объединенный
институт
ядерных
исследований
дубна

E2-84-654

I.V.Polubarinov

HIGHER HYPERCOMPLEX NUMBERS AND QUANTUM MECHANICS

Submitted to VII International Conference on the Problems of Quantum Field Theory (Alushta, Crimea, 19-25 April, 1984)

1984

1. In quantum mechanics observables (operators) are usually presented by some matrices or differential operators with the property of associativity. One can also construct nonassociative octonionic quantum mechanics, where octonions serve as observables [2,16]. One more quantum mechanics, which uses octonions, is the famous exceptional quantum mechanics by Jordan, von Neumann and Wigner [3] (see also [13]).

Here a series of quantum mechanics will be discussed, where observables belong to algebras A_q of hypercomplex numbers of order $p=2^q$ ($q=1,2,3,4,\ldots$) with the multiplication tables

$$e_0^2=e_0, \qquad e_j e_k=-\delta_{jk}e_0+\varepsilon_{jkl}e_l \quad (j,k,l=1,2,\ldots,p-1) \qquad (1)$$

for basis hypercomplex elements e_j, ε_{jkl} are totally antisymmetric "tensors" defined in what follows. Nonzero components ε_{jkl} equal ± 1 (see Appendix A). After the well-known algebras A_2 (of the quaternions) and A_3 (of the octonions) there follow algebra A_4 of 16 order (we call these hypercomplex numbers sedenions in what follows), algebra A_5 of 32 order, etc. The algebras A_q for $q \geqslant 4$ are not only noncommutative, nonassociative, but also nonalternative, being therefore not division algebras. They do not permit the composition of quadratic forms in the Hurwitz sense [15]. However, they remain to be flexible and power-associative and have unit element and involution.

The algebras A_q were defined by Albert [5,6] and Schafer [7] by the Cayley-Dickson process, originally proposed by Dickson [1] for passing from the quaternions to the octonions.

2. Note one more application of these algebras A_q ($q=2,3,4$), namely, for embeddings of classical and quantum dynamics into spaces of higher dimensionalities, using the Hopf maps (fiber bundles) $S^3\to S^2$, $S^7\to S^4$ and $S^{15}\to S^8$. These embeddings result [17-29], when in two-body or many-body Lagrangians or Hamiltonians in $(p+1)$-dimensional spaces we replace the Cartesian coordinates according to the maps $R^{2p}\to R^{p+1}$ ($p=1,2,4,8$)

$$\begin{aligned}
&x_0=2\alpha_\mu\alpha'_\mu=2\alpha\alpha', \\
&x_j=2(\alpha_0\alpha'_j-\alpha_j\alpha'_0\pm\varepsilon_{jkl}\alpha_k\alpha'_l) \quad (j,k,l=1,2,\ldots,p-1), \\
&x_p=\alpha_\mu\alpha_\mu+\alpha'_\mu\alpha'_\mu=\alpha\alpha+\alpha'\alpha', \qquad (2)\\
&\rho^2=\alpha_\mu\alpha_\mu+\alpha'_\mu\alpha'_\mu=\alpha\alpha+\alpha'\alpha', \quad r^2=|\vec{x}|^2=x_\mu x_\mu+x_p^2,
\end{aligned}$$

and use the following identities with one 2p-dimensional vector (spinor) $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_{p-1},\alpha'_0,\alpha'_1,\ldots,\alpha'_{p-1})$

$$(a^2+a'^2)^2=(a^2-a'^2)^2+4(aa')^2+4\sum_{j=1}^{p-1}(a_0a'_j-a_ja'_0\pm\varepsilon_{jkl}a_ka'_l)^2+X, \quad (3.a)$$

$$X=\begin{cases}0 & \text{for } p=1,2,4,8\\ -4\beta_{jklm}a_ja'_ka_la'_m & \text{for } p\geq 16\end{cases} \quad (3.b)$$

and with two such vectors a and b

$$(a^2+a'^2)(b^2+b'^2)=(ab-a'b')^2+\sum_{j=1}^{p-1}(a_0b_j-a_jb_0\mp\varepsilon_{jkl}a_kb_l+a'_0b'_j-a'_jb'_0\mp\varepsilon_{jkl}a'_kb'_l)^2+$$
$$+(ab'+a'b)^2+\sum_{j=1}^{p-1}(a_0b'_j-a_jb'_0\pm\varepsilon_{jkl}a_kb'_l-a'_0b_j+a'_jb_0\mp\varepsilon_{jkl}a'_kb_l)^2+Y, \quad (4.a)$$

$$Y=\begin{cases}0 & \text{for } p=1,2,4\\ -4\beta_{jklm}a_jb_ka'_lb'_m & \text{for } p=8\\ -\beta_{jklm}[(a_jb_k+a'_jb'_k)(a_lb_m+a'_lb'_m)+(a_jb'_k-a'_jb_k)(a_lb'_m-a'_lb_m)] & \text{for } p\geq 16.\end{cases} \quad (4.b)$$

In eqs.(2)-(4) $\mu=0,1,\ldots,p-1$; $j,k,l=1,2,\ldots,p-1$; ε_{ijk} is the same totally antisymmetric tensor as in eq. (1), and the "tensor" β_{jklm} is expressed via ε_{jkl} (see eqs. (36) and (37) below). Identity (3.a) follows from identity (4.a) if one puts $a=b$. In Appendix B a more general identity with 4 vectors will be given together with derivation of all these identities (in real terms).

For p=1,2,4,8 $X=0$ in eq. (3.a), and eqs. (2) correspond to the maps

A) $R^2\to R^2$, such that $S^1_\rho\to RP^1=S^1_\tau$ (fiber is Z_2, a pair of the points (a,a) and (-a,-a)), double covering, (5)

B) $R^4\to R^3$, such that $S^3_\rho\to CP^1=S^2_\tau$ (fiber is $S^1=SO(2)=U(1)$), (6)

C) $R^8\to R^5$, such that $S^7_\rho\to QP^1=S^4_\tau$ (fiber is $S^3=SU(2)=Sp(1)$), (7)

D) $R^{16}\to R^9$, such that $S^{15}_\rho\to OP^1=S^8_\tau$ (fiber is S^7) (8)

ρ and τ being radii of the spheres S^{2p-1}_ρ and S^p_τ, $\rho=\sqrt{\tau}$. At each fixed $\rho=\sqrt{\tau}$ these maps belong to the Hopf fiber bundles $S^{2p-1}\to S^p$ with the Hopf invariant (linking number) $H=1$, x_α ($\alpha=0,1,\ldots,p$) being coordinates of a point of the bases S^p. For $p\geq 16$ the connection with maps of spheres onto spheres disappears.

In eq. (4.a) the term Y equals zero only for p= 1,2 and 4, and eq. (4.a) is a needed form of the famous 2,4, and 8 square identities (the second due to Euler and the third due to Degen and Cayley). From p=8 the term Y does not vanish, and this is in accord with the Hurwitz theorem /15/ that forbids existence of pure 16 and more square identities.

In terms of hypercomplex numbers maps (2) and identities (3.a) and (4.a) take the form

$$\underset{\sim}{x}=2\underset{\sim}{a}'\bar{\underset{\sim}{a}},\quad x_p=\bar{\underset{\sim}{a}}\underset{\sim}{a}-\bar{\underset{\sim}{a}}'\underset{\sim}{a}',\quad \rho^2=\bar{\underset{\sim}{a}}\underset{\sim}{a}+\bar{\underset{\sim}{a}}'\underset{\sim}{a}' \quad \text{(upper signs)}, \quad (9.a)$$
$$\underset{\sim}{x}=2\bar{\underset{\sim}{a}}\underset{\sim}{a}',\quad x_p=\bar{\underset{\sim}{a}}\underset{\sim}{a}-\bar{\underset{\sim}{a}}'\underset{\sim}{a}',\quad \rho^2=\bar{\underset{\sim}{a}}\underset{\sim}{a}+\bar{\underset{\sim}{a}}'\underset{\sim}{a}' \quad \text{(lower signs)}, \quad (9.b)$$

$$(\bar{\underset{\sim}{a}}\underset{\sim}{a}+\bar{\underset{\sim}{a}}'\underset{\sim}{a}')^2=\bar{\underset{\sim}{x}}\underset{\sim}{x}+x_p^2+X, \qquad (\bar{\underset{\sim}{x}}\underset{\sim}{x}+x_p^2\equiv\tau^2) \quad (10)$$

$$(\bar{\underset{\sim}{a}}\underset{\sim}{a}+\bar{\underset{\sim}{a}}'\underset{\sim}{a}')(\bar{\underset{\sim}{b}}\underset{\sim}{b}+\bar{\underset{\sim}{b}}'\underset{\sim}{b}')=|\bar{\underset{\sim}{a}}\underset{\sim}{b}-\bar{\underset{\sim}{b}}'\underset{\sim}{a}'|^2+|\underset{\sim}{b}'\bar{\underset{\sim}{a}}+\underset{\sim}{a}'\bar{\underset{\sim}{b}}|^2+Y \quad \text{(upper signs)}, \quad (11.a)$$

$$(\bar{\underset{\sim}{a}}\underset{\sim}{a}+\bar{\underset{\sim}{a}}'\underset{\sim}{a}')(\bar{\underset{\sim}{b}}\underset{\sim}{b}+\bar{\underset{\sim}{b}}'\underset{\sim}{b}')=|\underset{\sim}{b}\bar{\underset{\sim}{a}}-\underset{\sim}{a}'\bar{\underset{\sim}{b}}'|^2+|\bar{\underset{\sim}{a}}\underset{\sim}{b}'+\bar{\underset{\sim}{b}}\underset{\sim}{a}'|^2+Y \quad \text{(lower signs)}, \quad (11.b)$$

where $\underset{\sim}{x}=x_\mu e_\mu$, $\underset{\sim}{a}=a_\mu e_\mu$, $\underset{\sim}{a}'=a'_\mu e_\mu$, ... ($\mu=0,1,\ldots,p-1$) are real numbers (p=1), complex numbers (p=2), quaternions (p=4), octonions (p=8), sedenions (p=16), etc., bar means the conjugation (involution operation): $\bar{\underset{\sim}{a}}=a_0e_0-a_me_m$. In the cases p=1,2 or 4 (real numbers, complex numbers, or quaternions) it is clear from eqs. (9), that each coordinate x_α of a base point is invariant under the following transformations Z_2, $U(1)$ or $Sp(1)$ on a fiber

$$\underset{\sim}{a}\to\tilde{\underset{\sim}{a}}=\underset{\sim}{a}\underset{\sim}{z},\quad \underset{\sim}{a}'\to\tilde{\underset{\sim}{a}}'=\underset{\sim}{a}'\underset{\sim}{z},\quad \bar{\underset{\sim}{z}}\underset{\sim}{z}=1, \quad \text{(upper signs)}, \quad (12.a)$$

$$\underset{\sim}{a}\to\tilde{\underset{\sim}{a}}=\underset{\sim}{z}\underset{\sim}{a},\quad \underset{\sim}{a}'\to\tilde{\underset{\sim}{a}}'=\underset{\sim}{z}\underset{\sim}{a}',\quad \bar{\underset{\sim}{z}}\underset{\sim}{z}=1, \quad \text{(lower signs)}, \quad (12.b)$$

where $\underset{\sim}{z}=n_\mu e_\mu$ is a real number, a complex number or quaternion, respectively. When $\underset{\sim}{z}$ runs over all possible values, these transformations generate the fiber, i.e., the inverse image of x (see eqs. (5)-(8)). For octonions (p=8) and for $p\geq 16$ there are no such invariances. However, for the octonions (p=8) a fiber can be also constructed according to eqs. (12.a) or (12.b), starting with a particular "gauge" $a'_j=0$ (j= 1,...7), or $a_j=0$ (see ref. /29/).

Renumbering the components of a and b (see Appendix A) we may rewrite identity (4.a) in terms of the next algebra A_{q+1}

$$(a_\mu a_\mu)(b_\nu b_\nu)=(a_\mu b_\mu)^2+\sum_{J=1}^{2p-1}(a_0b_J-a_Jb_0\pm\varepsilon_{JKL}a_Kb_L)^2-\beta_{JKLM}a_Jb_Ka_Lb_M, \quad (13)$$

where $\mu,\nu=0,1,\ldots,2p-1$; $J,K,L,M=1,2,\ldots,2p-1$, $a_\mu b_\mu=a_0b_0+a_Mb_M$, ε_{JKL}, again totally antisymmetric, corresponds to the multiplication table

$$e_0^2=e_0,\quad e_Je_K=-\delta_{JK}e_0+\varepsilon_{JKL}e_L\ (J,K,L=1,2,\ldots,p'-1), \quad (14)$$

$p'=2^{q+1}$. In terms of these hypercomplex numbers identity (4.a) can be be rewritten as follows

$$(\bar{\underset{\sim}{a}}\underset{\sim}{a})(\bar{\underset{\sim}{b}}\underset{\sim}{b})=(\underset{\sim}{b}\bar{\underset{\sim}{a}})(\underset{\sim}{a}\bar{\underset{\sim}{b}})-\beta_{JKLM}a_Jb_Ka_Lb_M \quad \text{(upper signs)}, \quad (15.a)$$
$$(\bar{\underset{\sim}{a}}\underset{\sim}{a})(\bar{\underset{\sim}{b}}\underset{\sim}{b})=(\bar{\underset{\sim}{a}}\underset{\sim}{b})(\bar{\underset{\sim}{b}}\underset{\sim}{a})-\beta_{JKLM}a_Jb_Ka_Lb_M \quad \text{(lower signs)}, \quad (15.b)$$

where $\underset{\sim}{a}=a_0e_0+a_Me_M$, bar means the conjugation (involution operation): $\bar{\underset{\sim}{a}}=a_0e_0-a_Me_M$. We do not alter the β– term. It equals zero for $p'=2,4$ and 8, and eqs. (14) are reduced to

$$(\bar{a}a)(\bar{b}b)=(\bar{a}b)(\bar{b}a)=(b\bar{a})(a\bar{b}), \qquad (16)$$

and this is a well-known form of the 2,4, and 8 square identities in terms of the complex numbers, quaternions and octonions.

3. Two forms (4.a) and (13) of the same identity can be used for obtaining of hypercomplex number algebras. Actually, starting with some known ε_{jkl} for an algebra A_q, we can write identity (4.a) and then bring it, by renumbering (see Appendix A) to form (13), thus obtaining the tensor ε_{JKL} for the next algebra A_{q+1} (and hence β_{JKLM}). Then we can repeat the process, etc. This iterating process is equivalent to the Dickson one /1/ in the forms

$$z_1z_2\equiv(a+e_{2p-1}a')(b+e_{2p-1}b')=ab-b'\bar{a}'+e_{2p-1}(\bar{a}b'+ba') \qquad (17.a)$$

(corresponding to the octonion algebra in ref./11/) or

$$z_1z_2\equiv(a+a'e_p)(b+b'e_p)=ab-\bar{b}'a'+(b'a+a'\bar{b})e_p. \qquad (17.b)$$

From these definitions we can extract, in particular, the relations

$$e_ie_J=-e_{2p-1}(e_ie_{J-p+1}),\quad 1\leq i\leq p-1,\quad p\leq J<2p-1, \qquad (18.a)$$

$$e_ie_J=(e_{J-p}e_i)e_p,\quad 1\leq i\leq p-1,\quad p<J\leq 2p-1, \qquad (18.b)$$

which permit us to find all the values ε_{JKL} [x], once all ε_{jkl} for the preceding algebra are known.

4. Basic properties of the algebra A_q of the hypercomplex numbers. Both the above processes permit us to conclude that

1) ε_{jkl} is totally antisymmetric in j,k,l;
2) $\varepsilon_{jkl}=\pm1$ or 0,
3) each pair jk uniquely defines a value l (the converse is not true), $\varepsilon_{jkl}=0$ for other l.

Using eq. (1), we find for the associator

$$(e_j,e_k,e_l)=(e_je_k)e_l-e_j(e_ke_l)=\alpha_{jklm}e_m \qquad (19)$$

$$\alpha_{jklm}=-\delta_{jk}\delta_{lm}+\delta_{kl}\delta_{jm}+\varepsilon_{jkn}\varepsilon_{nlm}+\varepsilon_{kln}\varepsilon_{njm}= =\beta_{jklm}+\beta_{kljm}, \qquad (20)$$

whence there follow the properties

---

[x] Identical with those, which are given by the first process, using identities (4.a) and (13).

1) α_{jklm} is antisymmetric under transpositions of 1st index with 3rd and 2nd index with 4th and changes its sign after each cyclic permutation of all four indices

$$\alpha_{jklm}=-\alpha_{lkjm}=-\alpha_{jmlk}=-\alpha_{mjkl}=\alpha_{lmjk}=-\alpha_{klmj}, \qquad (21)$$

2) $\alpha_{jjlm}=\alpha_{jkll}=0$ (no summation), (22)
3) $\alpha_{jklm}\neq0$ only with no coincident indices,
4) in general case all 4! components of α_{jklm} with fixed $jklm$ are expressed via three of them which differ in cyclic permutation of three indices, e.g., via

$$\alpha_{jklm},\quad \alpha_{ljkm},\quad \alpha_{kljm}. \qquad (23)$$

5) Since

$$(e_j,e_k,e_l)=-(e_l,e_k,e_j) \qquad (24)$$

then there are both

$$(e_j,e_k,e_j)=0 \quad \text{(no summation)} \qquad (25)$$

and

$$(a,b,a)=0, \qquad (26)$$

i.e., the flexible law.

6) In general case ($p\geq16$) there is no alternative law

$$(a,a,b)\neq0,\qquad (a,b,b)\neq0 \qquad (27)$$

however, the equalities for the basis units

$$(e_j,e_j,e_l)=0,\qquad (e_j,e_k,e_k)=0 \quad \text{(no summation)} \qquad (28)$$

are valid for all A_q ("quasialternativity").

For the quaternions ($p=4$) $\alpha_{jklm}=0$ and we have noncommutative, but associative division algebra. For the octonions ($p=8$)

$$\alpha_{jklm}=\alpha_{ljkm}=\alpha_{kljm}, \qquad (29)$$

α_{jklm} is totally antisymmetric (alternativity). (30)

The algebra A_3 is noncommutative, nonassociative, but alternative division algebra. For the sedenions (A_4) for many sets $jklm$ also there are equalities (29) and alternativity, while for others $jklm$ two of quantities (23) equal zero, and the third one +2 or –2. For a more detailed description of associators see Appendix A.

7) α_{jklm} is totally antisymmetric (alternative) if one of indices equals 15 in process (17.a) or 8 in process (17.b). In other words, all associators (a,b,e_{15}) or (a,b,e_8) are alternative in processes (17.a) or (17.b), respectively. This exceptional role of e_{15} or e_8

follows from the analysis of eqs. (17.a) and (17.b), and can be observed from tables 4 and 8 of Appendix A. Conversely, this property permits us to obtain the relations

$$a(e_{15}b') = (ae_{15})b' - (a,e_{15},b') = (e_{15}\bar{a})b' - (a,e_{15},b') =$$
$$= e_{15}(\bar{a}b') + (e_{15},\bar{a},b') - (a,e_{15},b') = e_{15}(\bar{a}b'),$$
$$(e_{15}a')b = (\bar{a'}e_{15})b = \bar{a'}(e_{15}b) + (\bar{a'},e_{15},b) = \bar{a'}(\bar{b}e_{15}) + (\bar{a'},e_{15},b) =$$
$$= (\bar{a'}\bar{b})e_{15} - (\bar{a'},\bar{b},e_{15}) + (\bar{a'},e_{15},b) = (\bar{a'}\bar{b})e_{15} = e_{15}(ba'),$$
$$(e_{15}a')(e_{15}b') = (e_{15}a')(\bar{b'}e_{15}) = e_{15}(a'\bar{b'})e_{15} = -b'\bar{a'}; \qquad (31.a)$$
$$a(b'e_8) = a(e_8\bar{b'}) = (ae_8)\bar{b'} - (a,e_8,\bar{b'}) = (e_8\bar{a})\bar{b'} - (a,e_8,\bar{b'}) =$$
$$= e_8(\bar{a}\bar{b'}) + (e_8,\bar{a},\bar{b'}) - (a,e_8,\bar{b'}) = e_8(\bar{a}\bar{b'}) = (b'a)e_8,$$
$$(a'e_8)b = a'(e_8b) + (a',e_8,b) = a'(\bar{b}e_8) + (a',e_8,b) =$$
$$= (a'\bar{b})e_8 - (a',\bar{b},e_8) + (a',e_8,b) = (a'\bar{b})e_8,$$
$$(a'e_8)(b'e_8) = (e_8\bar{a'})(b'e_8) = e_8(\bar{a'}b')e_8 = -\bar{b'}a' \qquad (31.b)$$

and to check eqs. (17.a) and (17.b) like in the octonion case (in the third eqs. (31.a) and (31.b) the Moufang identity /10/ is used). Similarly for higher algebras A_q the associators (a,b,e_{2p-1}) in process (17.a) and (a,b,e_p) in process (17.b) are alternative (together with $\alpha_{(2p-1)klm}$ or α_{pklm}) too. Using eqs. (19),(31),(1), and (36) we can obtain for associators with large index elements $e_J \equiv e_{j+7} = e_{15}e_j$ for process (17.a) and $e_J \equiv e_{j+8} = e_je_8$ for process (17.b) (j,k,l = = 1,...,7) the relations

$$(e_{15}e_j,e_k,e_l) = 2\varepsilon_{jkl}e_{15} - 2e_{15}(\delta_{jk}e_l - \delta_{jl}e_k),$$
$$(e_j,e_{15}e_k,e_l) = 2\varepsilon_{jkl}e_{15} + 2e_{15}(\delta_{jk}e_l - \delta_{kl}e_j) + e_{15}(e_j,e_k,e_l),$$
$$(e_j,e_k,e_{15}e_l) = 2\varepsilon_{jkl}e_{15} + 2e_{15}(\delta_{kl}e_j - \delta_{jl}e_k),$$
$$(e_{15}e_j,e_k,e_{15}e_l) = 2(\delta_{jk}e_l - \delta_{kl}e_j) + (e_j,e_k,e_l),$$
$$(e_{15}e_j,e_{15}e_k,e_l) = 2(\delta_{kl}e_j - \delta_{jl}e_k),$$
$$(e_j,e_{15}e_k,e_{15}e_l) = -2(\delta_{jk}e_l - \delta_{jl}e_k),$$
$$(e_{15}e_j,e_{15}e_k,e_{15}e_l) = -2\varepsilon_{jkl}e_{15} + e_{15}(e_j,e_k,e_l), \qquad (32.a)$$
$$(e_je_8,e_k,e_l) = -2\varepsilon_{jkl}e_8 - 2(\delta_{jk}e_l - \delta_{jl}e_k)e_8,$$
$$(e_j,e_ke_8,e_l) = -2\varepsilon_{jkl}e_8 + 2(\delta_{jk}e_l - \delta_{kl}e_j)e_8 + (e_j,e_k,e_l)e_8,$$
$$(e_j,e_k,e_le_8) = -2\varepsilon_{jkl}e_8 + 2(\delta_{lk}e_j - \delta_{jl}e_k)e_8,$$
$$(e_je_8,e_k,e_le_8) = 2(\delta_{jk}e_l - \delta_{kl}e_j) + (e_j,e_k,e_l),$$
$$(e_je_8,e_ke_8,e_l) = -2(\delta_{jl}e_k - \delta_{kl}e_j),$$
$$(e_j,e_ke_8,e_le_8) = 2(\delta_{jl}e_k - \delta_{jk}e_l),$$
$$(e_je_8,e_ke_8,e_le_8) = 2\varepsilon_{jkl}e_8 + (e_j,e_k,e_l)e_8. \qquad (32.b)$$

For nonalternative associators from eqs. (32) there follow

$$(e_j,e_k,e_{15}e_l) + (e_j,e_{15}e_l,e_k) = e_{15}(e_j,e_k,e_l),$$
$$(e_{15}e_j,e_k,e_{15}e_l) + (e_{15}e_j,e_{15}e_l,e_k) = (e_j,e_k,e_l), \qquad (33.a)$$
$$(e_j,e_k,e_le_8) + (e_j,e_le_8,e_k) = -(e_j,e_k,e_l)e_8,$$
$$(e_je_8,e_k,e_le_8) + (e_je_8,e_le_8,e_k) = (e_j,e_k,e_l), \qquad (33.b)$$

while (e_j,e_k,e_l) and $(e_{15}e_j,e_{15}e_k,e_{15}e_l)$ or (e_je_8,e_ke_8,e_le_8) are alternative.

Let us consider the double commutator

$$[a[bc]] = 4a_kb_lc_m\varepsilon_{jkn}\varepsilon_{nlm}e_j. \qquad (34)$$

From the identity (see Appendix D)

$$[a[bc]] = \{c\{ab\}\} - \{b\{ca\}\} - (a,b,c) + (c,a,b) - (b,c,a) +$$
$$+ (c,b,a) - (b,a,c) + (a,c,b) \qquad (35)$$

valid in any associative and nonassociative cases, it follows that

$$\varepsilon_{jkn}\varepsilon_{nlm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} + \beta_{jklm}, \qquad (36)$$

where

$$\beta_{jklm} = \tfrac{1}{2}(\alpha_{jklm} + \alpha_{ljkm} - \alpha_{kljm}). \qquad (37)$$

It is clear from eq. (36) (and from eq. (37) too) that β_{jklm} is antisymmetric in 1st and 2nd indices, antisymmetric in 3rd and 4th indices, and does not alter by interchange of the first pair of indices with the second one

$$\beta_{jklm} = -\beta_{kjlm} = -\beta_{jkml} = \beta_{lmjk}. \qquad (38)$$

All 4! components of β_{jklm} with fixed jklm can be expressed (using eqs. (38)) via three of them, e.g., via

$$\beta_{jklm} = \tfrac{1}{2}(\alpha_{jklm} + \alpha_{ljkm} - \alpha_{kljm}),$$
$$\beta_{ljkm} = \tfrac{1}{2}(-\alpha_{jklm} + \alpha_{ljkm} + \alpha_{kljm}),$$
$$\beta_{kljm} = \tfrac{1}{2}(\alpha_{jklm} - \alpha_{ljkm} + \alpha_{kljm}). \qquad (39)$$

These relations can be resolved with respect α to give eq. (20). For the quaternions β =0, and eq. (36) represents the identity

$$[\vec{a}[\vec{b}\vec{c}]] = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b}) \qquad (40)$$

for the usual double vector product in the 3-dimensional vector algebra. It demonstrates that three vectors $\vec{b}$, $\vec{c}$ and $[\vec{b}\vec{c}]$, being linearly independent, form a basis in the 3-dimensional space, and the double vector product cannot generate a new vector . In the 7-dimensional vector algebra corresponding to octonions three linearly independent vectors must be given, and then we obtain the basis

$\vec{b},\vec{c},\vec{d},[\vec{b}\vec{c}],[\vec{b}\vec{d}],[\vec{c}\vec{d}]$ and $[\vec{d}[\vec{b}\vec{c}]]$ /8/. Now only the triple vector product does not longer generate new vector. One can obtain for the triple vector product the following rather complicated identity

$$
\begin{aligned}
[[ab][cd]] = {} & \tfrac{1}{2}\{a\{b[cd]\}\} - \tfrac{1}{2}\{b\{a[cd]\}\} - \tfrac{1}{2}\{c\{d[ab]\}\} + \tfrac{1}{2}\{d\{c[ab]\}\} - \\
& -([ab],c,d) + (d,[ab],c) - (c,d,[ab]) + \\
& +([cd],a,b) - (b,[cd],a) + (a,b,[cd]) = \qquad (41.a)
\end{aligned}
$$

$$
\begin{aligned}
= {} & \tfrac{1}{2}\{a\{d[bc]\}\} + \tfrac{1}{2}\{d\{a[bc]\}\} + \tfrac{1}{2}\{b\{c[ad]\}\} + \tfrac{1}{2}\{c\{b[ad]\}\} - \\
& - \tfrac{1}{2}\{a\{c[bd]\}\} - \tfrac{1}{2}\{c\{a[bd]\}\} - \tfrac{1}{2}\{b\{d[ac]\}\} - \tfrac{1}{2}\{d\{b[ac]\}\} + \\
& + \{\{ac\}[bd]\} - \{\{ad\}[bc]\} + \{[ac]\{bd\}\} - \{[ad]\{bc\}\} + \\
& + 2(b,c,[ad]) + 2(c,b,[ad]) - 2(a,c,[bd]) - 2(c,a,[bd]) - \\
& - 2(b,d,[ac]) - 2(d,b,[ac]) + 2(a,d,[bc]) + 2(d,a,[bc]) - \\
& - [a,(b,c,d)+(d,b,c)+(c,d,b)] + [b,(a,c,d)+(d,a,c)+(c,d,a)] + \\
& + [c,(d,a,b)+(b,d,a)+(a,b,d)] - [d,(c,a,b)+(b,c,a)+(a,b,c)] - \\
& - ([ab],c,d) - (d,[ab],c) - (c,d,[ab]) + \\
& + (a,b,[cd]) + ([cd],a,b) + (b,[cd],a). \qquad (41.b)
\end{aligned}
$$

It is valid for all the hypercomplex algebras (but not for all the nonassociative quantities because in the derivation we use the fact that associators (a,b,c) are antisymmetric under interchange of 1st and 3rd elements). From eq. (41.b) it follows the identity

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{khn}\varepsilon_{nlm} = {} & \delta_{ih}\varepsilon_{jlm} - \delta_{jh}\varepsilon_{ilm} - \delta_{hl}\varepsilon_{ijm} + \delta_{hm}\varepsilon_{ijl} - \\
& - \delta_{il}\varepsilon_{jhm} + \delta_{im}\varepsilon_{jhl} + \delta_{jl}\varepsilon_{ihm} - \delta_{jm}\varepsilon_{ihl} + \gamma_{ijhlm}, \qquad (42)
\end{aligned}
$$

where γ_{ijhlm} represents the totality of terms with associators in eq. (41.b), whence it can be easily expressed. For quaternions $\gamma = 0$, since the associators are equal to zero. For the octonions $\gamma = 0$ too, since the associators in eq. (41.b) cancel. Hence the triple vector product generates no new vector. For the sedenions, however, $\gamma \neq 0$. Only a fourfold vector product produces no a new vector.

<u>Trace of a hypercomplex number</u> is defined to be the doubled its real part

$$\mathrm{tr}\,a = a + \bar{a} = 2\,\mathrm{Re}\,a \qquad (43)$$

with the properties

$$\mathrm{tr}(ab) = \mathrm{tr}(ba), \qquad (44)$$

$$\mathrm{tr}(a(bc)) = \mathrm{tr}((ab)c), \qquad (45)$$

the latter is due to the associator being always a purely imaginary quantity (see eq. (19)). Due to the identity

$$a^2 - \mathrm{tr}(a)a + n(a)e_0 = 0, \qquad n(a)e_0 = \bar{a}a = a\bar{a} \qquad (46)$$

for any hypercomplex number it is said that each algebra A_q is of power 2, and therefore is a power-associative algebra $(a^m a^n = a^{m+n})$ /7/.

<u>Derivation algebra of A_q and automorphism group</u>. A derivation D of a nonassociative algebra $\mathcal{A}$ is a linear transformation with the property

$$D(xy) = (Dx)y + x(Dy). \quad \text{for all } x, y \in \mathcal{A} \qquad (47)$$

A derivation Lie algebra $D(A_q)$ of any algebra A_q, obtained from the octonion algebra A_3 by iterations of the Cayley-Dickson process, is isomorphic to the derivation Lie algebra $D(A_3)$ of the octonions, $D(A_q) \cong D(A_3)$, i.e., to the 14-parameter Lie algebra G_2 /7/. The infinitesimal G_2 transformations, derivations of octonions, are known to be

$$\delta x = Dx = [[ab]x] - 3(a,b,x) \qquad a,b,x \in A_3 \qquad (48)$$

and a finite G_2 transformation for the octonions can be symbolically written as follows

$$x' = e^{ad}x, \qquad ad\,x = [[ab]x] - 3(a,b,x). \qquad (49)$$

If one represents an arbitrary hypercomplex number

$$x = x_1 + x_2 e_8 + (x_3 + x_4 e_8)e_{16} + (x_5 + x_6 e_8 + (x_7 + x_8 e_8)e_{16})e_{32} + \dots \qquad (50)$$

where $x_1, x_2, x_3, \dots$ are octonions. Then according to the Schafer theorem the infinitesimal and finite G_2 transformations of x can be written as follows

$$
\begin{aligned}
\delta x \equiv Dx = {} & [\alpha[\beta x_1]] - 3(\alpha,\beta,x_1) + ([\alpha[\beta x_2]] - 3(\alpha,\beta,x_2))e_8 + \\
& + ([\alpha[\beta x_3]] - 3(\alpha,\beta,x_3) + ([\alpha[\beta x_4]] - 3(\alpha,\beta,x_4))e_8)e_{16} + \cdots, \qquad (51)
\end{aligned}
$$

$$x' = e^{ad}x, \qquad ad\,x \equiv Dx. \qquad (52)$$

The group (algebra) G_2 is the automorphism group (algebra) of A_q. A linear transformation

$$\delta a_i = \omega_{ij}a_j \quad (\delta a = \omega_{ij}a_j e_i) \quad (i,j = 1,2,\dots,p-1) \qquad (53)$$

will be automorphism of A_q if $\omega_{ij} = -\omega_{ji}$,

$$\omega_{ii'}\varepsilon_{i'jk} + \omega_{jj'}\varepsilon_{ij'k} + \omega_{kk'}\varepsilon_{ijk'} = 0. \qquad (54)$$

The latter is satisfied by any $\omega_{ij} = -\omega_{ji}$ in the case of quaternions. In the case of octonions eqs. (54) may be reduced to the following seven equations (with the use of table 1 of Appendix A)

$$
\begin{aligned}
& -\omega_{23} + \omega_{74} - \omega_{65} = 0, && \omega_{16} - \omega_{27} + \omega_{43} = 0, \\
& -\omega_{31} - \omega_{57} + \omega_{64} = 0, && \omega_{15} + \omega_{42} + \omega_{37} = 0, \\
& \omega_{12} - \omega_{45} + \omega_{67} = 0, && -\omega_{14} - \omega_{25} + \omega_{63} = 0, \\
& \omega_{17} + \omega_{26} + \omega_{53} = 0, && \qquad (55)
\end{aligned}
$$

If we choose all ω_{ij} to be zero except for ω_{ij} entering into the first equation (55), we obtain an automorphism transformation leaving e_1 invariant (cf. refs.[8,9]). Analogously, for $e_2, e_3, \ldots e_7$. Among 21 parameters ω_{ij} entering into eqs. (55) only 14 are independent (the group G_2). A solution of eqs. (54) (with arbitrary choice of ε_{jkl}) can be written as follows

$$\omega_{mn} = \alpha_j \beta_k (4\varepsilon_{jkl}\varepsilon_{lmn} - 3\alpha_{jkmn}) =$$

$$= 2\alpha_j \beta_k (3\delta_{jm}\delta_{kn} - 3\delta_{jn}\delta_{kn} - \varepsilon_{jkl}\varepsilon_{lmn}), \tag{56}$$

where α_i and β_j are arbitrary constants, and it can be checked with the use of identity (42). Equations (53),(56) prove eq. (48).

In other cases (sedenions, ...) the totality of equations (54) make many of ω_{ij} to be zero and others are expressed via 14 independent ω_{ij} entering into eqs. (55) (again G_2). For sedenions this can be checked directly, thus illustrating the Schafer theorem[7].

The trace is invariant under the automorphism transformations

$$tr\, x' = tr\, x \ , \qquad tr\, \delta x = 0. \tag{57}$$

5. Quantum mechanics with A_q as algebras of observables.

Let as assume pure imaginary hypercomplex numbers $a_m e_m$, or better $i a_m e_m$ with the imaginary unit i , to be observables. The i makes the observables be Hermitian and eigenvalues and expectation values be real. The quaternionic quantum mechanics (quantum mechanics of spin 1/2) is well-known; it uses the Pauli matrices σ_j i.e., quaternions, multiplied by i as observables. An octonionic quantum mechanics [16] and quantum mechanics of higher hypercomplex numbers can be constructed analogously.

Eigenvalues and eigenstates. Let us solve the eigenvalue problem for e_1 (quanternion, octonion, sedenion or higher hypercomplex number)

$$e_1 \rho = \lambda_1 \rho \ , \qquad \rho e_1 = \mu_1 e_1 \ , \tag{58}$$

where $\rho = \rho_0 e_0 + \rho_m e_m$ is unknown, and λ_1 and μ_1 are eigenvalues. We have as solutions (according to table 3 of Appendix A)

$$\lambda_1 = \mu_1 = \pm i \quad \tfrac{1}{2}(e_0 \mp i e_1) \quad \left.\begin{array}{l} = \left\{ \begin{array}{l} |\lambda_1 = i\rangle\langle\mu_1 = i| \\ |\lambda_1 = -i\rangle\langle\mu_1 = -i| \end{array}\right. , \\ \\ = \left\{ \begin{array}{l} |\lambda_1 = i\rangle\langle\mu_1 = -i| \\ |\lambda_1 = -i\rangle\langle\mu_1 = i| \end{array}\right. , \end{array}\right\} \text{for quaternions}$$

$$\lambda_1 = -\mu_1 = \pm i \left\{ \begin{array}{ll} \tfrac{1}{2}(e_2 \mp i e_3) & \\ \tfrac{1}{2}(e_4 \mp i e_7), & \tfrac{1}{2}(e_9 \pm i e_{10}), \\ \tfrac{1}{2}(e_5 \pm i e_6), & \tfrac{1}{2}(e_{11} \pm i e_{14}), \\ \tfrac{1}{2}(e_8 \mp i e_{15}), & \tfrac{1}{2}(e_{12} \mp i e_{13}) \end{array}\right. \tag{59}$$

for the sedenions. There exist only four first solutions in the case of the quaternions. The first two of them are density operators orthogonal to each other. The latter two represent the nondiagonal direct products of bras and kets (see expressions in parentheses in eq. (59)). In other cases (octonions, sedenions, ...) we also have two such sets. First of them contains also two solutions, two mutually orthogonal density operators. The second set includes the remaining 6,14, ... solutions.

Among eigenstates of e_2 one can found the density operators

$$\rho(\lambda_2 = \pm i) = \tfrac{1}{2}(e_0 \mp i e_2) \tag{60}$$

and in the case of an arbitrary axis $\vec{c}$

$$\rho(\lambda_c = \pm i) = \tfrac{1}{2}(e_0 \mp i c_j e_j) \ , \qquad c_j c_j = 1 \ . \tag{61}$$

The set of operators (59) is complete and orthogonal, when the operation $tr = 2Re$ is used to form a scalar product. The set of the density operators $\tfrac{1}{2}(e_0 \mp i e_k)$ $(k = 1, 2, \ldots , p-1)$ is complete, but not orthogonal.

A probability for finding one state in another is given

$$w(\lambda_b = i, \lambda_a = i) = tr\left[\tfrac{1}{2}(e_0 - i\vec{b}\vec{e})\tfrac{1}{2}(e_0 - i\vec{a}\vec{e})\right] = \tfrac{1}{2}(1 + \vec{a}\vec{b}),$$

$$w(\lambda_b = i, \lambda_a = i) + w(\lambda_b = -i, \lambda_a = i) = \tfrac{1}{2}(1 + \vec{a}\vec{b}) + \tfrac{1}{2}(1 - \vec{a}\vec{b}) = 1 \tag{62}$$

for example,

$$w(\lambda_1 = i, \lambda_1 = i) = 1, \quad w(\lambda_1 = -i, \lambda_1 = -1) = 1, \quad w(\lambda_1 = -i, \lambda_1 = i) = 0$$

$$w(\lambda_1 = i, \lambda_c = i) = \tfrac{1}{2}(1 + c_1), \quad w(\lambda_1 = -i, \lambda_c = i) = \tfrac{1}{2}(1 - c_1) \ . \tag{63}$$

These are valid for both quaternion, octonion, sedenion, etc., cases. In the quaternion one the latter two probabilities are in fact the well-known Pauli result.

Equations (62) prove that transition probabilities w are always positive and their sum is equal to 1. These properties are conserved in the course of time evolution, since any transformation of the automorphism group preserves the form (61) ($\vec{c} \to \vec{c}\,'$ with $\vec{c}\,'\vec{c}\,' = 1$).

Expectation value of an operator F is defined as usual

$$tr(\rho F) \tag{64}$$

For example, for $F = e_1$ ($= i\sigma_2$ for quaternions) and $\rho = \tfrac{1}{2}(e_0 - i e_1)$

$$tr\left(e_1 \tfrac{1}{2}(e_0 - i e_1)\right) = i \tag{65}$$

Therefore, the quantities $i e_j$ can serve as Hermitian operators.

Equations of motion for density operator and for observables.

In the case of quaternions evolution of a density operator ρ in the Schrödinger picture and any operator F, which does not depend explicitly on time, in the Heisenberg picture is governed by the Neumann (Liouville) and Heisenberg-Born-Jordan-Dirac equations

$$\frac{d}{dt}\rho(t) = -[\gamma, \rho(t)], \qquad \frac{d}{dt}F = 0, \tag{66}$$

$$\frac{d}{dt}F(t) = [\gamma, F(t)], \qquad \frac{d}{dt}\rho = 0 \tag{67}$$

with the formal solutions

$$\rho(t) = e^{-\gamma t}\rho(0)\,e^{\gamma t}, \tag{68}$$

$$F(t) = e^{\gamma t}\,F(0)\,e^{-\gamma t}. \tag{69}$$

The Hamiltonian γ is a pure imaginary quaternion. The evolution laws (68) and (69) are transformations of the 1-parameter subgroup of automorphism group of the quaternion algebra.

Analogously for other algebras A_q we also assume 1-parameter subgroups of the automorphism group G_2 to be the evolution laws

$$\rho(t) = e^{-t\,\mathrm{ad}}\,\rho(0), \tag{70}$$

$$F(t) = e^{t\,\mathrm{ad}}\,F(0), \tag{71}$$

where ad is defined by eqs. (51),(52). As equations of motion in the octonion case we get the Lie group equations

$$\frac{d}{dt}\rho(t) = -[[\alpha\beta]\rho(t)] + 3(\alpha,\beta,\rho(t)), \tag{72}$$

$$\frac{d}{dt}F(t) = [[\alpha\beta]F(t)] - 3(\alpha,\beta,F(t)), \tag{73}$$

where α and β are two imaginary octonions, which together play the role of Hamiltonian. These equations generalize the Neumann (Liouville) and Heisenberg-Born-Jordan-Dirac equations to the octonion case. Note the condition of conservation in time

$$[[\alpha\beta]\rho] = 3(\alpha,\beta,\rho). \tag{74}$$

For other algebras A_q, formed by the Dickson process (17.b) according to Schafer /7/, we can represent ρ as follows

$$\rho = \rho_1 + \rho_2 e_8 + (\rho_3 + \rho_4 e_8)e_{16} + \cdots, \tag{75}$$

where $\rho_1, \rho_2, \rho_3, \rho_4, \ldots$ are octonions, and equations of motion can be written as follows

$$\frac{d}{dt}\rho = -[[\alpha\ \beta]\rho_1] + 3(\alpha,\beta,\rho_1) - ([[\alpha\beta]\rho_2] - 3(\alpha,\beta,\rho_2))e_8 - \\ -\big(([[\alpha\beta]\rho_3] - 3(\alpha,\beta,\rho_3)) + ([[\alpha\beta]\rho_4] - 3(\alpha,\beta,\rho_4))e_8\big)e_{16} - \cdots, \tag{76}$$

$$\frac{d}{dt}F = [[\alpha\beta]F_1] - 3(\alpha,\beta,F_1) + ([[\alpha\beta]F_2] - 3(\alpha,\beta,F_2))e_8 + \\ +\big(([[\alpha\beta]F_3] - 3(\alpha,\beta,F_3)) + ([[\alpha\beta]F_4] - 3(\alpha,\beta,F_4))e_8\big)e_{16} + \cdots. \tag{77}$$

For sedenions only ρ_1 and ρ_2, and F_1 and F_2 are nonzero.

All the above evolution laws obey the property

$$\mathrm{tr}(F_1(0)\ldots F_n(0)\rho(t)) = \mathrm{tr}(F_1(t)\ldots F_n(t)\rho(0)). \tag{78}$$

Note also the conservation of the total probability

$$\mathrm{tr}\,\dot{\rho}(t) = 0, \qquad \mathrm{tr}\,\rho(t) = \mathrm{tr}\,\rho(0). \tag{79}$$

6. Matrix representations. Any hypercomplex number F of some algebra A_q is completely defined by its representative

$$\mathrm{tr}(\bar{e}_\mu F) \qquad (\mu = 0, 1, \ldots, p-1), \tag{80}$$

thought to be a column;

$$F = e_\mu \mathrm{tr}(\bar{e}_\mu F). \tag{81}$$

Multiplications of F by a hypercomplex number a from the left or right may be written as operators (matrices) acting on representative (80)x)

$$\mathrm{tr}(\bar{e}_\mu(aF)) = (a^\ell)_{\mu\nu}\mathrm{tr}(\bar{e}_\nu F) \equiv a^\ell\,\mathrm{tr}(\bar{e}_\mu F), \tag{82}$$

$$\mathrm{tr}(\bar{e}_\mu(Fa)) = (a^r)_{\mu\nu}\mathrm{tr}(\bar{e}_\nu F) \equiv a^r\,\mathrm{tr}(\bar{e}_\mu F), \tag{83}$$

a^ℓ and a^r being called the left and right (matrix) representatives xx). This is clear since this is valid for the basis elements:

$$\mathrm{tr}(\bar{e}_\mu(e_j F)) = (\delta_{\mu j}\delta_{\nu 0} - \delta_{\mu 0}\delta_{\nu j} + \varepsilon_{0\mu j\nu})\mathrm{tr}(\bar{e}_\nu F) = (e_j^\ell)_{\mu\nu}\mathrm{tr}(\bar{e}_\nu F), \tag{84}$$

$$\mathrm{tr}(\bar{e}_\mu(F e_j)) = (\delta_{\mu j}\delta_{\nu 0} - \delta_{\mu 0}\delta_{\nu j} - \varepsilon_{0\mu j\nu})\mathrm{tr}(\bar{e}_\nu F) = (e_j^r)_{\mu\nu}\mathrm{tr}(\bar{e}_\nu F), \tag{85}$$

where eqs. (1) and (45) are used, $\varepsilon_{0mjn} = \varepsilon_{mjn}$. Hence due to arbitrariness of F the left and right matrix representatives of e_j are

$$\left(e_j^{\substack{\ell \\ r}}\right)_{\mu\nu} = \delta_{\mu j}\delta_{\nu 0} - \delta_{\mu 0}\delta_{\nu j} \pm \varepsilon_{0\mu j\nu}, \quad e_0^\ell = e_0^r = 1, \text{ the unit pxp matrix} \tag{86}$$

$$e_j^r = -\eta e_j^\ell \eta, \qquad \eta = \tfrac{1}{2}(e_0^\ell + e_j^\ell e_j^r), \qquad \eta^2 = e_0^\ell. \tag{87}$$

The matrix representatives of quaternion, octonion and sedenion basis elements are given in Appendix C. Left and right representatives commute mutually in all associative algebras (e.g., in the conventional quantum mechanics and quantum field theory, cf. ref. /30/), and, in particular, for quaternions

$$[e_j^\ell e_k^r] = 0. \tag{88}$$

However, they do not commute, in general, in nonassociative algebras, so

---

x) The last symbolic form will be used below.

xx) Our notation arose by following closely to the spirit of the Dirac representstion theory. They seem to be more indicative than notation R_a and L_a used in algebra and called the right and left multiplications (see, e.g., ref. /10/). Note that other realizations of e_j^l and e_j^r are also possible (see ref. /16/).

$$[e_j^\ell, e_k^r] = [e_j^r, e_k^\ell] \neq 0 \tag{89}$$

for octonions, sedenions, etc. However,

$$[e_j^\ell, e_j^r] = 0 \qquad \text{(no summation)}. \tag{90}$$

A repeated action on F is converted into the usual matrix multiplication of representatives:

$$\mathrm{tr}(\bar{e}_\mu(a(bF))) = a^\ell b^\ell \, \mathrm{tr}(\bar{e}_\mu F), \tag{91}$$

$$\mathrm{tr}(\bar{e}_\mu(a(Fb))) = a^\ell b^r \, \mathrm{tr}(\bar{e}_\mu F), \tag{92}$$

$$\mathrm{tr}(\bar{e}_\mu((Fa)b)) = b^r a^r \, \mathrm{tr}(\bar{e}_\mu F). \tag{93}$$

Representatives of a product of two hypercomplex numbers in terms of representatives of factors can be found as follows

$$\begin{aligned}(ab)^\ell \, \mathrm{tr}(\bar{e}_\mu F) &= \mathrm{tr}(\bar{e}_\mu((ab)F)) = \mathrm{tr}(\bar{e}_\mu(a(bF) + (a,b,F))) = \\ &= \mathrm{tr}(\bar{e}_\mu(a(bF) - (a,F,b) + (a,b,F) + (a,F,b))) = \\ &= (a^\ell b^\ell + [a^\ell b^r]) \, \mathrm{tr}(\bar{e}_\mu F) + \mathrm{tr}(\bar{e}_\mu((a,b,F) + (a,F,b))),\end{aligned} \tag{94}$$

$$\begin{aligned}(ab)^r \, \mathrm{tr}(\bar{e}_\mu F) &= \mathrm{tr}(\bar{e}_\mu(F(ab))) = \mathrm{tr}(\bar{e}_\mu((Fa)b - (F,a,b))) = \\ &= \mathrm{tr}(\bar{e}_\mu((Fa)b + (a,F,b) - (F,a,b) - (a,F,b))) = \\ &= (b^r a^r + [b^\ell a^r]) \, \mathrm{tr}(\bar{e}_\mu F) + \mathrm{tr}(\bar{e}_\mu((b,a,F) + (b,F,a))).\end{aligned} \tag{95}$$

Note that

$$[a^\ell b^r] = [a^r b^\ell] \tag{96}$$

due to eq. (24). For octonions the last term $\mathrm{tr}(\bar{e}_\mu((a,b,F)+(a,F,b)))$ vanishes due to the alternativity, and the desired representatives have been exhibited (see also ref. /16/). For sedenions the last term also vanishes if a or b equals e_{15} in process (17.a) or e_8 in process (17.b). In other situations it can be transformed, using the equations

$$\mathrm{tr}(\bar{e}_\mu((e_j,e_k,F)+(e_j,F,e_k))) =$$

$$= f_{\ell+7} \, \mathrm{tr}(\bar{e}_\mu(e_{15}(e_j,e_k,e_\ell))) = -e_{15}^\ell [e_j^\ell e_k^r] e_{15}^\ell \, \mathrm{tr}(\bar{e}_\mu f_{\ell+7}(e_{15} e_\ell)) =$$

$$= -\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} e_{15}^\ell [e_j^\ell e_k^r] e_{15}^\ell \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \mathrm{tr}(e_\mu F) = \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix} [e_j^\ell e_k^r] \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F),$$

$$\mathrm{tr}(\bar{e}_\mu((e_j,e_{15}e_k,F)+(e_j,F,e_{15}e_k))) =$$

$$= f_\ell \, \mathrm{tr}(\bar{e}_\mu(e_{15}(e_j,e_\ell,e_k))) = -e_{15}^\ell [e_j^\ell e_k^r] \, \mathrm{tr}(\bar{e}_\mu f_\ell e_\ell) =$$

$$= -\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} e_{15}^\ell [e_j^\ell e_k^r] \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F) = -\begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix} [e_j^\ell e_k^r] \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F),$$

$$\mathrm{tr}(\bar{e}_\mu((e_{15}e_j,e_k,F)+(e_{15}e_j,F,e_k))) =$$

$$= f_{\ell+7} \, \mathrm{tr}(\bar{e}_\mu(e_j,e_k,e_\ell)) = -[e_j^\ell e_k^r] e_{15}^\ell \, \mathrm{tr}(\bar{e}_\mu f_{\ell+7}(e_{15} e_\ell)) =$$

$$= -\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} [e_j^\ell e_k^r] e_{15}^\ell \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F) = \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} [e_j^\ell e_k^r] \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F),$$

$$\mathrm{tr}(\bar{e}_\mu((e_{15}e_j,e_{15}e_k,F)+(e_{15}e_j,F,e_{15}e_k))) =$$

$$= f_\ell \, \mathrm{tr}(\bar{e}_\mu(e_j,e_\ell,e_k)) = -[e_j^\ell e_k^r] \, \mathrm{tr}(\bar{e}_\mu f_\ell e_\ell) =$$

$$= -\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} [e_j^\ell e_k^r] \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F), \tag{97.a}$$

$$\mathrm{tr}(\bar{e}_\mu((e_j,e_k,F)+(e_j,F,e_k))) =$$

$$= f_{\ell+8} \, \mathrm{tr}(\bar{e}_\mu((e_j,e_\ell,e_k)e_8)) = -e_8^r [e_j^\ell e_k^r] e_8^\ell \, \mathrm{tr}(\bar{e}_\mu f_{\ell+8}(e_\ell e_8)) =$$

$$= -\begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix} e_8^r [e_j^\ell e_k^r] e_8^\ell \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F) = -\begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix} [e_j^\ell e_k^r] \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F),$$

$$\mathrm{tr}(\bar{e}_\mu((e_j,e_k e_8,F)+(e_j,F,e_k e_8))) =$$

$$= f_\ell \, \mathrm{tr}(\bar{e}_\mu((e_j,e_k,e_\ell)e_8)) = e_8^r [e_j^\ell e_k^r] \, \mathrm{tr}(\bar{e}_\mu f_\ell e_\ell) =$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix} e_8^r [e_j^\ell e_k^r] \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F) = \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix} [e_j^\ell e_k^r] \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F),$$

$$\mathrm{tr}(\bar{e}_\mu((e_j e_8,e_k,F)+(e_j e_8,F,e_k))) =$$

$$= f_{\ell+8} \, \mathrm{tr}(\bar{e}_\mu(e_j,e_k,e_\ell)) = [e_j^\ell e_k^r] e_8^\ell \, \mathrm{tr}(\bar{e}_\mu f_{\ell+8}(e_\ell e_8)) =$$

$$= \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} [e_j^\ell e_k^r] e_8^\ell \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F) = \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} [e_j^\ell e_k^r] \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F),$$

$$\mathrm{tr}(\bar{e}_\mu((e_j e_8,e_k e_8,F)+(e_j e_8,F,e_k e_8))) =$$

$$= f_\ell \, \mathrm{tr}(\bar{e}_\mu(e_j,e_\ell,e_k)) = -[e_j^\ell e_k^r] \, \mathrm{tr}(\bar{e}_\mu f_\ell e_\ell) =$$

$$= -\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} [e_j^\ell e_k^r] \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \mathrm{tr}(\bar{e}_\mu F). \tag{97.b}$$

To prove eqs. (97) the decompositions $F = f_0 e_0 + f_\ell e_\ell + f_{\ell+7} e_{15} e_\ell + f_{15} e_{15}$ and $F = f_0 e_0 + f_\ell e_\ell + f_8 e_8 + f_{\ell+8} e_\ell e_8$ ($\ell = 1,\ldots,7$) are inserted in processes (17.a) and (17.b), respectively, and eqs. (32) and (33) are used. The 16x16 matrices $\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}, \ldots$ have entries being 8x8 matrices. In turn,

$$u = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \\ 0 & & \end{bmatrix}, \quad v = \begin{bmatrix} & 1 & 0 \\ & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad w = \begin{bmatrix} 0 & & \\ \vdots & 1 & \\ 0 & \cdots & 0 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \vdots \\ & & 0 \end{bmatrix},$$

where 1 is the unit 7x7 matrix.

In further hypercomplex algebras associators can successively be reduced to those for lower algebras (after all to octonion associators) like in eqs. (32), This will also permit us to transform the last term of eqs. (94) and (95).

Once matrix representation is obtained all expressions and equations can be rewritten in these terms (see ref. /16/).

The author is deeply grateful for programming and computation of tables of Appendix A and those for further hypercomplex numbers to A.I.Polubarinova, and for useful discussions to A.B.Govorkov, who also considered higher hypercomplex numbers with a multiplication table identical to table 7.

Appendix A. The multiplication table of the quaternions (p=4) is defined by the only quantity $\varepsilon_{123} = 1$. In general there are $N_\varepsilon = \frac{1}{3} C^2_{p-1} = \frac{(p-1)(p-2)}{2\cdot 3}$ independent ε_{jkl} , 7 for the octonions (p=8), 35 for the sedenions (p=16), etc. Associators are defined by $N_\alpha = \frac{1}{4}\left(C^3_{p-1} - \frac{1}{3} C^2_{p-1}\right) = \frac{(p-1)(p-2)(p-4)}{1\cdot 2\cdot 3\cdot 4}$ quantities α_{jklm} $N_\alpha = 7$ for the octonions, $N_\alpha = 105$ for the sedenions, etc. Below tables of independent nonzero values of ε_{jkl} , α_{jklm} and β_{jklm} are given. The tables 1,2,3,4 correspond to process (17.a), and tables 5,6,7,8 to process (17.b).

Table 1. ε_{jkl} for octonions.

I,2,3	I
I,4,7	I
I,5,6	-I
2,4,6	I
2,5,7	I
3,4,5	-I
3,6,7	I

Table 2. α_{jklm} and β_{jklm} for octonions

I,2,4,5	-2	-I
I,2,6,7	2	I
I,3,4,6	-2	-I
I,3,5,7	-2	-I
2,3,4,7	2	I
2,3,5,6	-2	-I
4,5,6,7	-2	-I

Table 3. ε_{jkl} for sedenions

I, 2, 3	I	2, 9,I5	I	4, 8,I4	I	6, 8,I2	I
I, 4, 7	I	2,II,I3	-I	4, 9,I3	I	6, 9,II	-I
I, 5, 6	-I	2,I2,I4	-I	4,I0,I2	-I	6,I0,I4	I
I, 8,I5	I			4,II,I5	I	6,I3,I5	I
I, 9,I0	-I	3, 4, 5	-I				
I,II,I4	-I	3, 6, 7	I	5, 8,I3	-I	7, 8,II	-I
I,I2,I3	I	3, 8, 9	-I	5, 9,I4	I	7, 9,I2	-I
		3,I0,I5	I	5,I0,II	I	7,I0,I3	-I
2, 4, 6	I	3,II,I2	I	5,I2,I5	I	7,I4,I5	I
2, 5, 7	I	3,I3,I4	-I				
2, 8,I0	I						

Table 4. $\alpha_{jklm}, \alpha_{ljkm}, \alpha_{kljm}$ and β_{jklm} for sedenions x)

I, 2, 4, 5	-2			-I	2, 5, 8,II	0	0	2	-I	4, 7, 8,I5	2			I
I, 2, 6, 7	2			I	2, 5, 9,I2	-2			-I	4, 7, 9,I0	0	0	2	-I
I, 2, 8, 9	-2			-I	2, 5,I0,I3	0	0	2	-I	4, 7,II,I4	-2			-I
I, 2,I0,I5	2			I	2, 5,I4,I5	2			I	4, 7,I2,I3	0	0	-2	I
I, 2,II,I2	0	0	-2	I	2, 6, 8,I4	0	0	2	-I					
I, 2,I3,I4	0	0	2	-I	2, 6, 9,I3	-2			-I	5, 6, 8,I5	-2			-I
I, 3, 4, 6	-2			-I	2, 6,I0,I2	0	0	-2	I	5, 6, 9,I0	0	0	-2	I
I, 3, 5, 7	-2			-I	2, 6,II,I5	-2			-I	5, 6,II,I4	0	0	-2	I
I, 3, 8,I0	-2			-I	2, 7, 8,I3	0	0	-2	I	5, 6,I2,I3	-2			-I
I, 3, 9,I5	-2			-I	2, 7, 9,I4	-2			-I	5, 7, 8,I0	0	0	-2	I
I, 3,II,I3	0	0	-2	I	2, 7,I0,II	0	0	2	-I	5, 7, 9,I5	2			I
I, 3,I2,I4	0	0	-2	I	2, 7,I2,I5	-2			-I	5, 7,II,I3	0	0	2	-I
I, 4, 8,II	-2			-I						5, 7,I2,I4	-2			-I
I, 4, 9,I2	0	0	2	-I	3, 4, 8,I3	0	0	-2	I					
I, 4,I0,I3	0	0	2	-I	3, 4, 9,I4	0	0	2	-I	6, 7, 8, 9	0	0	2	-I
I, 4,I4,I5	2			I	3, 4,I0,II	-2			-I	6, 7,I0,I5	2			I
I, 5, 8,I2	-2			-I	3, 4,I2,I5	-2			-I	6, 7,II,I2	0	0	-2	I
I, 5, 9,II	0	0	-2	I	3, 5, 8,I4	0	0	-2	I	6, 7,I3,I4	-2			-I
I, 5,I0,I4	0	0	2	-I	3, 5, 9,I3	0	0	-2	I					
I, 5,I3,I5	-2			-I	3, 5,I0,I2	-2			-I	8, 9,I0,I5	-2			-I
I, 6, 8,I3	-2			-I	3, 5,II,I5	2			I	8, 9,II,I2	-2			-I
I, 6, 9,I4	0	0	-2	I	3, 6, 8,II	0	0	2	-I	8, 9,I3,I4	2			I
I, 6,I0,II	0	0	-2	I	3, 6, 9,I2	0	0	2	-I	8,I0,II,I3	-2			-I
I, 6,I2,I5	2			I	3, 6,I0,I3	-2			-I	8,I0,I2,I4	-2			-I
I, 7, 8,I4	-2			-I	3, 6,I4,I5	2			I	8,II,I4,I5	-2			-I
I, 7, 9,I3	0	0	2	-I	3, 7, 8,I2	0	0	2	-I	8,I2,I3,I5	2			I
I, 7,I0,I2	0	0	-2	I	3, 7, 9,II	0	0	-2	I					
I, 7,II,I5	-2			-I	3, 7,I0,I4	-2			-I	9,I0,II,I4	2			I
					3, 7,I3,I5	-2			-I	9,I0,I2,I3	-2			-I
2, 3, 4, 7	2			I						9,II,I3,I5	-2			-I
2, 3, 5, 6	-2			-I	4, 5, 6, 7	-2			-I	9,I2,I4,I5	-2			-I
2, 3, 8,I5	2			I	4, 5, 8, 9	0	0	-2	I					
2, 3, 9,I0	-2			-I	4, 5,I0,I5	-2			-I	I0,II,I2,I5	2			I
2, 3,II,I4	0	0	2	-I	4, 5,II,I2	-2			-I	I0,I3,I4,I5	-2			-I
2, 3,I2,I3	0	0	-2	I	4, 5,I3,I4	0	0	-2	I					
2, 4, 8,I2	0	0	-2	I	4, 6, 8,I0	0	0	-2	I	II,I2,I3,I4	-2			-I
2, 4, 9,II	-2			-I	4, 6, 9,I5	2			I					
2, 4,I0,I4	0	0	-2	I	4, 6,II,I3	-2			-I					
2, 4,I3,I5	2			I	4, 6,I2,I4	0	0	2	-I					

x) Only one value α_{jklm} is given, when $\alpha_{jklm} = \alpha_{ljkm} = \alpha_{kljm}$.

For α_{jklm} and β_{jklm} there are three possibilities: a) all indices are small ($1 \leq j,k,l,m \leq 7$), b) two indices are small and two are large (e.g., $1 \leq j,k \leq 7$; $8 \leq l,m \leq 15$), c) all indices are large ($8 \leq j,k,l,m \leq 15$). Nonalternativity arises only in situations b). The quantities

$$\alpha_{jklm},\ \alpha_{jkL(15)},\ \alpha_{JKL(15)},\ \alpha_{JKLM},\ \alpha_{jkj+7\,k+7}$$

(where j,k, l,m = 1,2,...,7; J,K,L,M = 8,9,...,14) are totally antisymmetric. To each $\alpha_{jklm} \neq 0$ there correspond

$$\alpha_{jkl+7m+7} = \alpha_{l+7jkm+7} = 0, \quad \alpha_{kl+7jm+7} = \alpha_{jklm}.$$

According to tables 3 and 4 one can construct the following

alternative triangles	nonalternative triangles
1,2,3, 4, 5, 6, 7	
1,2,3, 8, 9,10,15	1,2,3,11,12,13,14
1,4,7, 8,11,14,15	1,4,7, 9,10,12,13
1,5,6, 8,12,13,15	1,5,6, 9,10,11,14
2,4,6, 9,11,13,15	2,4,6, 8,10,12,14
2,5,7, 9,12,14,15	2,5,7, 8,10,11,13
3,4,5,10,11,12,15	3,4,5, 8, 9,13,14
3,6,7,10,13,14,15	3,6,7, 8, 9,11,12

Table 5. ε_{jkl} for octonions

I,2,3	I
I,4,5	I
I,6,7	-I
2,4,6	I
2,5,7	I
3,4,7	I
3,5,6	-I

Table 6. α_{jklm} and β_{jklm} for octonions

I,2,4,7	2	I
I,2,5,6	-2	-I
I,3,4,6	-2	-I
I,3,5,7	-2	-I
2,3,4,5	2	I
2,3,6,7	-2	-I
4,5,6,7	-2	-I

Table 7. ε_{jkl} for sedenions

I, 2, 3	I	2, 9,II	I	4, 8,I2	I	6, 8,I4	I
I, 4, 5	I	2,I2,I4	-I	4, 9,I3	I	6, 9,I5	-I
I, 6, 7	-I	2,I3,I5	-I	4,I0,I4	I	6,I0,I2	-I
I, 8, 9	I			4,II,I5	I	6,II,I3	I
I,I0,II	-I	3, 4, 7	I				
I,I2,I3	-I	3, 5, 6	-I	5, 8,I3	I	7, 8,I5	I
I,I4,I5	I	3, 8,II	I	5, 9,I2	-I	7, 9,I4	I
		3, 9,I0	-I	5,I0,I5	I	7,I0,I3	-I
2, 4, 6	I	3,I2,I5	-I	5,II,I4	-I	7,II,I2	-I
2, 5, 7	I	3,I3,I4	I				
2, 8,I0	I						

Table 8. $\alpha_{jklm}, \alpha_{ljkm}, \alpha_{kljm}$ and β_{jklm} for sedenions x)

I, 2, 4, 7	2			I	2, 5, 8,I5	2			I	4, 7, 8,II	2			I
I, 2, 5, 6	-2			-I	2, 5, 9,I4	0	0	-2	I	4, 7, 9,I0	0	0	2	-I
I, 2, 8,II	2			I	2, 5,I0,I3	-2			-I	4, 7,I2,I5	-2			-I
I, 2, 9,I0	-2			-I	2, 5,II,I2	0	0	2	-I	4, 7,I3,I4	0	0	-2	I
I, 2,I2,I5	0	0	2	-I	2, 6, 8,I2	-2			-I					
I, 2,I3,I4	0	0	-2	I	2, 6, 9,I3	0	0	2	-I	5, 6, 8,II	-2			-I
I, 3, 4, 6	-2			-I	2, 6,I0,I4	-2			-I	5, 6, 9,I0	0	0	-2	I
I, 3, 5, 7	-2			-I	2, 6,II,I5	0	0	2	-I	5, 6,I2,I5	0	0	-2	I
I, 3, 8,I0	-2			-I	2, 7, 8,I3	-2			-I	5, 6,I3,I4	-2			-I
I, 3, 9,II	-2			-I	2, 7, 9,I2	0	0	-2	I	5, 7, 8,I0	2			I
I, 3,I2,I4	0	0	-2	I	2, 7,I0,I5	-2			-I	5, 7, 9,II	0	0	-2	I
I, 3,I3,I5	0	0	-2	I	2, 7,II,I4	0	0	-2	I	5, 7,I2,I4	0	0	2	-I
I, 4, 8,I3	2			I						5, 7,I3,I5	-2			-I
I, 4, 9,I2	-2			-I	3, 4, 8,I5	2			I					
I, 4,I0,I5	0	0	-2	I	3, 4, 9,I4	0	0	-2	I	6, 7, 8, 9	-2			-I
I, 4,II,I4	0	0	2	-I	3, 4,I0,I3	0	0	2	-I	6, 7,I0,II	0	0	-2	I
I, 5, 8,I2	-2			-I	3, 4,II,I2	-2			-I	6, 7,I2,I3	0	0	-2	I
I, 5, 9,I3	-2			-I	3, 5, 8,I4	-2			-I	6, 7,I4,I5	-2			-I
I, 5,I0,I4	0	0	2	-I	3, 5, 9,I5	0	0	-2	I					
I, 5,II,I5	0	0	2	-I	3, 5,I0,I2	0	0	-2	I	8, 9,I0,II	-2			-I
I, 6, 8,I5	-2			-I	3, 5,II,I3	-2			-I	8, 9,I2,I3	-2			-I
I, 6, 9,I4	-2			-I	3, 6, 8,I3	2			I	8, 9,I4,I5	2			I
I, 6,I0,I3	0	0	-2	I	3, 6, 9,I2	0	0	2	-I	8,I0,I2,I4	-2			-I
I, 6,II,I2	0	0	-2	I	3, 6,I0,I5	0	0	-2	I	8,I0,I3,I5	-2			-I
I, 7, 8,I4	2			I	3, 6,II,I4	-2			-I	8,II,I2,I5	-2			-I
I, 7, 9,I5	-2			-I	3, 7, 8,I2	-2			-I	8,II,I3,I4	2			I
I, 7,I0,I2	0	0	2	-I	3, 7, 9,I3	0	0	2	-I					
I, 7,II,I3	0	0	-2	I	3, 7,I0,I4	0	0	2	-I	9,I0,I2,I5	2			I
					3, 7,II,I5	-2			-I	9,I0,I3,I4	-2			-I
2, 3, 4, 5	2			I						9,II,I2,I4	-2			-I
2, 3, 6, 7	-2			-I	4, 5, 6, 7	-2			-I	9,II,I3,I5	-2			-I
2, 3, 8, 9	2			I	4, 5, 8, 9	2			I					
2, 3,I0,II	-2			-I	4, 5,I0,II	0	0	2	-I	I0,II,I2,I3	2			I
2, 3,I2,I3	0	0	2	-I	4, 5,I2,I3	-2			-I	I0,II,I4,I5	-2			-I
2, 3,I4,I5	0	0	-2	I	4, 5,I4,I5	0	0	-2	I					
2, 4, 8,I4	2			I	4, 6, 8,I0	2			I	I2,I3,I4,I5	-2			-I
2, 4, 9,I5	0	0	2	-I	4, 6, 9,II	0	0	-2	I					
2, 4,I0,I2	-2			-I	4, 6,I2,I4	-2			-I					
2, 4,II,I3	0	0	-2	I	4, 6,I3,I5	0	0	2	-I					

x) Only one value α_{jklm} is given, when $\alpha_{jklm} = \alpha_{ljkm} = \alpha_{kljm}$.

Now the quantities

$$\alpha_{jklm}\,,\ \alpha_{jk8M}\,,\ \alpha_{8KLM}\,,\ \alpha_{JKLM}\,,\ \alpha_{jk\,j+8\,k+8}$$

(where $j,k,\ell,m = 1,2,\ldots,7$; $J,K,L,M = 9,10,\ldots,15$) are totally anti-symmetric. Moreover, to each $\alpha_{jklm} \neq 0$ there correspond

$$\alpha_{jk\,\ell+8\,m+8} = \alpha_{\ell+8\,jk\,m+8} = 0\,, \qquad \alpha_{k\,\ell+8\,j\,m+8} = \alpha_{jklm}$$

According to tables 7 and 8 one can form the following

alternative triangles	nonalternative triangles
1,2,3, 4, 5, 6, 7	
1,2,3, 8, 9,10,11	1,2,3,12,13,14,15
1,4,5, 8, 9,12,13	1,4,5,10,11,14,15
1,6,7, 8, 9,14,15	1,6,7,10,11,12,13
2,4,6, 8,10,12,14	2,4,6, 9,11,13,15
2,5,7, 8,10,13,15	2,5,7, 9,11,12,14
3,4,7, 8,11,12,15	3,4,7, 9,10,13,14
3,5,6, 8,11,13,14	3,5,6, 9,10,12,15

These and other observations are in accord with formulas (32).

Tables 4 and 8 contain β_{jklm} only for special sets jklm. The other two independent components $\beta_{\ell jkm}$ and $\beta_{k\ell jm}$ are given by

$$\beta_{\ell jkm} = \beta_{k\ell jm} = \beta_{jklm} = \tfrac{1}{2}\alpha_{jklm} \quad \text{in alternative cases ,}$$

$$\beta_{\ell jkm} = \beta_{k\ell jm} = -\beta_{jklm} = \tfrac{1}{2}\alpha_{k\ell jm} \quad \text{in nonalternative cases (see}$$

eqs. (39)), j,k being small, and l,m large indices.

Note that the multiplication table 1 is identical to that in ref. /11/. Table 5 corresponds to the Dickson multiplication table /1/ (p. 20) with one natural correction $e_6 \to -e_6$ to represent a general octonion as $a_0e_0+a_1e_1+\ldots+a_7e_7 = q + Qe_4$, where $q = a_0e_0+a_1e_1+a_2e_2+a_3e_3$, $Q = a_4e_0+a_5e_1+a_6e_2+a_7e_3$ unlike $Q = a_4e_0+a_5e_1-a_6e_2+a_7e_3$ (with the same q), following from the Dickson multiplication table.

It is well-known that the multiplication table of octonions can be represented by the Freudental triangle. Similarly, in the case of sedenions, points 1,...,15 can be placed on a terahedron : 4 at vertices, 6 at the midpoints of edges, 4 in the centers of faces, and 1 in the center of the tetrahedron. One can consider triangles of 7 points; the faces and triangles formed by one of the edges and the midpoint of the opposite edge. However, some other triangles of interest cannot be so simply represented in terms of the tetrahedron elements.

For p = 4 renumberings

$$a_{j+3} = a'_j\,,\quad a_7 = -a'_0\,,\quad b_{j+3} = -b'_j\,,\quad b_7 = b'_0 \quad (j=1,2,3),$$

$$a_{\mu+4} = -a'_\mu\,,\quad b_{\mu+4} = b'_\mu \quad (\mu = 0,1,2,3)$$

convert eq. (4.a) into eq. (13) and give tables 1 and 5 in processes

(17.a) and (17.b), respectively, thus, leading from quaternions to octonions. Similarly, for p = 8 renumberings

$$a_{j+7} = a'_j\,,\quad a_{15} = -a'_0\,,\quad b_{j+7} = -b'_j\,,\quad b_{15} = b'_0 \quad (j=1,\ldots,7),$$

$$a_{\mu+8} = -a'_\mu\,,\quad b_{\mu+7} = b'_\mu \quad (\mu = 0,1,\ldots,7)$$

generate tables 3 and 7 in processes (17.a) and (17.b), respectively, and lead from octonions to sedenions.

<u>Appendix B.</u> The 2,4 and 8 square identities contain two vectors. One can write a more general identity with 4 different vectors, namely:

$$(a_\mu b_\mu)(c_\nu d_\nu) = (a_\mu c_\mu)(b_\nu d_\nu) - \beta_{jklm} a_j d_k b_\ell c_m +$$
$$+ \sum_{j=1}^{p-1} (a_0 d_j - a_j d_0 \pm \varepsilon_{jk\ell} b_k c_\ell)(b_0 c_j - b_j c_0 \pm \varepsilon_{jk'\ell'} a_{k'} d_{\ell'}), \quad \text{(B.1)}$$

where $a = (a_0, a_1, \ldots, a_{p-1})$, b , c and d are vectors in R^p . The identity can be simply obtained as follows. In any space R^n we have the identity

$$(a_\varkappa b_\varkappa)(c_\lambda d_\lambda) = (a_\varkappa c_\varkappa)(b_\lambda d_\lambda) + \tfrac{1}{2}(a_\varkappa d_\lambda - a_\lambda d_\varkappa)(b_\varkappa c_\lambda - b_\lambda c_\varkappa) =$$
$$= (a_\varkappa c_\varkappa)(b_\lambda d_\lambda) + (a_0 d_j - a_j d_0)(b_0 c_j - b_j c_0) + \tfrac{1}{2}(a_k d_\ell - a_\ell d_k)(b_k c_\ell - b_\ell c_k) =$$
$$= (a_\varkappa c_\varkappa)(b_\lambda d_\lambda) + (a_0 d_j - a_j d_0)(b_0 c_j - b_j c_0) + (a_k b_k)(c_\ell d_\ell) - (a_k c_k)(b_\ell d_\ell), \quad \text{(B.2)}$$

where $\varkappa, \lambda = 0,1,2,\ldots,n-1$; $j,k,\ell = 1,2,\ldots,n-1$; $a_\varkappa b_\varkappa = a_0 b_0 + a_k b_k$. In the spaces R^p under consideration ($p = 2^q$) one can transform the last two terms, using eq. (36), to obtain the identity:

$$(a_\mu b_\mu)(c_\nu d_\nu) = (a_\mu c_\mu)(b_\nu d_\nu) + (a_0 d_j - a_j d_0)(b_0 c_j - b_j c_0) +$$
$$+ (\varepsilon_{jk\ell}\varepsilon_{jk'\ell'} - \beta_{k\ell k'\ell'}) a_k d_\ell b_{k'} c_{\ell'} \quad \text{(B.3)}$$

which can be easily reduced to eq. (B.1). In identity (B.1) $\beta = 0$ for $p = 2$ (complex numbers) and β =4 (quaternions). This term appears for $p \geqslant 8$ (octonions). Identifying $a = b$ and $c = d$ one obtains

$$(a_\mu a_\mu)(b_\nu b_\nu) = (a_\mu b_\mu)^2 + \sum_{j=1}^{p-1} (a_0 b_j - a_j b_0 \pm \varepsilon_{jk\ell} a_k b_\ell)^2 -$$
$$- \beta_{jklm} a_j b_k a_\ell b_m . \quad \text{(B.4)}$$

Now the β -term vanishes for $p = 8$ too due to the total antisymmetry of β for octonions (due to the alternativity). For $p \geqslant 16$ this term is present in addition to 16 squares, etc.

The identities with two vectors can be written in terms of half-vectors as eq. (4.a), whence there follows easily identity (3.a) with one vector (while equating $a = b$ in (B.4) gives a trivial result). One can check the validity of eq. (4.a) as follows. Let us separate terms with and without ε . The linear in ε terms cancel. The bilinear in ε terms and β -term can be removed by using eq.(36).

We are led to the identity

$$(\alpha^2+\alpha'^2)(\beta^2+\beta'^2)=(\alpha\beta-\alpha'\beta')^2+\tfrac{1}{2}(\alpha_\mu\beta_\nu-\alpha_\nu\beta_\mu+\alpha'_\mu\beta'_\nu-\alpha'_\nu\beta'_\mu)^2+$$
$$+(\alpha\beta'+\alpha'\beta)^2+\tfrac{1}{2}(\alpha_\mu\beta'_\nu-\alpha_\nu\beta'_\mu-\alpha'_\mu\beta_\nu+\alpha'_\nu\beta_\mu)^2 \qquad \text{(B.5)}$$

which can be trivially checked, and it is valid in any space R^n ($\mu, \nu = 0,1,\dots,n-1$).

Appendix C.

Matrix representatives of quaternions

$$e^{\substack{\ell\\ r}}_0=\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1\end{pmatrix},\quad e^{\substack{\ell\\ r}}_1=\begin{pmatrix}\cdot&-1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\mp1\\ \cdot&\cdot&\pm1&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_2=\begin{pmatrix}\cdot&\cdot&-1&\cdot\\ \cdot&\cdot&\cdot&\pm1\\ 1&\cdot&\cdot&\cdot\\ \cdot&\mp1&\cdot&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_3=\begin{pmatrix}\cdot&\cdot&\cdot&-1\\ \cdot&\cdot&\mp1&\cdot\\ \cdot&\pm1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot\end{pmatrix},\quad \eta=\begin{pmatrix}-1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1\end{pmatrix}.$$

Here and in what follows dots denote zeros.

Matrix representatives of octonions (according to table 1)

$$e^{\substack{\ell\\ r}}_1=\begin{pmatrix}\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_2=\begin{pmatrix}\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1\\ \cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_3=\begin{pmatrix}\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot\end{pmatrix},$$

$$e^{\substack{\ell\\ r}}_4=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_5=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1\\ \cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_6=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1\\ \cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},$$

$$e^{\substack{\ell\\ r}}_7=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\ \cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot\\ \cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad \eta=\begin{pmatrix}-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}.$$

Matrix representatives of octonions (according to table 5)

$$e^{\substack{\ell\\ r}}_1=\begin{pmatrix}\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_2=\begin{pmatrix}\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1\\ \cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_3=\begin{pmatrix}\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot\end{pmatrix},$$

$$e^{\substack{\ell\\ r}}_4=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_5=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot\\ \cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad e^{\substack{\ell\\ r}}_6=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1\\ \cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot\\ \cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},$$

$$e^{\substack{\ell\\ r}}_7=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad \eta=\begin{pmatrix}-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}.$$

Matrix representatives of sedenions (according to table 7)

$$e^{\ell}_1=\left(\begin{array}{cccccccc|cccccccc}
\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot
\end{array}\right),\quad
e^{\ell}_2=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot
\end{array}\right),$$

$$e^{\ell}_3=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot
\end{array}\right),\quad
e^{\ell}_4=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot
\end{array}\right),$$

$$e^{\ell}_{5}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right)
e^{\ell}_{6}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right),$$

$$e^{\ell}_{7}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right)
e^{\ell}_{8}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\hline
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right),$$

$$e^{\ell}_{9}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\hline
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right)
e^{\ell}_{10}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\
\hline
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right),$$

$$e^{\ell}_{11}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right)
e^{\ell}_{12}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right),$$

$$e^{\ell}_{13}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right)
e^{\ell}_{14}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right),$$

$$e^{\ell}_{15}=\left(\begin{array}{cccccccc|cccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right).$$

The corresponding right representatives have opposite signs throughout except for zero row and column as above.

Appendix D. Identities with commutators and anticommutators. For all nonassociative quantities there are the identities

$$[a[bc]]+[c[ab]]+[b[ca]]=-(a,b,c)-(c,a,b)-(b,c,a)+$$
$$+(c,b,a)+(b,a,c)+(a,c,b),$$
$$[a[bc]]-\{c\{ab\}\}+\{b\{ca\}\}=-(a,b,c)+(c,a,b)-(b,c,a)+$$
$$+(c,b,a)-(b,a,c)+(a,c,b),$$
$$[a\{bc\}]+[c\{ab\}]+[b\{ca\}]=-(a,b,c)-(c,a,b)-(b,c,a)-$$
$$-(c,b,a)-(b,a,c)-(a,c,b),$$
$$[a\{bc\}]-\{c[ab]\}+\{b[ca]\}=-(a,b,c)+(c,a,b)-(b,c,a)-$$
$$-(c,b,a)+(b,a,c)-(a,c,b),$$
$$\{a[bc]\}+\{c[ab]\}+\{b[ca]\}=$$
$$=2\sum_{\substack{\text{antisymm.}\\ \text{over } a,b,c}} a(bc)+(a,b,c)+(c,a,b)+(b,c,a)-(c,b,a)-(b,a,c)-(a,c,b)=$$
$$=2\sum_{\substack{\text{antisymm.}\\ \text{over } a,b,c}} (ab)c-(a,b,c)-(c,a,b)-(b,c,a)+(c,b,a)+(b,a,c)+(a,c,b),$$
$$\{a\{bc\}\}+\{c\{ab\}\}+\{b\{ca\}\}=$$
$$=2\sum_{\substack{\text{symm.}\\ \text{over } a,b,c}} a(bc)+(a,b,c)+(c,a,b)+(b,c,a)+(c,b,a)+(b,a,c)+(a,c,b)=$$
$$=2\sum_{\substack{\text{symm.}\\ \text{over } a,b,c}} (ab)c-(a,b,c)-(c,a,b)-(b,c,a)-(c,b,a)-(b,a,c)-(a,c,b).$$

They generalize the Jacobi identity and other ones for associative quantities, when all associators vanish. For higher hypercomplex numbers under consideration

$$(a,b,c)=-(c,b,a),$$

and the above identities are reduced to

$$[a[bc]]+[c[ab]]+[b[ca]]=-2(a,b,c)-2(c,a,b)-2(b,c,a),$$
$$[a[bc]]-\{c\{ab\}\}+\{b\{ca\}\}=-2(a,b,c)+2(c,a,b)-2(b,c,a),$$
$$[a\{bc\}]+[c\{ab\}]+[b\{ca\}]=0,$$
$$[a\{bc\}]-\{c[ab]\}+\{b[ca]\}=0,$$
$$\{a[bc]\}+\{c[ab]\}+\{b[ca]\}=$$
$$=2\sum_{\text{antisymm.}} a(bc)+2(a,b,c)+2(c,a,b)+2(b,c,a)=$$
$$=2\sum_{\text{antisymm.}} (ab)c-2(a,b,c)-2(c,a,b)-2(b,c,a),$$

$$\{a\{bc\}\}+\{c\{ab\}\}+\{b\{ca\}\}=2\sum_{\text{symm.}} a(bc)=2\sum_{\text{symm.}} (ab)c.$$

For octonions with the use of alternativity the identities take form

$$[a[bc]]+[c[ab]]+[b[ca]]=-6(a,b,c),$$
$$[a[bc]]-\{c\{ab\}\}+\{b\{ca\}\}=-2(a,b,c),$$
$$[a\{bc\}]+[c\{ab\}]+[b\{ca\}]=0,$$
$$[a\{bc\}]-\{c[ab]\}+\{b[ca]\}=0,$$
$$\{a[bc]\}+\{c[ab]\}+\{b[ca]\}=2\sum_{\text{antisymm.}} a(bc)+6(a,b,c)=$$
$$=2\sum_{\text{antisymm.}} (ab)c-6(a,b,c),$$
$$\{a\{bc\}\}+\{c\{ab\}\}+\{b\{ca\}\}=2\sum_{\text{symm.}} a(bc)=2\sum_{\text{symm.}} (ab)c.$$

REFERENCES

1. Диксон Л.Э. Линейные алгебры. ОНТИ ГНТИУ, Харьков, 1935.
2. Jordan P. Zs.für Phys., 1933, 80, p.285.
3. Jordan P., Neumann v.J., Wigner E. Ann. of Math.,1934,35,p.29.
4. Jacobson A.A. Ann. of Math.,1942, 43, p.161.
5. Albert A.A. Ann.of Math.,1942, 43, p.161.
6. Albert A.A. Transactions Am.Math.Soc.,1948, 64, p.552.
7. Schafer R.D. Amer.Journ.of Math., 1954, 76, p.435.
8. Zorn M. Proc.Nat.Acad.Sci (USA) 1935, 21, p.355.
9. Freudental H. Oktaven, Ausnahmengruppen und Oktavengeometrie. Mathematishe Institut der Rijksuniversiteit te Utrecht, 1951. Перевод в "Математика" 1957, I:I, стр.II7.
10. Schafer R.D. An introduction to nonassociative algebras. Academic Press. New York, London, 1966.
11. Gunyadin M., Gursey F. J.Math.Phys., 1973, 14, p.1651.
12. Ramond P. Introduction to exceptional Lie groups and algebras. Caltech preprint CALT-68-577, 1976.
13. Gunyadin M., Piron C., Ruegg H. Commun.Math.Phys.,1978, 61, p.69.
14. Жевлаков К.А., Слинько А.М., Шестаков И.П., Ширшов А.И. Кольца, близкие к ассоциативным. Наука, Москва, 1978.
15. Hurwitz A. Nachrichten von der k.Geselschaft der Wissenschaften zu Gottingen, Math.-phys.Kl. 1898, S.309.

16.Polubarinov I.V. JINR, E2-11169, Dubna, 1978.
17.Kustaanheimo P., Stiefel E. Journ.f.reine u.angew.Math. Berlin, 1965, 218, 204.
18.Штифель Е., Шейфеле Г. Линейная и регулярная небесная механика, 1975, "Наука", М.
19.Aarseth S.J., Zare K. Celestial Mechanics, 1974, 10, p.185.
20.Zare K. Celestial Mechanics, 1974, 10, p.207.
21.Heggie D.C. Celestial Mechanics, 1974, 10, p .217.
22.Duru I.H., Kleinert H. Phys.Lett., 1979, 84B, p.185.
23. Бакай А.С., Степановский Ю.П. Адиабатические инвариантны. "Наукова думка", Киев, 1981.
24.Ho R., Inomata A. Phys.Rev.Lett., 1982, 48, p.231.
25.Kennedy J. Proc.R.Irish.Acad.,1982, 82A, n.1, p.1.
26.Polubarinov I.V. JINR, E2-82-932, Dubna, 1982. Полубаринов И.В.
Полубаринов И.В. Квантовая механика и расслоения Хопфа. В кн.: "Труды П Международного семинара "Теоретико-групповые методы в физике" (Звенигород, 24-26 ноября, 1982 г.), "Наука", М., 1983, т.2, стр.3.
27.Polubarinov I.V. JINR, E2-83-384, Dubna, 1983.
28.Полубаринов И.В. ОИЯИ, Р2-83-872, Дубна, 1983.
29.Полубаринов И.В. В кн.: "Труды УП Международного совещания по проблемам квантовой теории поля. ОИЯИ , Д2-84-366,Дубна,1984, стр.100.
Polubarinov I.V. JINR, E2-84-607, Dubna, 1984.
30. Polubarinov I.V. JINR, E2-82-800, Dubna, 1982.
Полубаринов И.В. О представлениях фазового пространства в квантовой теории поля. В кн.:"Труды П Международного семинара "Теоретико-групповые методы в физике" (Звенигород, 24-26 ноября, 1982 г.). "Наука", М., 1983, т.2, стр.189.

Received by Publishing Department
on September 26, 1984.

Полубаринов И.В. E2-84-654
Дальнейшие системы гиперкомплексных чисел
и квантовая механика

Рассмотрены алгебры гиперкомплексных чисел, следующие за алгеброй октонионов. Дано элементарное изложение основных свойств этих алгебр и связанных с ними тождеств. Сформулированы соответствующие квантовые механики.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1984

Polubarinov I.V. E2-84-654
Higher Hypercomplex Numbers
and Quantum Mechanics

Algebras of hypercomplex numbers following octonion algebra are considered. Main properties of these algebras and relevant identities are exposed in an elementary way. Corresponding quantum mechanics are presented.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1984