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HARMONIC SUPERSPACES OF EXTENDED SUPERSYMMETRY.
The Calculus of Harmonic Variables

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1. Introduction:

Recently, the concept of harmonic superspace was proposed for obtaining an unconstrained formulation of N=2 matter, super Yang--Mills (SYM) and supergravity theories /1/. The same approach proved to be suitable for constructing an off-shell N=3 SYM theory /2/, thus circumventing the famous N=3 barrier /3/. The main idea consists of enlarging the ordinary superspace by some new even coordinates u^A_i that form a basis set of harmonic functions on some coset manifold G/H, G being the group of automorphism of the supersymmetry (SUSY) algebra (i.e., SU(2) in N=2 and SU(3) in N=3 cases etc.), H being one of its subgroups. Then it is possible to extract from this enlarged superspace a subspace, called analytic /1,2/ relevant for constructing unconstrained SUSY theories. The fundamental superfield objects of those theories appear very elegantly as analytic functions defined on this subspace.

The purpose of the present paper is to give a kind of glossary of harmonic calculus for the simplest groups and their cosets G/H. Only a bit of the examples considered here have been already used in constructing extended SUSY theories, the relevance of the remaining ones may be revealed later. There is a remarkable intimate connection between the geometric structure of a SUSY theory and the choice of the homogeneous space G/H used to define the harmonics u^A_i. We intend to list all harmonic super-spaces of interest and their analytic subspaces in another paper based on the matter given here.

The paper is organized as follows. Section 2 describes the general techniques of constructing harmonics on some coset space G/H. For the pedagogical reasons we illustrate these techniques by the familiar SU(2)/U(1) example extensively used in N=2 SUSY /1/. All other cases are treated analogously to this simplest one. Sect.2 treats cosets associated with G=SU(3) (both for H=SU(2)xU(1) and U(1)xU(1)), in Sect.3 we consider cosets of G=SU(4) for H=SU(3)xU(1).

SU(2)xSU(2)xU(1), SU(2)xU(1)xU(1) and U(1)xU(1)xU(1). Sect.4 is devoted to the case of G=USp(2) with H= SU(2)xSU(2), SU(2)xU(1) and U(1)xU(1). Some brief concluding remarks are given in Sect.5.

2. General techniques and SU(2)/U(1) example

We begin by introducing the set of harmonics u_i^A defined on the manifold G/H, H⊂G; H,G being compact groups. We take the matrix representation of G and H, and so we can define $u_i^j \in G/H$ to be, say, NxN matrix. For instance, if G=SU(2) and H=U(1) is its diagonal subgroup, then:

$$u_i^{\,j} = [\exp i(\varphi T^+ + \bar{\varphi} T^-)]_i^{\,j}, \tag{2.1}$$

where φ is complex variable, $T^+=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $T^-=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ are SU(2) generators, i,j=1,2.

The action of an element g_i^k of G on the coset element is defined as [4]

$$u_i^j \to u_i'^{\,j} = g_i^k u_k^\ell h_\ell^j(g,u), \tag{2.2}$$

where h ∈ H is a "compensating" right H transformation. Next, let us introduce a set of basis vectors q_i^A in the G-representation space (normally, the fundamental representation is considered) such that for any h ∈ H

$$q_i^A \to q_i'^A = h_i^\ell q_\ell^A \equiv q_i^B h_B^A \tag{2.3}$$

and h_B^A has a block-diagonal form. Here indices i,j refer to the fundamental irrep. of G and A,B,... run through all irreps of H which are contained in the fundamental irrep. of G. Thus h_B^A is the matrix in this reducible H representation.

Now, the "harmonics" are defined as:

$$u_i^A = u_i^j q_j^A$$
$$G:\ u_i^A \to u_i^{A\prime} = g_i^k u_k^B h_B^A \tag{2.4}$$

and they belong to the representation space of Gx representation space of H. So, they transform under GxH, where G acts from the left and H from the right.

According to (2.2) and (2.3) the right H transformation is not independent, it is completely fixed by the left G ones. However, we can in fact introduce u_i^A as "free" objects in the GxH representation space, i.e,as a kind of vielbeins converting G reps into H reps. The group G is originally realized on them by left multiplications (without compensating H-transformations). At the same time,

u_i^A are transformed from the right by new independent gauge group H_τ whose parameters are arbitrary functions of u_i^A themselves. If we then fix the H_τ gauge so as to reduce the number of independent parameters in u_i^A (equal originally to dim G) to dim G/H, we recover the standard coset formulation (2.4). Thus, requiring invariance under right gauge H_τ transformations one may adhere to the "vielbein" interpretation of u_i^A which is convenient in a number of aspects.

For the SU(2)/U(1) case the q_i^A vectors are evidently $q_i^+=\begin{pmatrix}1\\0\end{pmatrix}$, $q_i^-=\begin{pmatrix}0\\1\end{pmatrix}$, (+ and - label the U(1) irreps) that, according to (2.3) and (2.4), gives us the harmonics from [1], namely

$$u_i^+ = u_i^1, \quad u_i^- = u_i^2, \qquad u_i^j = \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix}. \tag{2.5}$$

Let us point out that, since u_i^j is a G matrix, this implies some relations between u_i^A, e.g., in the SU(2)/U(1) case

$$u^+u = uu^+ = 1_2 \qquad (a)$$

whence

$$u^{+i}u_i^- = 1 \Leftrightarrow u_i^+u_j^- - u_j^+u_i^- = \varepsilon_{ij} \quad (b) \tag{2.6}$$
$$u^{i+}u_i^+ = u^{i-}u_i^- = 0 \qquad (c)$$
$$u_i^- = \overline{(u^{+i})}.$$

Here (2.6) represents the unitarity and unimodularity properties of SU(2) matrices. We shall see that such constraints are prototypes of those for more complicated cases listed below. Moreover, (2.6) gives us the possibility to convert, as have been stated above for the general case, SU(2) indices into U(1) ones and vice versa, namely:

$$\psi_i = u_i^+\psi^- - u_i^-\psi^+$$
$$\psi^\pm = u_j^\pm \psi^j, \quad \psi^j = \varepsilon^{ji}\psi_i \tag{2.6'}$$
$$u^{j+} = \varepsilon^{ji}u_i^+, \text{ etc.}$$

Finally we note that with the help of harmonics u_i^A one can expand a function defined on G/H and belonging in external indices to an H representation, in powers of u. Namely

$$F^{(A\dots B)}_{(G/H)} = \sum u^A_i u_j \cdots u^B_\ell f^{ij\dots\ell}, \tag{2.7}$$

where $f^{ij\dots\ell}$ are G irrep. coefficients independent of u and the summation is over all (usually infinitely many) monomials in u^A_i belonging to the same H-representation as F(these are nothing but higher harmonics on G/H).For example, in the SU(2)/U(1) case one has

$$F^{(n+)}_{(SU(2)/U(1))} = \sum_{k=0}^{\infty} u^+_{i_1} \dots u^+_{i_{k+n}} u^-_{j_1} \dots u^-_{j_k} f^{(i_1 \dots j_k)}. \tag{2.8}$$

For such functions (they are exactly the reps. of G induced from H irreps) one may define covariant differentiation with respect to the coset parameters using the standard technique of Cartan's forms (see, e.g.,/4 /). We would not present a general formula but just illustrate this again by the example of SU(2)/U(1). The covariant derivatives in -- and ++ directions of SU(2)/U(1) have very simple form in terms of harmonics $u^\pm_i$:

$$D^{++} = u^+_i \frac{\partial}{\partial u^-_i}, \quad D^{--} = u^-_i \frac{\partial}{\partial u^+_i}. \tag{2.9}$$

Together with the operator

$$D^3 = \frac{1}{2}\left(u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}\right)$$

(which is just the generator of right U(1) transformations and is equal to overall U(1) charge when applied to any function of the type (2.8)) they constitute an SU(2) algebra:

$$[D^{++}, D^{--}] = 2D^3, \quad [D^{\pm\pm}, D^3] = \mp D^{\pm\pm}.$$

The last property can be understood from the fact that D^{++}, D^{--}, D^3 can be alternatively defined as generators of right SU(2) transformations of the coset SU(2)/U(1) (which are realized on indices +,- of harmonics). One more remark concerning that case is in order. Besides the usual complex conjugation (-)

$$u^{\pm i} \xrightarrow{(-)} \overline{u^{\pm i}} = \pm u^\mp_i \tag{2.10}$$

one can define another involution (*)

$$u^\pm_i \xrightarrow{(*)} (u^\pm_i)^* = \pm u^\mp_i \tag{2.11}$$

allowing, together with (2.10) to define self-conjugated charged objects, say

$$F^{(n+)} = \overline{\left(F^{(n+)}\right)^*} \qquad (n = 2k). \tag{2.12}$$

The geometric meaning of * is very simple: it takes any point of the sphere SU(2)/U(1) to the opposite one, i.e.,is the antipodal mapping of this sphere. We shall see that such an operation is not always possible and respectively the reality in the sense of(2.12) can be defined only for certain G/H. This places a strong restriction on the choice of subgroup H.

3. <u>The harmonics for G=SU(3)</u>

Now we are ready to collect useful formulas for harmonics of G/H, G=SU(3), H=SU(2)xU(1) and H=U(1)xU(1)

Table 1.

G = SU(3)	H=SU(2)xU(1)	H=U(1)xU(1)
Generators of H in the 3x3 matrix form	$SU(2)$; $T_i = \begin{pmatrix} \tau_i & 0 \\ 0 & 0 \end{pmatrix}$ where τ_i are 2x2 SU(2) generators $U(1)$; $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$U_1(1)$; $T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $U_2(1)$; $T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
q^A_i	SU(2) doublet $q^{+a}_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ SU(2) singlet $q^-_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$q^{(1,1)}_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $q^{(0,-2)}_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $q^{(-1,1)}_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in what follows, a couple of indices(a,b) represents T_1 and T_2 charges,respectively
Harmonics	u^{+a}_i, u^{--}_i and their conjugates $u^{i,-a}$, u^{i++}	$u^{(1,1)}_i$, $u^{(-1,1)}_i$, $u^{(0,-2)}_i$ and their conjugates $u^{i(-1,-1)}$, $u^{i(1,-1)}$, $u^{i(0,2)}$

Unitarity $u^+u = uu^+ = 1$	$u^+_{ia} u^{-i}_b = \varepsilon_{ab}$ $u^{--}_i u^{i++} = 1$ $u^+_{ia} u^{i++} = u^{i-}_b u^{--}_i = 0$ $-\varepsilon^{ab} u^+_{ai} u^{j-}_b + u^{--}_i u^{j++} = \delta_i^j$	$u_i^{(1,1)} u^{i(-1,-1)} = 1$ $u_i^{(-1,1)} u^{i(1,-1)} = 1$ $u_i^{(0,-2)} u^{i(0,2)} = 0$ $u_i^{(1,1)} u^{j(-1,-1)} + u_i^{(-1,1)} u^{j(1,-1)} + u_i^{(0,-2)} u^{j(0,2)} = \delta_i^j$
Unimodularity condition $\det\lVert u \rVert = 1$	$\varepsilon^{ab} u^+_{ai} u^+_{bj} = \varepsilon_{ijk} u^{k++}$ $\varepsilon^{ijk} u^+_{ai} u^+_{bj} u^{--}_k = \varepsilon_{ab}$ and their conjugates	$u_i^{(0,-2)} = \varepsilon_{ijk} u^{(-1,-1)j} u^{(1,-1)k}$ and their conjugates
Harmonic derivatives preserving unitarity and unimodularity	$D^{3+}_a = u^{++i}\frac{\partial}{\partial u^{-ai}} + u^+_{ai}\frac{\partial}{\partial u^{--}_i}$ $D^{3-}_b = u^{--}_i\frac{\partial}{\partial u^{+b}_i} + u^{-i}_b\frac{\partial}{\partial u^{++i}}$	$D^{(1,3)} = -u^{(0,2)i}\frac{\partial}{\partial u^{(-1,-1)i}} + u_i^{(1,1)}\frac{\partial}{\partial u_i^{(0,-2)}}$ $D^{(-1,3)} = -u^{(0,2)i}\frac{\partial}{\partial u^{(1,-1)i}} + u_i^{(-1,1)}\frac{\partial}{\partial u_i^{(0,-2)}}$ $D^{(2,0)} = -u^{(1,1)i}\frac{\partial}{\partial u^{(-1,-1)i}} + u_i^{(1,1)}\frac{\partial}{\partial u_i^{(-1,1)}}$ and their conjugates $D^{(1,-3)} = \overline{D}^{(-1,3)}$ $D^{(-1,-3)} = \overline{D}^{(1,3)}$ $D^{(-2,0)} = \overline{D}^{(2,0)}$
Converting indices	$\psi_i = -\varepsilon_{ab} u^{+a}_i \psi^{-b} + u^{--}_i \psi^{++}$ $\psi^{-b} = u^{-bj}\psi_j$ $\psi^{++} = u^{++j}\psi_j$	$\psi_i = u_i^{(1,1)}\psi^{(-1,-1)} + u_i^{(-1,1)}\psi^{(1,-1)} + u_i^{(0,-2)}\psi^{(0,2)}$ $\psi^{(\pm1,-1)} = u^{(\pm1,-1)j}\psi_j$ $\psi^{(0,2)} = u^{(0,2)j}\psi_j$

Note that in the case of H=SU(2)xU(1) there is no proper involution between u_i^A harmonics allowing us to define an appropriate notion of reality for the harmonic decomposition.

On the other hand, in the case of H=U(1)xU(1) along with the usual complex conjungation.

$$u_i^{(1,1)} \longleftrightarrow u^{(-1,-1)i}, \qquad \text{etc.}$$

there exists another involution generalizing the operation (2.11)

$$u_i^{(1,1)} \overset{*}{\longleftrightarrow} u_i^{(0,-2)}$$
$$u^{(1,-1)i} \overset{*}{\longleftrightarrow} -u^{(-1,1)i}$$
$$u^{(-1,-1)i} \overset{*}{\longleftrightarrow} u^{(0,2)i}.$$

Together with the usual conjugation the latter involution provides a tool for defining reality in the N=3 SYM theory /2/.

4. <u>Harmonics for G=SU(4)</u>

Now we briefly list all the results in our analysis with G=SU(4) for H=SU(3)xU(1), H=SU(2)xSU(2)xU(1), H=SU(2)xU(1)xU(1), and H=U(1)xU(1)xU(1) (see Table 2)

Note that for the case H=SU(2)xSU(2)xU(1) the involution (*) is defined as

$$u_{ia}^{(+)} \overset{*}{\longleftrightarrow} u_{ip}^{(-)}$$

which together with the usual conjugation (‾)

$$\overline{u}_{ia}^{(+)} = u^{ia(-)}$$
$$\overline{u}_{ip}^{(-)} = u^{ip(+)}$$

allows us to define reality in N=4 SUSY theory

5. <u>Harmonics for G=USp (2)</u>

Before establishing the harmonic analysis for G=USp(2) let us briefly recall the definition and introduce the matrix representation of USp(2)

The Lie algebra corresponding to USp(2) is formed by 2x2 quaternionic matrices

$$\begin{pmatrix} a & c \\ -\bar{c} & b \end{pmatrix} \in USp(2) \tag{5.1}$$

a,b are pure imaginary quaternions ($\bar{a} = -a$, $\bar{b} = -b$), c is an arbitrary quaternion. Representing a,b as SU(2) matrices, we obtain the following 4x4 matrix representation

$$\begin{pmatrix} SU_1(2) & c \\ -\bar{c} & SU_2(2) \end{pmatrix} \in USp(2), \tag{5.2}$$

c is a general GL(2,R) matrix. Matrices belonging to USp(2) group are unitary and have unit determinant as can be checked easily from (5.1) and (5.2). USp(2) is a subgroup of SU(4). The USp(2) group is also known to possess an invariant antisymmetric tensor of second rank Ω_{ij} (symplectic metrics), that is

$$\Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{5.3}$$

$$\Omega^{ij} \equiv \Omega^{T} = -\Omega = \Omega^{-1}.$$

We shall list below the harmonic analysis on USp(2) cosets with $H=SU_1(2)\times SU_2(2)$; $H=SU_1(2)\times U_2(1)$; $H=U_1(1)\times U_2(1)$, U(1) being the diagonal subgroups of $SU_I(2)$, $I=1,2$. (Table 3).

6. Conclusion

So far we had constructed objects that can connect the representation space of some group with the representation space of some of its subgroups. This situation will be especially useful in N-extended SYM theories. In those theories, by adding the set of harmonics, we come to the concept of harmonic superspace. It turns out that in the harmonic superspace there is a hypersurface spanned by the so-called analytic basis. All the fundamental objects (e.g. superfields) appear naturally in this basis. Roughly speaking, the appearance of analytic basis is connected with the existence of Cauchy-Riemann's structure/6/. With the help of harmonics SU(N) indices of the spinor covariant derivatives fall into the H ⊂ SU(N) ones. After the Yang-Mills covariantization of those spinor derivatives by adding to them connections, we can pick out a subset fullfilling the "flat" algebra. This condition is equivalent to the integrability condition of the Cauchy-Riemann's structure. Its presence crucially depends on the choice of H. For the N=3 case, e.g., H=SU(2)xU(1) does not allow the existence of CR-structure as H=U(1)xU(1) does. In the case N=4, C R-structure does not exist with H=SU(3)xU(1), SU(2)xSU(2)xU(1), SU(2)xU(1)xU(1), while it can be constructed for H=U(1)x U(1)xU(1).

Table 2.

G= SU(4)	H=SU(3)xU(1)	$H=SU_1(2)\times SU_2(2)\times U(1)$	$H=SU(2)\times U_1(1)\times U_2(1)$	$H=U_1(1)\times U_2(1)\times U_3(1)$
Generators of H in the 4x4 matrix form	$SU(3);\ T_i=\begin{pmatrix}\lambda_i & 0\\ 0 & 0\end{pmatrix}$ where λ_i are 3x3 SU(3) generators $U(1);\ T=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{pmatrix}$	$SU_1(2);\ T_a=\begin{pmatrix}\tau_a & 0\\ 0 & 0\end{pmatrix}$ $SU_2(2);\ T_p=\begin{pmatrix}0 & 0\\ 0 & \tau_p\end{pmatrix}$ $U(1);\ T=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$ τ_a, τ_p are 2x2 $SU_1(2)$ and $SU_2(2)$ generators	$SU(2);\ T_a=\begin{pmatrix}\tau_a & 0\\ 0 & 0\end{pmatrix}$ $U_1(1);\ T_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$ $U_2(1);\ T_2=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$	$U_1(1);\ T_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$ $U_2(1);\ T_2=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$ $U_3(1);\ T_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$
q_i^A	SU(3) triplet $q_i^{+a}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ SU(3) singlet $q_i^{(3-)}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$SU_1(2)$ doublet $q_i^{+a}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $SU_2(2)$ doublet $q_i^{-p}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	SU(2) doublet $q_{ia}^{(0,1)}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ SU(2) singlets $q_i^{(1,-1)}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $q_i^{(-1,-1)}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$q_i^{(1,0,1)}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $q_i^{(-1,0,1)}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $q_i^{(0,1,-1)}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $q_i^{(0,-1,-1)}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$
Harmonics	$u_i^{a(+)}, u_i^{(3-)}$ and their conjugates $u_a^{i(-)}, u^{i(3+)}$	$u_i^{a(+)}, u_i^{p(-)}$ and their conjugates $u_a^{i(-)}, u_p^{i(+)}$	$u_{ia}^{(0,1)}, u_i^{(1,-1)}, u_i^{(-1,-1)}$ and their conjugates $u^{ia(0,-1)}, u^{i(-1,1)}, u^{i(1,1)}$	$u_i^{(1,0,1)}, u_i^{(-1,0,1)}, u_i^{(0,1,-1)}$ $u_i^{(0,-1,-1)}$ and their conjugates $u^{i(-1,0,-1)}, u^{i(1,0,-1)}, u^{i(0,-1,1)}$ $u^{i(0,1,1)}$

Table 2 (continued)

Unitarity $u^+u = uu^+ = 1$	$u_i^{a(+)} u_b^{i(-)} = \delta_a^b$ $u_i^{a(+)} u^{i(3-)} = u_i^{(3-)} u_a^{i(+)} = 0$ $u_i^{(3-)} u^{i(3+)} = 1$ $u_i^{(+)a} u_a^{(-)j} + u_i^{(3-)} u^{(3+)j} = \delta_i^j$	$u_{ia}^{(+)} u_b^{i(-)} = \varepsilon_{ab}$ $u_{ip}^{(-)} u_q^{(+)i} = \varepsilon_{pq}$ $u_{ia}^{+} u^{ja(-)} + u_{iq}^{(-)} u^{jq(+)} = -\delta_i^j$	$u_{ia}^{(0,1)} u_b^{i(0,-1)} = \varepsilon_{ab}$ $u_i^{(1,-1)} u^{i(-1,1)} = 1$ $u_i^{(-1,-1)} u^{i(1,1)} = 1$ all other SU(4) contractions vanish $-u_{ia}^{(0,1)} u^{ja(0,-1)} + u_i^{(1,-1)} \cdot$ $\cdot u^{j(-1,1)} + u_i^{(-1,-1)} u^{j(1,1)} = \delta_i^j$	$u_i^{(1,0,1)} u^{i(-1,0,-1)} = 1$ $u_i^{(-1,0,1)} u^{i(1,0,-1)} = 1$ $u_i^{(0,1,-1)} u^{i(0,-1,1)} = 1$ $u_i^{(0,-1,-1)} u^{i(0,1,1)} = 1$ all other SU(4) contractions vanish $u_i^{(1,0,1)} u^{j(-1,0,-1)} +$ $+ u_i^{(1,0,-1)} u^{j(-1,0,1)} +$ $+ u_i^{(0,-1,-1)} u^{j(0,1,1)} +$ $+ u_i^{(0,1,-1)} u^{j(0,-1,1)} = \delta_i^j$
Unimodularity conditions det $\Vert u \Vert = 1$	$\varepsilon^{ijkl} u_i^{a(+)} u_j^{b(+)} u_k^{(3-)} \cdot$ $\cdot u_l^{c(+)} = -\varepsilon^{abc}$	$\varepsilon^{ab} \varepsilon^{ijkl} u_{ia}^{+} u_{jb}^{+} =$ $= u_p^{+k} u_q^{+l} \varepsilon^{pq}$	$\varepsilon^{ab} \varepsilon^{ijkl} u_{ia}^{(0,1)} u_{jb}^{(0,1)} u_k^{(1,-1)} \cdot$ $\cdot u_l^{(-1,-1)} = 1$ and its conjugate	$u_i^{(1,0,1)} u_j^{(-1,0,1)} u_k^{(0,1,-1)} u_l^{(0,-1,-1)} \cdot$ $\cdot \varepsilon^{ijkl} = 1$ and its conjugate
Harmonic derivatives preserving unitarity and unimodularity	$D^{(4+)a} = u^{(3+)i} \frac{\partial}{\partial u_a^{(-)i}} +$ $+ u_i^{(+)a} \frac{\partial}{\partial u_i^{(3-)}}$ $D_a^{(4-)} = u_i^{(3-)} \frac{\partial}{\partial u_i^{(+)a}} +$ $+ u_a^{(-)i} \frac{\partial}{\partial u^{(3+)i}}$	$D_{ap}^{(2+)} = u_{ai}^{(+)} \frac{\partial}{\partial u_i^{p(-)}}$ $- u_p^{(+)i} \frac{\partial}{\partial u^{ai(-)}}$ $D_{ap}^{(2-)} = u_a^{i(-)} \frac{\partial}{\partial u^{(+)pi}} -$ $- u_{pi}^{(-)} \frac{\partial}{\partial u_i^{a(-)}}$	$D_a^{(1,2)} = u_{ai}^{(0,1)} \frac{\partial}{\partial u_i^{(-1,-1)}} -$ $- u^{(1,1)i} \frac{\partial}{\partial u^{ai(0,-1)}}$ $D_a^{(1,-2)} = u_a^{i(0,-1)} \frac{\partial}{\partial u^{(-1,1)i}} -$ $- u_i^{(1,-1)} \frac{\partial}{\partial u_i^{a(0,1)}}$	$D^{(1,1,2)} = u_i^{(1,0,1)} \frac{\partial}{\partial u_i^{(0,-1,-1)}} - u^{(0,1,1)i} \frac{\partial}{\partial u^{i(-1,0,-1)}}$ $D^{(-1,1,2)} = u_i^{(-1,0,1)} \frac{\partial}{\partial u_i^{(0,-1,-1)}} - u^{(0,1,1)i} \frac{\partial}{\partial u^{i(1,0,-1)}}$ $D^{(1,-1,2)} = u_i^{(1,0,1)} \frac{\partial}{\partial u_i^{(0,1,-1)}} - u^{(0,-1,1)i} \frac{\partial}{\partial u^{i(-1,0,1)}}$ $D^{(-1,-1,2)} = u_i^{(-1,0,1)} \frac{\partial}{\partial u_i^{(0,1,-1)}} - u^{i(0,-1,1)} \frac{\partial}{\partial u^{i(1,0,-1)}}$

Table 2 (continued)

			$D_a^{(-1,2)} = u_{ai}^{(0,1)} \frac{\partial}{\partial u_i^{(1,-1)}} -$ $- u^{(-1,1)i} \frac{\partial}{\partial u^{ai(0,-1)}}$ $D_a^{(-1,-2)} = \overline{D}_a^{(1,2)}$ $D_a^{(-1,2)} = \overline{D}_a^{(1,-2)}$ $D^{(2,0)} = u_i^{(1,-1)} \frac{\partial}{\partial u_i^{(-1,-1)}} -$ $- u^{i(1,1)} \frac{\partial}{\partial u^{(-1,1)i}}$ $D^{(-2,0)} = \overline{D}^{(2,0)}$	$D^{(0,2,0)} = u_i^{(0,1,-1)} \frac{\partial}{\partial u_i^{(0,-1,-1)}} - u^{i(0,1,1)} \frac{\partial}{\partial u^{i(0,-1,1)}}$ $D^{(2,0,0)} = -u_i^{(1,0,1)} \frac{\partial}{\partial u_i^{(-1,0,1)}} +$ $+ u^{i(1,0,-1)} \frac{\partial}{\partial u^{i(-1,0,-1)}}$
Converting indices	$\psi_i = u_i^{+a} \psi_a^- - u_i^{(3-)} \psi^{(3+)}$ $\psi_a^- = \psi_i u_a^{i-}$ $\psi^{(3+)} = \psi_i u^{i(3+)}$	$\psi_i = u_i^{+a} \psi_a^- - u_i^{-p} \psi_p^+$ $\psi_a^- = \psi_i u_a^{-i}$ $\psi_p^+ = \psi_i u_p^{+i}$	$\psi_i = -u_{ia}^{(0,1)} \psi^{a(0,-1)} + u_i^{(1,-1)} \psi^{(-1,1)} +$ $+ u_i^{(-1,-1)} \psi^{(1,1)}$ $\psi^{a(0,-1)} = u^{ia(0,-1)} \psi_i$ $\psi^{(-1,1)} = u^{(-1,1)i} \psi_i$ $\psi^{(1,1)} = u^{(1,1)i} \psi_i$	$\psi_i = u_i^{(1,0,1)} \psi^{(-1,0,-1)} +$ $+ u_i^{(1,0,-1)} \psi^{(-1,0,1)} +$ $+ u_i^{(0,-1,-1)} \psi^{(0,1,1)}$ $\psi^{(\alpha\beta\gamma)} = \psi_i u^{i(\alpha\beta\gamma)}$
Reality and involution(*) (if exists)	No	Yes $u_{ia}^{(+)} \overset{*}{\longleftrightarrow} u_{ip}^{(-)}$	No	Yes $u_i^{(1,0,1)} \overset{*}{\longleftrightarrow} u_i^{(0,-1,-1)}$ $u_i^{(-1,0,1)} \overset{*}{\longleftrightarrow} u_i^{(0,1,-1)}$ $u_i^{(0,1,-1)} \overset{*}{\longleftrightarrow} u_i^{(-1,0,1)}$ $u_i^{(0,-1,-1)} \overset{*}{\longleftrightarrow} u_i^{(1,0,1)}$

Table 3.

G=USp(2)	$H=SU_1(2)\times SU_2(2)$	$H=SU(2)\times U(1)$	$H=U_1(1)\times U_2(1)$
Generators of H in the 4x4 matrix form	$SU_1(2):\ T_a = \begin{pmatrix} \tau_a & 0 \\ 0 & 0 \end{pmatrix}$ $SU_2(2):\ T_p = \begin{pmatrix} 0 & 0 \\ 0 & \tau_p \end{pmatrix}$ where τ_a, τ_p are 2x2 $SU_1(2)$ and $SU_2(2)$ algebra's generators	$SU(2):\ T_a = \begin{pmatrix} \tau_a & 0 \\ 0 & 0 \end{pmatrix}$ $U(1)\quad T = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \end{pmatrix}$	$U_1(1):\ T = \begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ $U_2(1)\quad T' = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \end{pmatrix}$
q_i^A	$q_i^a = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}$ $q_i^p = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}$ a,b,c are doublet indices of $SU_1(2)$ p,q,r are doublet indices of $SU_2(2)$	$q_i^a = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}$ is SU(2) doublet $q_i^+ = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}$ SU(2) singlet $q_i^- = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}$ SU(2)singlet	$q_i^{(1,0)} = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}$ $q_i^{(-1,0)} = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}$ $q_i^{(0,1)} = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}$ $q_i^{(0,-1)} = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}$

Table 3 (continued)

Harmonics	u_i^a, u_i^p and their conjugates u^{ia}, u^{jp} satisfied $u^{ia} = \Omega^{ij} u_j^a$ $u^{jp} = \Omega^{ji} u_i^p$	u_i^a, $u_i^+\ u_i^-$ and their conjugates $u^i_a\ u^{-i}\ u^{+i}$ satisfied $\Omega_{ij} u^{-j} = u_i^-$ $\Omega_{ij} u^{+j} = u_i^+$ $\Omega_{ij} u^{aj} = u_i^a$	$u_i^{(1,0)}, u_i^{(-1,0)}, u_i^{(0,1)}, u_i^{(0,-1)}$ and their conjugates $u^{i(1,0)} = \Omega^{ij} u_j^{(-1,0)}$ $u^{i(0,1)} = \Omega^{ij} u_j^{(0,-1)}$ $u^{i(1,0)} = \Omega^{ij} u_j^{(1,0)}$ $u^{i(0,1)} = \Omega^{ij} u_j^{(0,1)}$
Unitarity $uu^+ = uu = 1$	$\Omega^{ij} u_{ip} u_{jq} = \varepsilon_{pq}$ $\Omega^{ij} u_{ia} u_{jb} = \varepsilon_{ab}$ $u_{ia} u_{jb} \varepsilon^{ab} + u_{ip} u_{jq} \varepsilon^{pq} =$ $= \Omega_{ij}$	$\Omega^{ij} u_{ai} u_{bj} = \varepsilon_{ab}$ $\Omega^{ij} u_i^+ u_j^- = 1$ $\Omega^{ij} u_i^+ u_j^+ = \Omega^{ij} u_i^- \cdot$ $\cdot u_j^- = 0$ $-\varepsilon^{ab} u_{ai} u_{bj} + u_i^+ u_j^- -$ $- u_j^+ u_i^- = -\Omega_{ij}$	$u_i^{(1,0)} u^{i(-1,0)} = u_i^{(0,1)} u^{i(0,-1)} =$ $= 1$ $u_i^{(1,0)} u^{i(0,1)} = u_i^{(1,0)} u^{i(0,-1)} = 0$ $u_i^{(1,0)} u_j^{(-1,0)} - u_j^{(1,0)} u_i^{(-1,0)} +$ $+ u_i^{(0,1)} u_j^{(0,-1)} - u_j^{(0,1)} u_i^{(0,-1)} = \Omega_{ij}$
Unimodularity det $\Vert u \Vert = 1$	$\varepsilon^{ijkl} u_{ia} u_{jb} u_{kp} u_{lq} \varepsilon^{ab} \varepsilon^{pq} = 1$	$\varepsilon^{ab} u_{ai} u_{bj} u_k^+ u_l^- \varepsilon^{ijkl} = 1$	$\varepsilon^{ijkl} u_i^{(1,0)} u_j^{(-1,0)} u_k^{(0,1)} u_l^{(0,-1)} = 1$
Harmonic derivatives preserving unitarity and unimodularity	$D_{pa} = u_{ia} \frac{\partial}{\partial u_i^p} - u_{ip} \frac{\partial}{\partial u_i^a}$	$D_a^+ = u_{ia} \frac{\partial}{\partial u_i^-} - u_i^+ \frac{\partial}{\partial u_i^a}$ $D_a^- = u_{ia} \frac{\partial}{\partial u_i^+} - u_i^- \frac{\partial}{\partial u_i^a}$	$D^{(2,0)} = u_i^{(1,0)} \frac{\partial}{\partial u_i^{(-1,0)}}$ $D^{(-2,0)} = u_i^{(-1,0)} \frac{\partial}{\partial u_i^{(1,0)}}$

Table 3 (continued)

		$D^{++} = u_i^{+} \frac{\partial}{\partial u_i^{-}}$ $D^{--} = u_i^{-} \frac{\partial}{\partial u_i^{+}}$	$D^{(0,2)} = u_i^{(0,1)} \frac{\partial}{\partial u_i^{(0,-1)}}$ $D^{(0,-2)} = u_i^{(0,-1)} \frac{\partial}{\partial u_i^{(0,1)}}$ $D^{(1,1)} = u_i^{(1,0)} \frac{\partial}{\partial u_i^{(0,-1)}} - u_i^{(0,1)} \frac{\partial}{\partial u_i^{(-1,0)}}$ $D^{(1,-1)} = u_i^{(1,0)} \frac{\partial}{\partial u_i^{(0,1)}} - u_i^{(0,-1)} \frac{\partial}{\partial u_i^{(-1,0)}}$ $D^{(-1,1)} = \overline{D}^{(1,-1)}$ $D^{(-1,-1)} = \overline{D}^{(1,1)}$
Converting indices	$\psi_i = -u_i^{a}\psi_a - u_i^{p}\psi_p$ $\psi_a = u_a^{i}\psi_i$ $\psi_p = u_p^{i}\psi_i$	$-\psi_i = -u_i^{a}\psi_a + u_i^{+}\psi^{-} - u_i^{-}\psi^{+}$ $\psi_a = u_a^{i}\psi_i$ $\psi^{\pm} = u^{\pm i}\psi_i$ $\psi^{i} = \Omega^{ij}\psi_j$	$\psi_i = u_i^{(1,0)}\psi^{(-1,0)} - u_i^{(-1,0)}\psi^{(1,0)}$ $+ u_i^{(0,1)}\psi^{(0,-1)} - u_i^{(0,-1)}\psi^{(0,1)}$ $\psi^{(\pm 1,0)} = -u^{(\pm 1,0)i}\psi_i$ $\psi^{(0,\pm 1)} = -u^{(0,\pm 1)i}\psi_i$
Involution besides the complex conju gation	$u_{ia} \xleftrightarrow{*} u_{ip}$	no	$u_i^{(\pm 1,0)} \xleftrightarrow{*} u_i^{(0,\pm 1)}$

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СООБЩЕНИЯ, КРАТКИЕ СООБЩЕНИЯ, ПРЕПРИНТЫ И СБОРНИКИ ТРУДОВ КОНФЕРЕНЦИЙ, ИЗДАВАЕМЫЕ ОБЪЕДИНЕННЫМ ИНСТИТУТОМ ЯДЕРНЫХ ИССЛЕДОВАНИЙ, ЯВЛЯЮТСЯ ОФИЦИАЛЬНЫМИ ПУБЛИКАЦИЯМИ.

Ссылки на СООБЩЕНИЯ и ПРЕПРИНТЫ ОИЯИ должны содержать следующие элементы:

- фамилии и инициалы авторов,
- сокращенное название Института /ОИЯИ/ и индекс публикации,
- место издания /Дубна/,
- год издания,
- номер страницы /при необходимости/.

Пример:

1. *Первушин В.Н. и др. ОИЯИ, Р2-84-649, Дубна, 1984.*

Ссылки на конкретную СТАТЬЮ, помещенную в сборнике, должны содержать:

- фамилии и инициалы авторов,
- заглавие сборника, перед которым приводятся сокращенные слова: "В кн."
- сокращенное название Института /ОИЯИ/ и индекс издания,
- место издания /Дубна/,
- год издания,
- номер страницы.

Пример:

Колпаков И.Ф. В кн. XI Международный симпозиум по ядерной электронике, ОИЯИ, Д13-84-53, Дубна, 1984, с.26.

Савин И.А., Смирнов Г.И. В сб."Краткие сообщения ОИЯИ", № 2-84, Дубна,1984,с.3.

Иванов Е. и др. E2-84-720
Гармонические суперпространства расширенной суперсимметрии. Исчисление гармонических переменных

Основной технический аппарат нового подхода к расширенным суперсимметриям, анализ гармонических переменных на однородных пространствах групп автоморфизмов соответствующих суперальгебр, подробно описан для случаев N = 2,3,4. Мы строим базисные гармоники на однородных пространствах G/H с G = SU(2), H = U(1); G = SU(3), H = SU(2)xU(1), H = U(1)xU(1); G = SU(4), H = SU(3)xU(1), H = SU(2)xSU(2)xU(1), H = SU(2)xU(1)xU(1), H = U(1)xU(1)xU(1); G = USp(2), H = SU(2)xSU(2), H = SU(2)xU(1), H = U(1)xU(1) и приводим ряд полезных соотношений между ними.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1984

Ivanov E. et al. E2-84-720
Harmonic Superspaces of Extended Supersymmetry. The Calculus of Harmonic Variables

The main technical apparatus of the harmonic superspace approach to extended SUSY, the calculus of harmonic variables on homogeneous spaces of the SUSY automorphism groups, is presented in detail for N = 2,3,4. We construct the basic harmonics for the coset manifolds G/H with G = SU(2), H = U(1); G = SU(3), H = SU(2)xU(1) and H = U(1)xU(1); G = SU(4), H = SU(3)xU(1), H = SU(2)xSU(2)xU(1), H = SU(2)xU(1)xU(1) and H = U(1)xU(1)xU(1); G = USp(2), H = SU(2)xSU(2), H = SU(2)xU(1) and H = U(1)xU(1) and tabulate a number of useful relations among them.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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