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REALIZATIONS OF THE REAL SIMPLE LIE ALGEBRAS: THE METHOD OF CONSTRUCTION

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1. Introduction

1.1 A realization (also canonical realization or boson representation) of a Lie algebra g denotes an expression of elements of g by means of polynomials in quantum canonical variables p_i, q_i, which preserves the commutation relation of g. Several types of the boson representation are used in physical applications (e.g., Holstein-Primakoff's type /1/, Dyson's type /2/ or Schwinger's type /3/). The amount of publications on this subject increases rapidly in the last years in connection with the introduction of the interacting boson model in nuclear physics/4-8/. In papers /9-11/ explicit forms of realizations are given for the symplectic algebras sp(2d,R). These realizations are physically interesting in connection with a microscopic model for the system of N nucleons in d (= 1,2 or 3) dimensions; for a particular survey of this model see Ref. 12 . Canonical realizations have many applications in the representation theory/13/.

1.2 In the papers /14-22/ extensive families of realizations for all complex classical Lie algebras and for most of their real forms were constructed. All these realizations have two interesting properties with respect to an application in the representation theory. They are Schurean (i.e. they have the Casimir operators realised by multiples of unity), and in the case of real forms, they are skew-Hermitean.

1.3 In this paper we present an algebraic method of constructing realizations for any real semisimple Lie algebra g. It is shown that any induced representation of g can be rewritten as the boson one. This fact is a starting point of our construction. The resulting realizations are Schurean and skew-Hermitean. The realizations of the algebra gl(3,R), which we calculate as an illustration of this method, are of the same type as the realization obtained in the paper [14]. It is likely that application of the presented method to classical simple Lie algebras will lead to realizations similar to those given explicitly in the mentioned papers. However, we expect our method to give other realizations, too, which are not constructed there. Some positive indication has been obtained already; the results will be presented in a forthcoming series of papers.

2. Preliminaries

Here we shall resume briefly some notions needed in the following.

2.1 Let g be a real or complex Lie algebra. By $\tilde{g}$ we denote its complexification; there holds $g = \tilde{g}$ if g is complex.

2.2 Let g and g_0 be Lie algebras, let further $U(\tilde{g})$ and $U(\tilde{g}_0)$ be the enveloping algebras of their complexifications, and finally, let W_{2n} be the complex Weyl algebra in n canonical pairs p_i, q_i, $i=1,2,\ldots,n$ which fulfil the usual canonical commutation relations

$$[p_i,p_j] = [q_i,q_j] = 0, \quad [p_i,q_j] = \delta_{ij}1.$$

A realization of a Lie algebra g is a homomorphism

$$\tau . g \rightarrow W_{2n} \otimes U(\tilde{g}_0).$$

For more details about the definition and properties of the canonical realizations see Ref. 22.

2.3 The homomorfism τ extends naturally to the homomorphic mapping (denoted by the same symbol τ of the enveloping algebra $U(\tilde{g})$ into $W_{2n} \otimes U(\tilde{g}_0)$). Let $Z(\tilde{g})$ be the centre of $U(\tilde{g})$. A realization τ is called Schurean or Schur-realization if all central elements $C \in Z(\tilde{g})$ are realized by $1 \otimes C_0$ where the C_0's are central elements of the enveloping algebra $U(\tilde{g}_0)$.

2.4 Recall that the involution on an associative algebra g is the mapping "+" : $g \rightarrow g$ obeing the relations

$$(\alpha X + \beta Y)^+ = \bar{\alpha} X^+ + \bar{\beta} Y^+$$
$$(XY)^+ = Y^+ X^+, \; (X^+)^+ = X.$$

Let g_0 be a real Lie algebra. An involution on W_{2n} together with an involution on the enveloping algebra $U(\tilde{g}_0)$ defines naturally an involution on $W_{2n} \otimes U(\tilde{g}_0)$ by

$$\left(\sum_j \alpha_j \Pi_j \otimes g_j\right)^+ \equiv \sum_j \bar{\alpha}_j \Pi_j^+ \otimes g_j^+ .$$

2.5 In what follows, we consider this involution determined by the involutions on W_{2n} and $U(\tilde{g}_0)$, respectively, which are generated by the following relations

$$(q_i)^+ = -q_i \tag{1a}$$

$$(p_i)^+ = p_i \tag{1b}$$

on the algebra W_{2n}, and

$$Y^+ = -Y, \; Y \in g_0 \tag{1c}$$

on the algebra $U(\tilde{g}_0)$.

Let g be a real Lie algebra and let "+" be an involution on $W_{2n} \otimes U(\tilde{g}_0)$ described above. A realization of g on $W_{2n} \otimes U(\tilde{g}_0)$ is called skew-Hermitean, if for all elements $X \in g$, the following relation holds

$$(\tau(X))^+ = -\tau(X).$$

3. The General Construction

3.1 The construction presented in this section is based on an induced representation of Lie algebras. We suppose that the reader is familiar with the basic notions of the theory of the induced representations, whose exposition can be found, e.g., in Ref. 23. Below we resume briefly only the main definitions.

3.2 Let g' be a Lie subalgebra of g and g_0 be a Lie subalgebra of g'; i.e., $g_0 \subset g' \subset g$. In the following, we shall use a basis $X_1, X_2, \ldots, X_m$ in the algebra g, with properties: $X_{n+1}, X_{n+2}, \ldots, X_s$ is a basis for g_0 and $X_{n+1}, X_{n+2}, \ldots, X_m$ is a basis for g'.

Furthermore, we shall assume that an auxiliary representation σ of the algebra g' on the space V is given such that

$$\sigma(X_{s+j}) = 0 \quad j = 1,2,\ldots,m-s \tag{2a}$$

$$\sigma/g_0 \text{ is faithful.} \tag{2b}$$

3.3 On the space $U(\tilde{g})\otimes V$, we can simply define the representation ℓ of g by

$$\ell(X)(u\otimes v) = Xu\otimes v$$

for all $X\in g$, $u\in U(\tilde{g})$ and $v\in V$. This representation is called the left regular representation.

3.4 Consider the subspace L in $U(\tilde{g})\otimes V$ generated by

$(u\,y\otimes v) - (u\otimes \sigma(y)v)$, $u\in U(\tilde{g})$, $y\in U(\tilde{g}')$ and $v\in V$.

It is easy to see that L is invariant with respect to ℓ. Hence we may define quotient representation of g on the space $W=(U(\tilde{g})\otimes V)/_L$. This representation is called induced representation and denoted by ind (g,σ).

If $\{v_j\}$ is a basis in the space V, then the vectors

$$|\tilde{k}\rangle\otimes v_j \equiv |(k_1,\dots,k_n)\rangle \otimes v_j \equiv X_1^{k_1}\dots X_n^{k_n}\otimes v_j$$

$k_1,k_2,\dots,k_n\in N_o$, where N_o is the set of all non-negative integers, form a basis in W.

3.5 We define the creation and annihilation operators $\bar{a}_i$, a_i, $i = 1,2,\dots,n$ and the operators $\tilde{X}_r$, $r = n+1,\dots,s$ in the following way:

$$\bar{a}_i\,|k_1,\dots,k_i,\dots,k_n\rangle\otimes v \equiv |k_1,\dots,k_i+1,\dots,k_n\rangle\otimes v \tag{3a}$$

$$a_i\,|k_1,\dots,k_i,\dots,k_n\rangle\otimes v \equiv k_i|k_1,\dots,k_i-1,\dots,k_n\rangle\otimes v \tag{3b}$$

(notice the normalization convention), and

$$\tilde{X}_r\,|k_1,\dots,k_n\rangle\otimes v \equiv |k_1,\dots,k_n\rangle\otimes\sigma(X_r)\,v. \tag{3c}$$

They obey obviously the commutation relations

$$[a_i,a_j] = [\bar{a}_i,\bar{a}_j] = 0\,, \qquad [a_i,\bar{a}_j] = \delta_{ij}\,1$$

$$[a_i,\tilde{X}_r] = [\bar{a}_i,\tilde{X}_r] = 0$$

3.6 <u>Theorem</u>: Let $\varrho\equiv$ ind (g,σ), then all the operators $\varrho(X)$, $X\in g$, can be rewritten in the form

$$\varrho(X) = \sum_{i=1}^{n}\bar{a}_i\,\alpha_i^X\otimes 1 + \sum_{r=n+1}^{s}\alpha_r^X\otimes\tilde{X}_r\,, \tag{4}$$

where α_t^X , $t = 1,2,\dots,s$ are, in general, the infinite sums in the operators $a_1,\dots,a_n$.

<u>Proof</u>: The formula

$$X\,Y^k = \sum_{i=o}^{k}\binom{k}{i}Y^{k-i}\,\underbrace{[\dots[[X,Y],Y],]\dots,Y]}_{i\text{ - times}} \tag{5}$$

implies for $X\,X_1^{k_1}\dots X_n^{k_n}$ the following equality

$$X\,X_1^{k_1}\dots X_n^{k_n} = \sum_{j=1}^{n}\sum_{\tilde{i}=\tilde{o}}^{\tilde{k}} c^X_{j\,\tilde{i}}\binom{\tilde{k}}{\tilde{i}}X_1^{k_1-i_1}\dots X_j^{k_j-i_j+1}\dots X_n^{k_n-i_n} +$$

$$+\sum_{r=n+1}^{m}\sum_{\tilde{i}=\tilde{o}}^{\tilde{k}} c^X_{r\,\tilde{i}}\binom{\tilde{k}}{\tilde{i}}X_1^{k_1-i_1}\dots X_n^{k_n-i_n}X_r\,, \tag{6}$$

where $c^X_{t\,\tilde{i}}$, $t = 1,2,\dots,m$, are constants independent of $\tilde{k}$ and $\binom{\tilde{k}}{\tilde{i}} = \binom{k_1}{i_1}\dots\binom{k_n}{i_n}$. Of course, $\tilde{o}$ means $(\tilde{o},\dots,\tilde{o})$.

For any $\tilde{k}$, we define

$${}^{\tilde{k}}\alpha_t^X = \sum_{\tilde{i}=\tilde{o}}^{\tilde{k}} c^X_{t\,\tilde{i}}\,a^{\tilde{i}} \quad\text{where } a^{\tilde{i}}\equiv a_1^{i_1}\dots a_n^{i_n}. \tag{7}$$

According to the definition (3a,b), $a^{\tilde{k}'}(|\tilde{k}\rangle\otimes v) = 0$ for any $\tilde{k}'$ such that $\sum_{i=1}^{n}k'_i > \sum_{i=1}^{n}k_i$. This implies the equality

${}^{\tilde{k}'}\alpha_t^X(|\tilde{k}\rangle\otimes v) = {}^{\tilde{k}}\alpha_t^X(|\tilde{k}\rangle\otimes v)$ if $\sum_{i=1}^{n}k'_i > \sum_{i=1}^{n}k_i$, which further means that the infinite sums

$$\alpha_t^X = \sum_{\tilde{i}=\tilde{o}} c^X_{t\,\tilde{i}}\,a^{\tilde{i}} \tag{8}$$

are well defined. It is obvious that $\alpha_t^X(|\tilde{k}\rangle\otimes v) = {}^{\tilde{k}}\alpha_t^X(|\tilde{k}\rangle\otimes v)$.

For any $(|\tilde{k}\rangle\otimes v)\in W$, $\varrho(X)(|\tilde{k}\rangle\otimes v) = (X\,X_1^{k_1}\dots X_n^{k_n})\otimes v$ and according to (3a, b, c), (6), (7) and (8) the rhs further equals

$$\Big(\sum_{j=1}^{n} \bar{a}_j\, {}^{\tilde{k}}\alpha_j^X \otimes 1 + \sum_{r=n+1}^{s} {}^{\tilde{k}}\alpha_r^X \otimes \tilde{X}_r\Big)(|\tilde{k}\rangle \otimes v) =$$

$$= \Big(\sum_{j=1}^{n} \bar{a}_j\, \alpha_j^X \otimes 1 + \sum_{r=n+1}^{s} \alpha_r^X \otimes \tilde{X}_r\Big)(|\tilde{k}\rangle \otimes v).$$

This proves the theorem. ■

3.7 We give now conditions on $g' \subset g$ under which α_t^X, t=1,2,...,s defined by (8) are polynomials. Recall that this is required if one wants to obtain realizations in the sense of our definition. Let S be the subspace in g which is spanned by $X_1, \dots, X_n$. For any $Y \in g$, we define by induction $M_i^Y = [S, M_{i-1}^Y]$, $M_o^Y = \mathbb{C}\{Y\}$.

<u>Proposition</u>: If there exists $k \in N_o$ such that $M_k^Y = \{0\}$ for any $Y \in g$, then α_t^Y, t = 1,2,...,m, are finite polynomials.

<u>Proof</u>: Obviously, if $\sum_{i=1}^{n} j_i > k$, then $c^{X}_{t\,\gamma_{j}} = 0$ for any $Y \in g$, t = 1,2,...,m, and according to the definition (8), the expressions of α_t^Y become finite polynomials. ■

3.8 Now the sought realizations are obtained easily by replacing operators in the above expressions by suitable algebraic objects. The mapping

$$\varphi(p_i) = a_i \tag{9a}$$

$$\varphi(q_i) = \bar{a}_i \qquad i = 1,2,\dots, n \tag{9b}$$

$$\varphi(X_r) = \tilde{X}_r \qquad r = n+1, \dots, s \tag{9c}$$

extends naturally to a faithful representation $W_{2n} \otimes U(\tilde{g}_o)$ on W. Thus there exist φ^{-1}, and if α_j^X, α_r^X in (7) are polynomials. Then the mapping τ :

$$\tau(X) = \varphi^{-1} \circ \rho(X)$$

is consistently defined. We denote $\beta_j^X = \varphi^{-1} \circ \alpha_j^X$.

<u>Proposition</u>: τ is a realization of the algebra g in $W_{2n} \otimes U(\tilde{g}_o)$.

<u>Proof</u>: $\tau[X,Y] = \varphi^{-1} \circ \rho([X,Y]) = \varphi^{-1} \circ ([\rho(X), \rho(Y)]) =$

$$= [\varphi^{-1} \circ \rho(X), \varphi^{-1} \circ \rho(Y)] = [\tau(X), \tau(Y)].$$ ■

3.9 Let τ be a realization of the algebra g in $W_{2n} \otimes U(\tilde{g}_o)$, which is of the type

$$\tau(X) = \sum_{j=1}^{n} q_j\, \beta_j^X \otimes 1 + \sum_{r=n+1}^{s} \beta_r^X \otimes X_r, \quad X_r \in g_o, \tag{10}$$

where β_t^X are polynomials in the variables $p_1, \dots, p_n$. Then the following expressions are well-defined

$$\tau'(X) = \tau(X) + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial \beta_j^X}{\partial p_j} \otimes 1. \tag{11}$$

<u>Theorem</u>: τ' is a skew-Hermitean realization of g in $W_{2n} \otimes U(\tilde{g}_o)$.

<u>Proof</u>: Using the relations $[\beta_j^X, q_i] = \dfrac{\partial \beta_j^X}{\partial p_i}$ and (1a-c), one can check easily the Skew-Hermiticity.

$$(\tau'(X))^+ = \Big(\sum_{j=1}^{n} q_j\, \beta_j^X \otimes 1 + \sum_{r=n+1}^{s} \beta_r^X \otimes X_r + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial \beta_j^X}{\partial p_j} \otimes 1\Big)^+ =$$

$$= -\sum_{j=1}^{n} q_j\, \beta_j^X \otimes 1 - \sum_{r=n+1}^{s} \beta_r^X \otimes X_r - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial \beta_j^X}{\partial p_j} \otimes 1 = -\tau'(X).$$

Further $[\tau(X), \tau(Y)] = \Big[\sum_{j=1}^{n} q_j\, \beta_j^X \otimes 1, \sum_{k=1}^{n} q_k\, \beta_k^Y \otimes 1\Big] + P =$

$$= \sum_{j=1}^{n} q_j \Big(\sum_{k=1}^{n} \frac{\partial \beta_j^X}{\partial p_k}\, \beta_k^Y - \frac{\partial \beta_j^Y}{\partial p_k}\, \beta_k^X\Big) \otimes 1 + P,$$

where the P is independent of q_i, i = 1,2,...,n.
Since $[\tau(X), \tau(Y)] = \tau([X, Y])$, then according to formula (10)

$$\beta_j^{[X,Y]} = \sum_{k=1}^{n} \Big(\frac{\partial \beta_j^X}{\partial p_k}\, \beta_k^Y - \frac{\partial \beta_j^Y}{\partial p_k}\, \beta_k^X\Big). \tag{12}$$

Now a direct calculation gives

$$[\tau'(X),\ \tau'(Y)] = [\tau(X),\ \tau(Y)] + \left[\sum_{k=1}^{n} q_k \beta_k^X,\ \frac{1}{2}\sum_{j=1}^{n}\frac{\partial \beta_j^Y}{\partial p_j}\right]\otimes 1 +$$

$$+ \left[\frac{1}{2}\sum_{j=1}^{n}\frac{\partial \beta_j^X}{\partial p_j},\ \sum_{k=1}^{n} q_k \beta_k^Y\right]\otimes 1 =$$

$$= \tau([X,Y]) - \frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 \beta_j^Y}{\partial p_j \partial p_k}\beta_k^X - \frac{\partial^2 \beta_j^X}{\partial p_j \partial p_k}\beta_k^Y\right)\right)\otimes 1 =$$

$$= \tau([X,Y]) + \left(\frac{1}{2}\sum_{j=1}^{n}\frac{\partial \beta_j^{[X,Y]}}{\partial p_j}\right)\otimes 1 = \tau'([X,Y]) .$$

Hence τ' is a realization of g, and the proof is completed. ∎

4. The Construction for the Real Semisimple Lie Algebra

4.1 In this section, we apply the general construction described above to the case of a real semisimple Lie algebra.
Suppose that g is a real semisimple Lie algebra and that $b \in g$ and $B = \{X_1,\dots,X_n,X_{n+1},\dots, X_s, X_{s+1},\dots,X_{s+n}\}$ is a basis in g for which

$$[b,\ X_j] = \gamma_j X_j \qquad (13a)$$

$$[b,\ X_{s+j}] = -\gamma_j X_{s+j}, \qquad (13b)$$

where $\gamma_j > 0$ for any $j = 1,2,\dots,n$ and further

$$[b,\ X_r] = 0 \ ,\quad r = n+1,\dots,s. \qquad (13c)$$

The triangle decomposition of g implies the existence of such basis in g. For details see Ref. 23. In the example $gl(3,R)$ we give such a basis explicitly and for other real forms of classical Lie algebras we will construct explicitly these bases in a forthcoming series of papers.

4.2 The element b and the basis B define for g the following direct decomposition

$$g = n_+^b \oplus g_0^b \oplus n_-^b ,$$

where

$$n_+^b = R\{X_j,\ j = 1,\dots,n\}, \qquad (14a)$$

$$n_-^b = R\{X_{s+j},\ j=1,\dots,n\}, \qquad (14b)$$

$$g_0^b = R\{X_r,\ r = n+1,\dots,s\}. \qquad (14c)$$

The following lemma results directly from the Jacobi indentity, the formulae(13a-c) and the definitions(14a-c).

<u>Lemma</u>: (i) n_+^b , n_-^b are subalgebras of g.

(ii) $[n_+^b,\ g_0^b] \subset n_+^b$, $[n_-^b,\ g_0^b] \subset n_-^b$,

(iii) If $M_i^Y = [n_+^b,\ M_{i-1}^Y]$, $M_0^Y = \mathbb{C}\{Y\}$ then exist k such that for any $i > k$ and $Y \in g$ $M_i^Y = \{0\}$.

4.3 We now specify the g', g_0 described in 3.1 in this way:

$$g_0 \equiv g_0^b ,$$

$$g' \equiv g_0^b \oplus n_-^b$$

and further the auxiliar representation σ fulfills

$$\sigma | g_0^b \text{ is faithful ,} \qquad (15a)$$

$$\sigma(n_-^b) = 0 . \qquad (15b)$$

According to the assertion (iii) in the lemma 4.2 we can construct the realizations τ_b and τ_b' by means of the method from section 3. These have the following property.

<u>Theorem</u>: τ_b and τ_b' are Schur-realizations of g in the $W_{2n} \otimes U(\tilde{g}_0^b)$.

<u>Proof</u>: First we prove one proposition which specifies forms of elements $Z(\tilde{g})$.

<u>Proposition</u>: Any $C \in Z(\tilde{g})$ can be written in the form

$$C = \sum_{\tilde{n},\tilde{n}' \neq \tilde{0}} X_1^{n_1} \dots X_n^{n_n}\, Y_{\tilde{n},\tilde{n}'}\, X_{s+1}^{n_1'} \dots X_{s+n}^{n_n'} + C_0 , \qquad (16)$$

where $Y_{\tilde{n},\tilde{n}'} \in U(\tilde{g}_o^b)$ and $c_o \in Z(\tilde{g}_o^b)$.

Proof: Write $C \in Z(\tilde{g})$ in the form

$$C = \sum_{\substack{\tilde{n},\tilde{n}'=\tilde{o} \\ \tilde{n}\vee\tilde{n}'\neq\tilde{o}}} X_1^{n_1} \dots X_n^{n_n} Y_{\tilde{n},\tilde{n}'} X_{s+1}^{n_1'} \dots X_{s+n}^{n_n'} + c_o , \qquad (17)$$

where $Y_{\tilde{n},\tilde{n}'} \in U(\tilde{g}_o^b)$ and $c_o \in U(\tilde{g}_o^b)$. The facts, that the element C commutes with b and the formulae(13 a-c) imply that, if

$\sum_{i=1}^{n} n_i \gamma_i \neq \sum_{i=1}^{n} n_i' \gamma_i$ then $Y_{\tilde{n},\tilde{n}'} = 0$. Since $\gamma_i > 0$ then, if

$\sum_{i=1}^{n} n_i > 0$ also $\sum_{i=1}^{n} n_i' > 0$ and, therefore, the summation in (17)

runs only over $\tilde{n},\tilde{n}'$ for which $n,\tilde{n}' \neq 0$. This and the condition (ii) in lemma 4.2 imply that if $X \in g_o^b$, then $[X, c_o] = 0$. This completes the proof of proposition. ∎

Using (16), (15b) and the definition of induced representation we calculate now explicitly the operator $\varrho(C)$.

$$\varrho(C)(|\tilde{k}\rangle \otimes v) = \varrho(C) \; (\varrho(X_1))^{k_1} \dots (\varrho(X_n))^{k_n} (|\tilde{o}\rangle \otimes v) =$$
$$= (\varrho(X_1))^{k_1} \dots (\varrho(X_n))^{k_n} \varrho(C)(|\tilde{o}\rangle \otimes v) = |\tilde{k}\rangle \otimes \sigma(c_o) \; v.$$

This proves the Schur-property of τ_b, because $\tau(C) = \varphi^{-1} \circ \varrho(C) = 1 \otimes c_o$.

Now we are going to check the same property for τ_b'. We define $\varrho'(X) = \varphi \circ \tau'(X)$. Before continuing the proof of the theorem 4.3 we shall prove the following lemma.

<u>Lemma</u>: (i) $\varrho'(X_j) = \varrho(X_j)$ holds for any $j = 1,2,\dots,n$.

(ii) For any $X_r, r=n+1,\dots,s$ $\varrho(X_r) = \sum_{j=1}^{n} \bar{a}_j \alpha_j^{X} \otimes 1 + 1 \otimes \tilde{X}_r$.

(iii) For any $X_r, r=n+1,\dots,s$ $\varrho'(X_r)(|\tilde{o}\rangle \otimes v) = |\tilde{o}\rangle \otimes (c_r + \sigma(X_r)) \; v$, where $c_r \in \mathbb{C}$.

(iv) $\varrho'(X_{s+j})(|\tilde{o}\rangle \otimes v) = 0$ holds for any $j=1,2,\dots,n$.

<u>Proof</u>:

(i) Since γ_j, γ_i are positive, $\gamma_j + \gamma_i \neq \gamma_i$ holds for any $j,i = 1,2,\dots,n$. Therefore $c_{i\,\tilde{f}}^{X_k} = 0$ for $f_i > 0$, and further according to the definition (8) $\frac{1}{2} \sum_{i=1}^{n} \frac{\partial \alpha_i^{X_k}}{\partial a_i} = 0$ for any $X_k, k=1,2,\dots,n$ and also $\varrho'(X_k) = \varrho(X_k)$, $k = 1,2,\dots, n$.

(ii) The assertions (i) and (ii) from the lemma 4.2 imply $c_{t\,\tilde{f}}^{X_r} = 0$, $t = n+1,\dots,s$ and $\tilde{f}$ for which $\sum_{i=1}^{n} f_i > 0$ and, obviously, $c_{s\,\tilde{o}}^{X_r} = 0$ for $s \neq r$. This gives the assertion directly according to the definition (8).

(iii) By a direct calculation we get

$$\varrho'(X_r)(|\tilde{o}\rangle \otimes v) = \Big(\varrho(X_r) + \frac{1}{2}\sum \frac{\partial \alpha_j^{X_r}}{\partial a_j} \otimes 1\Big)(|\tilde{o}\rangle \otimes v) =$$
$$= (\varrho(X_r) + c_r)(|\tilde{o}\rangle \otimes v) = |\tilde{o}\rangle \otimes (c_r + \sigma(X_r)) \; v.$$

(iv) As in (i), any two γ_j, γ_i fulfil $\gamma_j - \gamma_i \neq \gamma_j$. This implies that $c_{i\,\tilde{1}_i}^{X_{s+k}} = 0$ for any $k,i=1,2,\dots,n$ where $\tilde{1}_i = (0,\dots,0,1,0,\dots,0)$ with 1 on i-the place. This and (15b) give that $\varrho'(X_{s+k})(|\tilde{o}\rangle \otimes v) =$

$$= \Big(\Big(\varrho(X_{s+k}) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial \alpha_i^{X_{s+k}}}{\partial a_i}\Big) \otimes 1\Big)(|\tilde{o}\rangle \otimes v) = 0. \qquad ∎$$

Using the lemma just proved we shall evaluate now $\varrho'(C)$ explicitly

$$\varrho'(C)(|\tilde{k}\rangle \otimes v) = \varrho'(C) \; (\varrho(X_1))^{k_1} \dots (\varrho(X_n))^{k_n} (|\tilde{o}\rangle \otimes v) =$$
$$= \varrho'(C) \; (\varrho'(X_1))^{k_1} \dots (\varrho'(X_n))^{k_n} (|\tilde{o}\rangle \otimes v) =$$
$$= (\varrho(X_1))^{k_1} \dots (\varrho(X_n))^{k_n} \varrho'(c_o)(|\tilde{o}\rangle \otimes v) = |\tilde{k}\rangle \otimes \sigma(c_o') v ,$$

where $c_o' \in U(\tilde{g}_o^b)$. Further we prove that $c_o' \in Z(\tilde{g}_o^b)$. Let X_r be any element of g_o^b , then $\left[1\otimes C_o' , 1\otimes X_r\right]$ =

$$= \left[1\otimes C'_o , \left(\sum_{j=1}^{n} q_j \,\beta_j^{X_r} + \frac{1}{2}\sum_{j=1}^{n}\frac{\partial \beta_j^{X_r}}{\partial p_j}\right)\otimes 1 + 1\otimes X_r\right] = \varphi^{-1}\left[\varsigma'(C), \varsigma'(X_r)\right] = 0 .$$

This completes the proof of the theorem. ∎

5. The Realizations for the Algebra gl(3,R)

5.1 Now we are going to illustrate the construction by the example of the algebra $gl(3,R)$.
The algebra $gl(3,R)$ is the 9-dimensional with the standard basis $\{E_{ij} : i,j = 1,2,3\}$, the elements of which obey

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} . \tag{18}$$

If we denote $b \equiv E_{11}$ and $B = \{E_{12}, E_{13}, E_{11}, E_{22}, E_{33}, E_{23}, E_{32}, E_{21}, E_{31}\}$ we obtain according to (13a-c) and definitions (14a-c)

$$n_+^b = R\{E_{12}, E_{13}\} ,$$

$$g_o^b = R\{E_{11}, E_{22}, E_{33}, E_{23}, E_{32}\} ,$$

$$n_-^b = R\{E_{21}, E_{31}\} .$$

5.2 The explicit form of the realizations follows from the equalities (6) for $X\, E_{12}^{k_{12}}\, E_{13}^{k_{13}}$ with an arbitrary $X \in B$, which are calculated in the Appendix. In this way, one obtains coefficients $c^X_{t\,\gamma}$ in the definition (8) which further give the formulae:

$$\tau(E_{11}) = q_{12}p_{12} + q_{13}p_{13} + E_{11}$$

$$\tau(E_{ij}) = -q_{1j}p_{1i} + E_{ij} \qquad i,j = 2,3$$

$$\tau(E_{12}) = q_{12}$$

$$\tau(E_{13}) = q_{13}$$

$$\tau(E_{21}) = (-q_{12}p_{12} - q_{13}p_{13} + E_{22} - E_{11})\, p_{12} + E_{23}p_{13}$$

$$\tau(E_{31}) = (-q_{12}p_{12} - q_{13}p_{13} + E_{33} - E_{11})\, p_{13} + E_{32}p_{12}$$

and

$$\tau'(E_{11}) = q_{12}p_{12} + q_{13}p_{13} + E_{11} + 1$$

$$\tau'(E_{ij}) = -q_{1j}p_{1i} + E_{ij} - \tfrac{1}{2}\,\delta_{ij} \qquad i,j = 2,3$$

$$\tau'(E_{12}) = q_{12}$$

$$\tau'(E_{13}) = q_{13}$$

$$\tau'(E_{21}) = (-q_{12}p_{12} - q_{13}p_{13} - \tfrac{3}{2} + E_{22} - E_{11})\, p_{12} + E_{23}p_{13}$$

$$\tau'(E_{31}) = (-q_{12}p_{12} - q_{13}p_{13} - \tfrac{3}{2} + E_{33} - E_{11})\, p_{13} + E_{32}p_{12} .$$

According to Theorems 4.2 and 3.9 these realizations are Schur-realizations of the algebra $gl(3,R)$ in $W_4 \otimes U(\tilde{g}_o^b)$; furthermore, the realization τ' is skew-Hermitean.

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Appendix

Using the formulae (5) and (18) we obtain:

$$E_{12}E_{12}^{k_{12}}\,E_{13}^{k_{13}} = E_{12}^{k_{12}+1}\,E_{13}^{k_{13}}$$

$$E_{13}E_{12}^{k_{12}}\,E_{13}^{k_{13}} = E_{12}^{k_{12}}\,E_{13}^{k_{13}+1}$$

$$E_{11}E_{12}^{k_{12}}\,E_{13}^{k_{13}} = k_{12}E^{k_{12}-1+1}\,E_{13}^{k_{13}} + k_{13}E_{12}^{k_{12}}\,E_{13}^{k_{13}-1+1} +$$

$$+ E_{12}^{k_{12}}\,E_{13}^{k_{13}}\,E_{11}$$

$$E_{22}E_{12}^{k_{12}}\,E_{13}^{k_{13}} = -k_{12}E_{12}^{k_{12}-1+1}\,E_{13}^{k_{13}} + E_{12}^{k_{12}}\,E_{13}^{k_{13}}\,E_{22}$$

$$E_{33}E_{12}^{k_{12}}\,E_{13}^{k_{13}} = -k_{13}E_{12}^{k_{12}}\,E_{13}^{k_{13}-1+1} + E_{12}^{k_{12}}\,E_{13}^{k_{13}}\,E_{33}$$

$$E_{23}E_{12}^{k_{12}}\,E_{13}^{k_{13}} = -k_{12}\,E_{12}^{k_{12}-1}\,E_{13}^{k_{13}+1} + E_{12}^{k_{12}}\,E_{13}^{k_{13}}\,E_{23}$$

$$E_{32}E_{12}^{k_{12}}\,E_{13}^{k_{13}} = -k_{13}\,E_{12}^{k_{12}+1}\,E_{13}^{k_{13}-1} + E_{12}^{k_{12}}\,E_{13}^{k_{13}}\,E_{32}$$

$$E_{21}E_{12}^{k_{12}}E_{13}^{k_{13}} = k_{12}E_{12}^{k_{12}-1}E_{13}^{k_{13}}(E_{22}-E_{11}) - k_{12}(k_{12}-1)E_{12}^{k_{12}-2+1} -$$

$$- k_{12}k_{13}E_{12}^{k_{12}-1}E_{13}^{k_{13}-1+1} + k_{13}E_{12}^{k_{12}}E_{13}^{k_{13}-1}E_{23} + E_{12}^{k_{12}}E_{13}^{k_{13}}E_{21}$$

$$E_{31}E_{12}^{k_{12}}E_{13}^{k_{13}} = k_{12}E_{12}^{k_{12}-1}E_{13}^{k_{13}}E_{32} - k_{12}k_{13}E_{12}^{k_{12}-1+1}E_{13}^{k_{13}-1} +$$

$$+ k_{13}E_{12}^{k_{12}}E_{13}^{k_{13}-1}(E_{33}-E_{11}) - k_{13}(k_{13}-1)E_{12}^{k_{12}}E_{13}^{k_{13}-1+1} + E_{12}^{k_{12}}E_{13}^{k_{13}}E_{31}.$$

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Бурдик Ч. E2-84-680

Реализации вещественных полупростых алгебр Ли: метод конструкции

Построен метод конструкции антиэрмитовых реализаций для любой вещественной полупростой алгебры Ли g. Метод основан на разложении алгебры Ли g в виде $g = n_+^b \oplus g_0^b \oplus n_-^b$, которое является простым обобщением треугольного разложения, и на представлении g, индуцированном с помощью представления σ параболической подалгебры $g_0^b \oplus n_-^b$. Показано, что каждое такое представление может быть превращено в реализацию /бозонное представление/. Доказано, что метод дает реализации, которые обладают двумя хорошими свойствами, позволяющими их применение в теории представлений. Они антиэрмитовы и шуровские. В качестве примера построен явный вид реализаций для алгебры $gl(3,R)$.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Realizations of the Real Simple Lie Algebras: the Method of Construction

The method of constructing skew-Hermitean realizations for an arbitrary real semisimple Lie algebra g is presented. The construction starts with a decomposition $g = n_+^b \oplus g_0^b \oplus n_-^b$ of g, which is a simple generalization of the triangle decomposition, employs substantially an induced representation of g with respect to a suitable representation σ of the subalgebra $g_0^b \oplus n_-^b$. It is shown that each such representation could be transformed into realization /bozon representation/. It is proved that the method gives the realizations which possess two good properties permitting their application in theory of representations. These are skew-Hermitean and Schurean. As an example, the realizations for the algebra $gl(3,R)$ are calculated explicitly.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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