mathcal{A}_H$.

A finite-dimensional conformal canonical group (c.c.g.) of transformations in $\mathcal{A}_H$ may be constructed, if we start with the general conformal transformations in C_n. These maps are given by analytical functions as $z_i' = f_i(z_j)$. Consider the broken linear transformations

$$z_i' = \frac{a_{il} \cdot z_l + b_i}{c_l \cdot z_l + d} \quad), \tag{III.4}$$

this is the group of projective transformations of C_n . The $(n+1)^2$ complex coefficients in (III.4) may be arranged as matrix A_{ij} using the notations $A_{i\,n+1} = b_i$, $A_{n+1\,l} = c_l$, $d = A_{n+1\,n+1}$ giving

$$(A)_{ij} = \left(\begin{array}{c|c} a_{ij} & A_{i\,n+1} = b_i \\ \hline A_{n+1\,j} = c_j & A_{n+1\,n+1} = d \end{array} \right) . \tag{III.5}$$

The transformations (III.4) are nonlinear in the variables and have some structural similarities with the group of conformal coordinate transformations in Minkowski space. The analytical functions f(z) defined on Cn form a commutative subalgebra $\mathcal{A}_0$ of $\mathcal{A}_H$ (this follows directly using (II.17)). The transformation (III.4) maps the algebra $\mathcal{A}_0$ onto itself. The generating vector fields of (III.4) may be calculated using standard techniques. We get (n^2+2n) independent basic generators as follows

$$-i \cdot \frac{\partial}{\partial z_i} \qquad \text{n translation generators} \tag{III.6a}$$

$$-i \cdot z_k \cdot \left(z_i \cdot \frac{\partial}{\partial z_i} \right) \qquad \text{n generators of special conformal transformations} \tag{III.6b}$$

$$-i \cdot z_i \cdot \frac{\partial}{\partial z_j} \qquad n^2 \text{ generators.} \tag{III.6c}$$

The connection (II.17), (Appendix A) of generating monomials and vector fields gives possibility of expressing the generators (III.6) defined on $\mathcal{A}_0$ by generating monomials. Using $L_{z_k^*} = -i \cdot \frac{\partial}{\partial z_k}$, we obtain for the generators (III.6a-c)

$$z_k^* \tag{III.7a}$$

$$z_k \cdot (z_i \cdot z_i^*) = z_k \cdot E_0^0 \tag{III.7b}$$

$$z_i \cdot z_j^* . \tag{III.7c}$$

The monomials (III.7) form together with the Poisson bracket operation (II.17) a Lie-algebra, which is also isomorphic to a Lie-algebra of the vector fields (III.6) with the usual commutator operation (Appendix B).

Using (A.11) (A.16) the monomials (III.7) generate vector fields defined on all elements $f(z, z^x)$ of $\mathcal{A}_H$. Especially, for the elements of $\mathcal{A}_0$ we get expressions (III. 6).

Using for monomials (III.7) the notations

$$z_i \cdot z_j^* = A_i{}^j , \quad z_i^* = A_{n+1}{}^i , \quad z_i \cdot E_0^0 = A_i{}^{n+1} , \quad E_0^0 = A_{n+1}{}^{n+1} , \tag{III.8}$$

one obtains the commutation relation as follows (Appendix B)

$$\{ A_i{}^j , A_l{}^m \} = i \cdot (g_l{}^j \cdot A_i{}^m - g_i{}^m \cdot A_l{}^j) , \tag{III.9}$$

$$i, k, l, m = 1, 2, \cdots, n+1 ; \quad g_k{}^l = \delta_k{}^l \ (l, k = 1, \cdots, n) , \ g_{n+1}{}^{n+1} = -1 .$$

The generating monomials (III.7) form a Lie algebra (III.9), which is locally isomorphic to the algebra of the Lie group SU(n,1). Vector fields defined by the Lie algebra (III.7) generate finite canonical transformations on all observables f(q,p) or $f(z, z^x)$, respectively.

Comparing the Lie algebra (III.7), (III.8) with the well-known realizations of the Lie algebra su(n, 1) as a spectrum generating algebra of the harmonic oscillator, one finds an essential difference, originating from that the generators (III.7a) and (III.7b) are not Hermitean-conjugated (cf. Appendix B).

As a consequence, the generated group of canonical transformations maps the subalgebra $\mathfrak{a}_0$ of $\mathfrak{a}_H$ consisting of the analytical functions f(z) into itself. Note that the subalgebra of antianalytical functions $f(z^x)$ does not have this property.

The Lie-algebra (III.7) is a finite-dimensional subalgebra of that infinite-dimensional algebra of canonical transformations on $\mathfrak{a}_H$, which may be constructed from the conformal transformations of Cn (see for some details Appendix B). We call therefore the Lie--group of canonical transformations, which is generated by the monomials (III.7), the conformal canonical group (c.c.g) of the Heisenberg algebra $\mathfrak{a}_H$.

IV. Proof of a theorem concerning the algebra k(2 n, R)

We proof in this chapter the following theorem: The generating function $f(z,z^x)$ (or f(q,p) respectively) for any generator of the algebra k(2n, R) may be expressed as linear combination of repeated commutators of generators (IV.1a), (IV.1b) of the two subgroups Sp(2n, R) and c.c.g.. The algebra k(2n, R) is the closure of these two Lie-algebras of canonical transformations in $\mathfrak{a}_H$.

Consider the infinite-dimensional algebra of canonical transformations k(2n, R) of the Hamiltonian system $\mathfrak{a}_H$. The generators of k(2n, R) may be expressed by vector fields or by the generating elements $f \in \mathfrak{a}_H$ (chapter II). The "Lie product" between any two elements of k(2n, R) is realized as commutators of Poisson brackets, respectively. Using the homeomorphism between these two realizations of k(2n, R), we consider below the monomials in (z, z^x) as explicit expressions for the basis generators in k(2n, R). This notation allows us to simplify the proof of our theorem.

Consider the generators of Lie-algebras for the two canonical transformation groups SP(2n, R) and c.c.g., e.g., the monomials (III.2) and (III.7), respectively

$$z_i \cdot z_j^*,\ -z_i \cdot z_j,\ z_i^* \cdot z_j^* \qquad (n^2 + 2n) \text{ generators} \qquad \text{(IV.1a)}$$

$$z_i \cdot z_j^*,\ z_i^*,\ z_i \cdot E_0^\circ,\ E_0^\circ \qquad (n + 1)^2 \text{generators.} \qquad \text{(IV.1b)}$$

Instead of (IV.1b) the generators ($z_1 \cdot z_j^x$, z_1, $z_j^x \cdot E^0_m$, E^0_0) may be used what does not change the content of our theorem (see also Appendix B).

We proof the theorem by induction to the degree l of the generating monomials (II.13). The proof proceeds similar to the Ogievetsky -theorem of generators (II.2) /1,2/ . Calling the closure of the two Lie-algebras (IV.1) as g we show, that g contains any monomial

in z, z^x of third degree. Assume, that g contains the monomials of some fixed degree l (l fixed integer, l > 3) - then it follows that g contains all monomials of degree (l+1) , too. Consequently, g contains any monomial (II.13), i.e., g contains the same elements as k (2n,R) = $\{L_f\}$.

Firstly, g contains the elements

$$2 \cdot i \cdot (z_i \cdot z_j \cdot z_k) = \{z_k \cdot E_0^\circ,\ z_i \cdot z_j\} \qquad \text{(IV.2a)}$$

$$-i \cdot (\delta_{ik} \cdot z_j^* \cdot E_0^\circ + \delta_{jk} \cdot z_i \cdot E_0^\circ + 2 \cdot z_i^* \cdot z_j^* \cdot z_k) = \{z_k \cdot E_0^\circ,\ z_i^* \cdot z_j^*\} \qquad \text{(IV.2b)}$$

$$i \cdot z_j = \{z_i^*,\ z_i \cdot z_j\}. \qquad \text{(IV.2c)}$$

The algebra g contains the inhomogeneous symplectic algebra (III.2) as result of (IV.2c). Taking in (IV.2a) all combinations of indices i, j, k we conclude, that g contains all monomials in the variables z of third degree.

The algebra g contains all commutators.
Taking all index combinations in $(z)^l$ it follows that g contains also all monomials of the type $(z)^{l+1}$.

Using (IV.8) and (IV.1a) we see, that the generator

$$\{z_i^{l+1},\ z_i^{*2}\} = -2 \cdot i \cdot (l+1) \cdot z_i^l \cdot z_i^* \qquad \text{(IV.9)}$$

is an element of g. Taking use of the repeated commutators $\{z_i^l \cdot z_i^*,\ z_i \cdot z_k^*\}$ it follows, that all monomials of the type $(z)^l \cdot (z^x)$ are elements g. The algebra g contains also all commutators of the form

$$\{(z)^l \cdot z_i^*,\ z_j^* \cdot z_k^*\} \subset (z)^{l-1} \cdot (z^*)^2, \qquad \text{(IV.10)}$$

and consequently all monomials of the type $(z)^{l-1} \cdot (z^x)^2$ of degree (l+1). Applying the generators $z_i^* \cdot z_j^*$ on the monomials of the type $(z)^{l_1+1} \cdot (z^x)^{l_2}$ one gets the result, that all monomials $z^{l_1} \cdot (z\ )^{l_2+1}$ are contained in g, too.

This procedure ends with the conclusion, that the monomials of the type $(z^x)^{l+1}$ are contained in g, e.g., the algebra g agrees with k(2n,R). This proves our theorem. We summarize our results and give some comments.

We have shown, that any generator (II.11), (II.13) can be expressed as linear combination of repeated commutators of generators (IV.1) of the symplectic group Sp(2n,R) and those of the conformal canonical group c.c.g.. The algebra k(2n,R) is the closure of the two Lie-algebras of these canonical transformation groups in the Heisenberg algebra $\mathfrak{a}_H$. The structural relations between the four algebras of the canonical transformation groups discussed above may be written in the following form

$$k(2n,R) = sp(2n,R) \cup c.c.g. , \qquad \text{(IV.11)}$$
$$u(n) = sp(2n,R) \cap c.c.g. .$$

The generators (III.7b) of special conformal transformations act as "raising operators" (IV.2a), (IV.2b), (IV.8) on the observables $f(z,z^x)$ or f(q.p), respectively.

Any continuous conformal transformation $z_i' = f(z_j)$ may be extended to canonical transformations on the Heisenberg algebra $\mathfrak{a}_H$ (see (III.4)-(III.7), Appendix B). This conformal group leaves the subalgebra $\mathfrak{a}_o \subset \mathfrak{a}_H$ invariant. This result is similar to that obtained for the light-cone invariance with respect to conformal transformations in the theory of general relativity. The action of the finite-dimensional conformal canonical group c.c.g. in phase space may be compared with the transformations of the special conformal group C_{15} in relativistic space-time.

The symmetry group u(n) of the harmonic oscillator is a common subgroup in both considered Lie-groups (IV.11). The remaining generators form in Sp(2n, R) and in c.c.g. two Abelian subgroups. The Ogievetsky-theorem describes the structure of the algebra for the general relativistic covariance group, while our theorem concerns the algebra of canonical transformations in the Hamiltonian system of classical mechanics. Both theorems deal with mappings but in quite different geometrical structures. However there exist some interesting similarities concerning the two considered subgroups in both geometries, e.g., the actions of these subgroups as transformation groups in the Minkowski space or in the phase space, respectively. Both subgroups have therefore common features.

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Appendix A

The usual basic observables of the algebra $\mathfrak{a}_H$ are in the classical Hamiltonian mechanics the canonical observables (q, p)

satisfying the relations (II.1). The measuring process for this physical system is a one to one mapping of the observables onto the real functions defined in the (2n) dimensional phase space (q, p) (see (II.9)). The function algebra is a true representation of $\mathfrak{a}_H$, i.e., the algebraical operation structure (II.2) is isomorphically mapped on operations which are defined within the function algebra. For instance the Poisson bracket operation on $\mathfrak{a}_H$ is mapped onto the analytical operation (II.10) which is defined between any elements of the function algebra.

For physical interpretations the following notation for phase space variables (q, p) is very useful. This is because it makes the geometry of phase and the structure of transformations very transparent. In doing so collects the 2n variables (q, p) into a combined set of variables (x) = (x_1 ... x_{2n}) by the rule

$$x_i = q_i \;, \quad x_{n+i} = p_i \;, \quad i = 1,2,\dots,n. \qquad \text{(A.1)}$$

In terms of x_s the Poisson bracket rules (II.1) become

$$\{x_i , x_j\} = J_{ij} = (J), \qquad \text{(A.2)}$$

where (J) denotes is the antisymmetric (2n) x (2n) matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \text{(A.3)}$$

for with I being the n dimensional unit matrix.

The Poisson bracket operation (II.10) is given in terms of x_s by the expression

$$\{f(x), g(x)\} = \sum_{k,l=1}^{2n} \left(\frac{\partial f}{\partial x_k}\right) \cdot J_{kl} \cdot \left(\frac{\partial g}{\partial x_l}\right) , \qquad \text{(A.4)}$$

i.e., it is a skew-symmetric operation between two vectors. The coefficients of these vectors are gradients of function f(x) and g(x), respectively. Eq. (A.4) is called "the symplectic product" of these vectors. Note, that (A.4) . determines a function for any two vectors. This is an essential difference compared to the usual scalar product in a Riemannian geometry.

A transformation $(x) \longrightarrow (X)$ (see in comparison with (II.4) - (II.6)) is canonical iff

$$\{X_i, X_j\} = \sum_{k,l} \left(\frac{\partial X_i}{\partial x_k}\right) \cdot J_{kl} \cdot \left(\frac{\partial X_j}{\partial x_l}\right) = \{x_i , x_j\}. \qquad \text{(A.5)}$$

In order to get this equation in a more compact form we denote the Jacobian matrix of the transformation $(x) \longrightarrow (X)$ by the $(2n)x(2n)$ matrix M, i.e.,

$$M_{ik}(x) = \frac{\partial X_i}{\partial x_k} \quad . \tag{A.6}$$

Condition (A.5) leads together with (A.6) to the block form

$$M \cdot J \cdot M^T = J , \tag{A.7}$$

i.e., any canonical transformation leaves the symplectic tensor J invariant. After applying the rule (A.1), we may define for any observable $f(x) \in \mathfrak{a}_H$ a(2n)-dimensional vector with components $f_k = \frac{\partial}{\partial x_k} f(x)$. Using (A.6) the transformation law for observable is now written as

$$\{z_i \cdot z_j \cdot z_k , z^*_\ell \cdot z^*_m\} = -i \cdot (\delta_{i\ell} \cdot z_j \cdot z_k \cdot z^*_m + \delta_{jm} \cdot z_i \cdot z_k \cdot z^*_\ell + \\ + \delta_{km} \cdot z_i \cdot z_j \cdot z^*_\ell + \delta_{j\ell} \cdot z_i \cdot z_k \cdot z^*_m + \delta_{k\ell} \cdot z_i \cdot z_j \cdot z^*_m + \delta_{im} \cdot z_j \cdot z_k \cdot z^*_\ell) . \tag{IV.3}$$

Taking the indices in (IV.3) as i=l, j≠K, j≠m there follows, that g contains the monomials $z_j \cdot z_k \cdot z^*_m \; (j \neq k \neq m)$. From i=l, j=k≠l follows that g contains $z_j^2 \cdot z^*_m$.

Since the commutator

$$\{z_i^3 , z_i^{*2}\} = -6 \cdot i \cdot z_i^2 \cdot z_i^* \tag{IV.4}$$

is an element of g it follows that g contains also all monomials of second degree in z and linear in z^x. Furthermore the commutators

$$\{z_i \cdot z_j \cdot z^*_k , z^*_\ell \cdot z_m\} = \\ = -i \cdot (\delta_{i\ell} \cdot z_j \cdot z^*_k \cdot z^*_m + \delta_{j\ell} \cdot z_i \cdot z^*_k \cdot z^*_m + \delta_{im} \cdot z_j \cdot z^*_k \cdot z^*_\ell + \delta_{jm} \cdot z_i \cdot z^*_k \cdot z^*_\ell) \tag{IV.5}$$

are also elements of g.

Repeating with (IV.5) the steps leading from (IV.3) to (IV.4) we conclude, that g contains all monomials of second degree in the variables z^x but linear in z.

The Lie algebra g contains the commutators (any combination i, j, l)

$$\{z_k \cdot z^*_i \cdot z^*_j , z^*_k \cdot z_\ell\} \quad , \tag{IV.6}$$

all monomials of third degree in the variables z^x are elements of g. Summarizing (IV.2)-(IV.6) we conclude, that the algebra g contains all elements of third degree in (z, z^x).

Assume then, that g contains all monomials of degree l in the variables (z, z^x), e.g., all elements of the form

$$(Z)^{\ell_1} \cdot (Z^*)^{\ell_2} , \tag{IV.7}$$

$\ell_1 + \ell_2 = \ell ; \; \ell_1 = 0, 1, \cdots, \ell .$

Taking $l_1 = 0,1, \ldots l$ there follows $(l+1)$ different types of monomials (IV.7) for each degree l. For the further proof we use the fact, that the algebra $\mathfrak{a}_H$ is -2 graded (II.14) and that the commutation relations of the two subalgebras (IV.1) act on the monomials (IV.7). Therefore, Lie-algebra g contains together with elements (II.7) of the type $(Z)^1$ also the commutators

$$\{(Z)^\ell , z_j \cdot E^\circ_\circ\} = -i \cdot z_j \cdot \Big(\sum_m z_m \cdot \frac{\partial}{\partial z_m} (Z)^\ell \Big) c \, (Z)^{\ell+1} . \tag{IV.8}$$

$$L_{\hat h}(\tau_1 \cdot \hat f + \tau_2 \cdot \hat g) = \tau_1 \cdot L_{\hat h}(\hat f) + \tau_2 \cdot L_{\hat h}(\hat g) , \\ L_{\hat h}(\hat f \cdot \hat g) = \hat f \cdot L_{\hat h}(\hat g) + L_{\hat h}(\hat f) \cdot \hat g , \\ L_{\hat h}(\{\hat f , \hat g\}) = \{L_{\hat h}(\hat f), \hat g\} + \{\hat f , L_{\hat h}(\hat g)\} , \tag{II.8}$$

the generators (II.7) act as derivations in the algebraical structure of $\mathfrak{a}_H$. The canonical transformations (II.4), (II.5) are therefore the continuous (inner) automorphisms of the Heisenberg-algebra $\mathfrak{a}_H$.

Equations (II.7), (II.8) are the algebraical formulation of the duality principle, valid in the Hamilton algebra of the classical mechanical particle system: The algebraical structures on observables of $\mathfrak{a}_H$ and on generators of canonical transformations on $\mathfrak{a}_H$ are realized by a skew-symmetric bilinear operation on the same real vector space structure of elements $\hat f(\hat q, \hat p)$, which are functions of the canonical observables (II.1).

A consequence of the axioms for the classical mechanical particle system is the very specific structure of the measuring process. At an arbitrary instant of time any canonical variable has exact one expectation value. The set of 2n variables $(\hat q, \hat p)$ is one-one mapped by the measuring process on the set of the 2n real numbers (q, p)

$$(\hat q, \hat p) = (\hat q_1, \hat q_2, \cdots, \hat q_n, \hat p_1, \hat p_2, \cdots, \hat p_n) \longrightarrow (q_1, \cdots, q_n, p_1, \cdots p_n) = (q, p) . \tag{II.9}$$

The map (II.9) holds for any instant of time, the resulting set (q, p) defines the 2n dimensional real phase space of the system. The Heisenberg algebra $\mathfrak{a}_H$ of dynamical observables is mapped by (II.9) into the real valued functions defined on the phase space. This set of functions has obviously a linear vector space structure, because it is closed under the operations of addition and scalar multiplication. The product•(II.2) of observables is mapped onto the commutative and associative product of related functions of expectation values. The classical mechanical measuring process (II.9) may be viewed as diffeomorphism between the algebra $\mathfrak{a}_H$ of dynamical observables $\hat{f}(\hat{q}, \hat{p})$ and the algebra of real valued functions f(q,p). Because both algebras are isomorph we shall use the same symbol $\mathfrak{a}_H$ for them in this paper if the concrete context is not misleading.

$$f_k(x) = M_{ik}(x) \cdot f_i(X). \tag{A.8}$$

The geometrical properties of the Poisson bracket (A.4) between any two functions and the transformation relations (A.7), (A.8) define the structure of this classical Hamiltonian system as that of a symplectic geometry. Note that the transformation matrices (A.8) depended in general case on the argument (x).

Canonical transformations of some finite dimensional groups as U(n), Sp (2n,R) and the c.c.g. get a transparent mathematical structure after using a complex phase space basis. Taking as a basis the linear combinations

$$z_i = \frac{1}{\sqrt{2}} \cdot (q_i + i \cdot p_i), \quad z_i^x = \frac{1}{\sqrt{2}} \cdot (q_i - i \cdot p_i), \tag{A.9}$$

it follows that the structure of phase space is very similar to the calculus of annihilation and creation operators well known in quantum mechanics /7/.

The transformation (A.9) maps the (2n) dimensional real phase space of variables (A.1) into the complex space Cn. Any real function f(x) is mapped on a function defined on Cn. The observables of $\mathfrak{a}_H$ fulfil the reality condition as

$$f(z, z^x) = (f(z, z^x))^x = f^x(z^x, z). \tag{A.10}$$

The algebraical operations (II.2) between any elements of $\mathfrak{a}_H$ are mapped onto similar operations in the function space defined over Cn. The Poisson bracket operations (II.1), (II.12) are, for instance, given as follows

$$\{z_i, z_j^x\} = -i \cdot \delta_{ij}, \quad \{f, g\} = -i \cdot \sum_{k=1}^{n} \left(\frac{\partial f}{\partial z_k} \cdot \frac{\partial g}{\partial z_k^x} - \frac{\partial f}{\partial z_k^x} \cdot \frac{\partial g}{\partial z_k} \right). \tag{A.11}$$

Writing the (2n) variables (A.1, (A.9) as column vectors $\begin{pmatrix} q_i \\ p_i \end{pmatrix}$ and $\begin{pmatrix} z_i \\ z_i^x \end{pmatrix}$, we can write (A.9) in a compact matrix form as

$$\begin{pmatrix} z_i \\ z_i^x \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} I & i \cdot I \\ I & -i \cdot I \end{pmatrix} \cdot \begin{pmatrix} q_i \\ p_i \end{pmatrix}, \tag{A.12}$$

I is the n dimensional unit matrix. Let us define for any function (A.10) the column vector $f_{z_i} = \left| \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial z_i^x} \right|^T$, then the real geometry (A.2)-(A.8) is mapped onto similar complex sympleotical expressions. The matrix (A.3) takes now the complex form J^c,

$$J^c = \begin{pmatrix} 0 & -i \cdot I \\ i \cdot I & 0 \end{pmatrix}. \tag{A.13}$$

The equation (A.11) may be written similar to (A.4) as matrix equation in the form

$$\{f, g\} = \sum_{i,j=1}^{2n} f_{z_i} J_{ij} \, g_{z_j}. \tag{A.14}$$

For any function $f(z, z^x)$ satisfying (A.10) the infinitesimal canonical transformations (II.6) for real variables can be written in the basis (A.9) as

$$\frac{dz_\alpha}{d\varepsilon} = -i \cdot \frac{\partial f}{\partial z_\alpha} \qquad \frac{dz_\alpha^x}{d\varepsilon} = i \cdot \frac{\partial f}{\partial z_\alpha^x}. \tag{A.15}$$

Using equation (A.11) the functions $f(z, z^x)$ define complex vector fields

$$L_f(g) = \{g, f\}, \tag{A.16}$$

which are similar to those introduced by equations (II.7) and (II.11). Using eq. (A.16), we obtain for the generating monomials (III.1), (III.2) of the Lie-algebra of the symplectic group Sp(2n, R) the expressions

$$L_{-z_i\cdot z_j} = -i\cdot\left(z_j\cdot\frac{\partial}{\partial z_i^*} + z_i\cdot\frac{\partial}{\partial z_j^*}\right),$$

$$L_{z_i^*\cdot z_j^*} = -i\cdot\left(z_i^*\cdot\frac{\partial}{\partial z_j} + z_j^*\cdot\frac{\partial}{\partial z_i}\right), \qquad \text{(A.17)}$$

$$L_{z_i\cdot z_j^*} = -i\cdot\left(z_i\cdot\frac{\partial}{\partial z_j} - z_j^*\cdot\frac{\partial}{\partial z_i^*}\right).$$

The generators (A.17) act as matrices (with numbers as entires) in each subspace $\mathfrak{a}^{\ell}_{H}$ of $\mathfrak{a}_{H}$. For completeness we introduce the vector fields $L_{z_k^*} = -i\cdot\frac{\partial}{\partial z_k}$ and $L_{z_k} = i\cdot\frac{\partial}{\partial z_k^*}$.

The generating monomials of special conformal transformations (III.6b) give rise to vector fields with higher order coefficient functions. For instance the vector fields $L_{z_\ell\cdot E_0^\circ}$ have the form

$$L_{z_\ell\cdot E_0^\circ} = -i\cdot\sum_k z_\ell\cdot z_k\cdot\frac{\partial}{\partial z_k} + i\cdot\sum_k\left(\delta_{\ell k}\cdot E_0^\circ + z_\ell\cdot z_k^*\right)\cdot\frac{\partial}{\partial z_k^*}. \qquad \text{(A.18)}$$

Appendix B

Using eq. (II.17) the commutation relations between the generators of the two Lie-algebras (III.2) and (III.6) may evaluated directly. The result is

$$\{z_k^*, z_\ell\cdot E_0^\circ\} = i\cdot\left(\delta_{\ell k}\cdot E_0^\circ + z_\ell\cdot z_k^*\right)$$
$$\{z_\ell^*, E_i^{\,j}\} = i\cdot\delta_{i\ell}\cdot z_j^* \qquad \text{(B.1)}$$
$$\{z_\ell^*, E_0^\circ\} = i\cdot z_\ell^*$$
$$\{z_k\cdot E_0^\circ, E_i^{\,j}\} = -i\cdot\delta_{jk}\cdot z_i\cdot E_0^\circ$$
$$\{z_k\cdot E_0^\circ, E_0^\circ\} = -i\cdot z_k\cdot E_0^\circ .$$

Using the notation

$$z_i\cdot z_j^* = A_i^{\,j}, \quad z_i^* = A_{n+1}^{\;\;i}, \qquad \text{(B.2)}$$
$$z_i\cdot E_0^\circ = A_i^{\,n+1}, \quad E_0^\circ = A_{n+1}^{\;\;n+1},$$

the equations (B.1) may be written in a compact form as follows

$$\{A_i^{\,j}, A_\ell^{\,m}\} = i\cdot\left(g_\ell^{\,j}\cdot A_i^{\,m} - g_i^{\,m}\cdot A_\ell^{\,j}\right), \qquad \text{(B.3)}$$

with $g_\ell^{\,j} = \delta_\ell^{\,j}$, $g_{n+1}^{\;\;n+1} = -1$.

The generating monomials(III.6) of the group c.c.g. are close to a Lie-algebra, which is locally isomorphic to the Lie-algebra of the group SU(n, 1) (B.3).

The equations (B.1), (B.2) are not the usual realization of the Lie-algebra SU(n,1), known in the context of dynamical algebras for the harmonic oscillator [8]. The generators of translations and of special conformal transformations are not represented by Hermitean-conjugated operators. In the well-known approaches [8] these operators take the form $z_i\cdot(E_0^\circ)^{\frac{1}{2}}$ and $z_i^*\cdot(E_0^\circ)^{\frac{1}{2}}$, respectively, and are of second degree in their arguments [8]. The generators (III.7) correspond to special conformal transformations (III.4) in Cn. Following (III.6), (III.7) any generator of continuous conformal transformations $z'_\mu = f_\mu(z_\nu)$ may be extended to canonical transformations defined on $\mathfrak{a}_H$. This subalgebra of k(2n,R) is infinite-dimensional. The generating elements have the form h(z), $E_0^\circ\cdot f(z)$, $z_\mu^*\cdot g(z)$, where $f(z)$, $g(z)$, $h(z)$ are functions defined on Cn.

The antianalytic transformations $z^{*\prime}_\mu = f_\mu(z_\nu^*)$ may be extended to canonical transformations, which are generated by expressions $\tilde{h}(z^*)$, $E_0^\circ\cdot\tilde{f}(z^*)$, $z_\mu\cdot\tilde{g}(z^*)$ similar to those above.

Let us add two remarks. The generating function E_0° in correlated to a vectorfield , which is the generator of dilatation transformations in the phase space. The commutators of $z_\mu\cdot E_0^\circ$ and $z_\mu^*\cdot E_0^\circ$ contain elements of fourth degree, therefore they are not elements of the same finite-dimensional Lie-algebra. The generating monomials (II.2) fulfill the commutation relations of the Lie-algebra for the group Sp(2n,R). Additional to (III.3) we get the Poisson bracket relations [7]

$$\{E^{k\ell}, E_{mn}\} = i\cdot\left(\delta_n^{\,k}\cdot E_m^{\,\ell} + \delta_n^{\,\ell}\cdot E_m^{\,k} + \delta_m^{\,\ell}\cdot E_n^{\,k} + \delta_m^{\,k}\cdot E_n^{\,\ell}\right), \qquad \text{(B.4)}$$
$$\{E_0^\circ, E^{k\ell}\} = -2\cdot i\cdot E^{k\ell}, \quad \{E_0^\circ, E_{k\ell}\} = 2\cdot i\cdot E_{k\ell},$$
$$\{E_i^{\,j}, E_0^\circ\} = 0,$$

$$\{E_i{}^j, E^{k\ell}\} = -i\cdot(\delta_i{}^k \cdot E^{j\ell} + \delta_i{}^\ell \cdot E^{jk}),$$
$$\{E_i{}^j, E_{k\ell}\} = i\cdot(\delta_\ell{}^j \cdot E_{ik} + \delta_k{}^j \cdot E_{ik}).$$

Consider the transformation behaviour of the generators (B.1), (B.4) for both Lie-algebras according to the action of the common subgroup U(n). Both sets of generators decompose obviously into a direct sum consisting of three parts, namely the algebra $u(n)$ itself and two Abelian subalgebras. The Abelian subalgebras are different for the generators (B.1) or (B.4), respectively.

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Алгебра классической гамильтоновой механики как замыкание двух конечномерных алгебр

Рассмотрена структура канонической алгебры классической механической системы со степенями свободы. Элементы этой бесконечномерной алгебры являются генераторами непрерывных автоморфизмов в алгебре Гейзенберга этой системы. Базис канонической алгебры выражается с помощью скобки Пуассона через производящие мономиалы. Рассмотрены элементы двух конечномерных подалгебр Ли и указаны их действия как генераторов канонических преобразований. Показано, что возможно представить каждый генератор бесконечномерной канонической алгебры в виде линейной комбинации вторичных генераторов из этих двух конечномерных подалгебр. В частности выяснена связь между точечными каноническими преобразованиями и конформными преобразованиями комплексного пространства и указаны некоторые подобия теореме Огиевецкого по структуре алгебр общековариантной группы преобразований координат.

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The Algebra of the Classical Hamiltonian Mechanics as the Closure of Two Finite-Dimensional Algebras

The canonical algebra of phase space transformations for a classical mechanical system is an infinite-dimensional one. It is shown, that any generator of this algebra can be expressed as linear combination of repeated commutators of generators of two finite-dimensional subgroups - the linear symplectic group and those of some conformal (point) transformation group. The result is related to the Ogievetsky-theorem concerning the structure of the algebra for the general coordinate transformation group in the theory of relativity.

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