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ON APPLICATION OF HOPF FIBER BUNDLES IN QUANTUM THEORY

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1. In most of the models of the contemporary quantum field theory fiber bundles enter one way or another. All the gauge theories are of this kind. Interesting examples of fiber bundles are generated by maps of spheres onto spheres. These maps are studied in topology /1-6/ and encountered in many field theoretical models /7-17/. Properties of the sphere S^7 are now of interest in supersymmetric theories treated according to the Kaluza-Klein approach /18-20/ (see further references therein). We consider applications of a double covering of a circle by a circle and Hopf fiber bundles $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$ in a simpler situation, namely, in quantum mechanics, and, in passing, in classical one. These examples can serve as models of non-linear gauge theories, illustrating, e.g., the role of constraints. A map $S^1 \to S^1$ was in fact used for a regularization of celestial mechanics equations (to eliminate the Newton potential singularity) on the 2-dimensional plane by Levi-Civita /21/. And only in 1965 Kustaanheimo and Stiefel succeeded in generalizing to the 3-dimensionsl case, using the Hopf fiber bundle $S^3 \to S^2$ /22,23/. For further developments see refs. /24-33/.

2. Transformations (changes of variables) under consideration map 2p-dimensional space onto (p+1)-dimensional one, $R^{2p} \to R^{p+1}$, $(p=2^q=1,2,4,8)$.

A) $R^2 \to R^2$ (p=1): $x_0 = u\sigma_3 u = u_1^2 - u_2^2,\ x_1 = u\sigma_1 u = 2u_1u_2$ (1)

$$\tau \equiv |\vec{x}| = uu = u_\mu u_\mu \equiv \rho^2 \quad (|\vec{x}| = \sqrt{x_m x_m}).$$

where u is a real 2-dimensional spinor $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and σ_m are the Pauli matrices. For each fixed $\tau = \rho^2$ we have the double covering of a circle by a circle x)

$$S^1_\rho \to RP^1 = S^1_\tau \quad \text{(fiber is a pair of points u and −u)}. \tag{2}$$

Expressions (1) are coordinates of a point of the base S^1_τ. They are invariant under the transformations

$$u \to \tilde{u} = \zeta u, \quad \zeta^2 = 1, \ \zeta \text{ is real}, \ \zeta = \pm 1 \tag{3}$$

(the group Z_2). This pair of points with ζ =+1 and −1 forms a fiber, the inverse image of point (1) of the base.

x) RP^1 here and CP^1, QP^1 and OP^1 in what follows denote real, complex, quaternion and octonion projective spaces (lines, planes).

B) $R^4 \to R^3$ (p=2): $x_m = \xi^* \sigma_m \xi$ (m=1,2,3), $\tau \equiv |\vec{x}| = \xi^*\xi = u_\mu u_\mu \equiv \rho^2$. (4)

where σ_m are the Pauli matrices, and ξ is a complex 2-component spinor

$$\xi = \begin{bmatrix} a_0 + i a_1 \\ a_0' + i a_1' \end{bmatrix} = \begin{bmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{bmatrix} \tag{5}$$

(If one parametrizes ξ by the Euler angles, then vector $\vec{x}$ occurs to be parametrized by the usual spherical angles, see, e.g., refs. /31/).
For each fixed $\tau = \rho^2$ eqs. (4) embody the Hopf fiber bundle

$$S^3_\rho \to CP^1 = S^2_\tau \quad (\text{fiber is } S^1 = SO(2) = U(1)). \tag{6}$$

The sphere S^2_τ with coordinates (4) serves as a base, and the transformation group U(1) on a fiber

$$\xi \to \tilde{\xi} = z\xi, \quad |z|^2 = 1, \quad z \text{ is complex}, \ z = e^{i\lambda} \tag{7}$$

generates from one value of ξ the whole fiber ($e^{i\lambda}\xi$ with all $0 \leq \lambda \leq 2\pi$), which is a great circle on S^3_ρ and maps into one point of the base (that is sketched on Fig. 1).

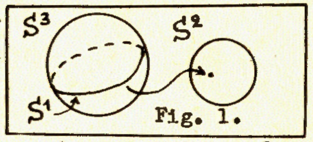

Fig. 1.

C) $R^8 \to R^5$ (p=4): $x_m = \psi^* \gamma_m \psi$ (m=1,2,3,4,5), $\tau \equiv |\vec{x}| = \psi^*\psi = u_\mu u_\mu = \rho^2$, (8)

where γ_m are the Dirac 4x4 matrices, and ψ is a complex 4-component spinor

$$\psi = \begin{bmatrix} a_0 + ia_1 \\ a_2 + ia_3 \\ a_0' + ia_1' \\ a_2' + ia_3' \end{bmatrix} = \begin{bmatrix} u_1 + iu_2 \\ u_3 + iu_4 \\ u_5 + iu_6 \\ u_7 + iu_8 \end{bmatrix}. \tag{9}$$

Asterisk means the complex conjugate. Now, for each fixed $\tau = \rho^2$ the Hopf fiber bundle

$$S^7_\rho \to QP^1 = S^4_\tau \quad (\text{fiber is } S^3 = SU(2) = Sp(1)) \tag{10}$$

takes place. The transformation group SU(2) on fiber is given by /33/

$$\psi \to \tilde{\psi} = z_1^* \psi - z_2 B \psi^*, \quad |z_1|^2 + |z_2|^2 = 1, \quad z_1, z_2 \text{ are complex}, \tag{11}$$

where $B = \gamma_1\gamma_3$ in the Pauli representation for the γ-matrices. Each of the coordinates (8) of a point of the base S^4_τ is invariant under these transformations.

D) $R^{16} \to R^9$ (p=8): $x_m = \Psi^* \Gamma_m \Psi$ (m = 1,2,...7),

$$x_8 = \mathrm{Re}\, \Psi\Gamma\Psi, \quad x_9 = \mathrm{Im}\, \Psi\Gamma\Psi, \quad \tau \equiv |\vec{x}| = \Psi^*\Psi = u_\mu u_\mu \equiv \rho^2. \tag{12}$$

Here Ψ is a complex 8-component spinor

$$\Psi = \begin{bmatrix} a_0 + ia_1 \\ \vdots \\ a_6 + ia_7 \\ a_0' + ia_1' \\ \vdots \\ a_6' + ia_7' \end{bmatrix} = \begin{bmatrix} u_1 + iu_2 \\ \vdots \\ \vdots \\ u_{15} + iu_{16} \end{bmatrix} \tag{13}$$

and Γ_m and Γ are Hermitian 8x8 γ-matrices (Γ is real one)

$$\{\Gamma_m,\Gamma_n\}=2\delta_{mn}\,(m,n=1,2,\dots 7),\quad \Gamma^{-1}=\Gamma^T=\Gamma,\quad \Gamma\Gamma_m=-\Gamma_m^T\Gamma,$$
$$\Gamma_7=i\,\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6 \qquad (14)$$

(see Appendix A). Equations (12) for each fixed $r=\rho^2$ correspond to the Hopf map (fiber bundle)

$$S^{15}_{\rho}\longrightarrow OP^1 = S^8_r \quad (\text{fiber is } S^7). \qquad (15)$$

However, now some complications arise. So, there is no such a group on a fiber similar to the above ones, and a fiber bundle can be built otherwise /3/. This is because of nonassociativity octonions. Moreover, S^{15} is not a parallelizable sphere, unlike S^1, S^3 and S^7. However, as it will be shown in Sect. 6 the fiber bundle $S^{15}\to S^8$ can be used for our purpose almost in the same manner as the previous ones.

Maps (4), (8) and (12) belong to Hopf maps with the Hopf invariant (linking number) H, equal to 1. This value of H, and hence all integer values are possible only for Hopf maps $S^3\to S^2$, $S^7\to S^4$ and $S^{15}\to S^8$. For instance, for other Hopf maps $S^{4n-1}\to S^{2n}$ ($n=3,5,6,7,\dots$) H can have only any even value. Whitehead /2/ has represented the Hopf invariant as an integral. In such a form H as well as degrees of maps $S^1\to S^1$, $S^2\to S^2,\dots$ have important applications in field-theoretical models /10,11,15-17/.

The above realization of the fiber bundles in terms of spinors is attractive due to the well-known Fierz method to obtain, starting with completeness relations, relevant identities needed when we transform Lagrangians, Hamiltonians, equations of motion, commutation relations, etc. /31-33/. In particular, this way leads to the identities*)

$$(u\sigma_3u)^2+(u\sigma_1u)^2=(uu)^2, \quad (a)$$
$$(uu)(vv)^2=\sum_{m=1,3}(u\sigma_mv)^2=\frac{1}{4}\sum_{m=1,3}(u\sigma_mv+v\sigma_mu)^2, \quad (b) \qquad (16)$$

$$(\xi^*\xi)^2=\sum_{m=1}^{3}(\xi^*\sigma_m\xi)^2, \quad (a) \qquad (\xi^*\eta)^2=\sum_{m=1}^{3}(\xi^*\sigma_m\eta)^2, \quad (b)$$
$$4(\xi^*\xi)(\eta^*\eta)=\sum_{m=1}^{3}(\eta^*\sigma_m\xi+\xi^*\sigma_m\eta)^2-(\eta^*\xi-\xi^*\eta)^2, \quad (c) \qquad (17)$$

$$(\psi^*\psi)^2=\sum_{m=1}^{5}(\psi^*\gamma_m\psi)^2, \quad (a) \qquad (\psi^*\psi')^2=\sum_{m=1}^{5}(\psi^*\gamma_m\psi')^2, \quad (b)$$
$$4(\psi^*\psi)(\psi'^*\psi')=\sum_{m=1}^{5}(\psi'^*\gamma_m\psi+\psi^*\gamma_m\psi')^2-(\psi'^*\psi-\psi^*\psi')^2+4(\psi^*B\psi'^*)(\psi B\psi'), \quad (c) \qquad (18)$$

$$(\Psi^*\Psi)^2=\sum_{m=1}^{7}(\Psi^*\Gamma_m\Psi)^2+(\Psi^*\Gamma\Psi^*)(\Psi\Gamma\Psi). \qquad (19)$$

*) See Appendices A and B.

They are just the identities (16a), (17a), (18a) and (19) which demonstrate that transformations (1), (4), (8) and (12) actually realize maps (2), (6), (10) and (15) with $r=\rho^2$. The approach considered can be of interest in connection with a growing importance of spinors in the contemporary theory (supersymmetries, twistor formalism) and proposals of global passing from Cartesian coordinates to spinors /35/.

3. <u>Classical mechanics</u>. Let us translate two-body problem Lagrangians in 2-, 3- and 5-dimensional spaces (we restrict ourselves here to the cases p=1,2 and 4).

A) $$L=\frac{m}{2}\dot{\vec{x}}\dot{\vec{x}}-V=2m(uu)(\dot u\dot u)-V, \qquad (20)$$

where V is a potential, $V(\vec{x})=V(u\vec{\sigma}u)$ or $V(r)=V(uu)$. The new Lagrangian is invariant under "gauge" group Z_2 (in addition to the symmetry group of the original Lagrangian). In the Coulomb case $V(r)=-\frac{e^2}{r}=-\frac{e^2}{uu}$, and equations of motion can be written as follows:

$$2m\overset{\circ\circ}{u}-Hu=0 \qquad \left(\overset{\circ}{u}=\frac{du}{ds},\ ds=\frac{dt}{r}=\frac{dt}{uu}\right), \qquad (21)$$

where $H=2m(uu)(\dot u\dot u)+V$ is Hamiltonian ($\dot H=0$). This is the regularized Levi-Civita form of celestial mechanics equations in the plane case. Actually, when energy H is fixed, eq. (21) is linear, and for $H<0$ it is the equation for a 2-dimensional oscillator.

B) $$L=\frac{m}{2}\dot{\vec{x}}\dot{\vec{x}}-V=2m(\xi^*\xi)(\dot\xi^*\dot\xi)+\frac{m}{2}(\dot\xi^*\xi-\xi^*\dot\xi)^2-V, \qquad (22)$$

$V=V(\vec{x})=V(\xi^*\vec{\sigma}\xi)$ or $V(r)=V(\xi^*\xi)$. In terms of new variables the Lagrangian possesses an additional invariance under gauge $U(1)_l$ transformations (7), where λ is an arbitrary function of t (l (local) indicates this fact), $\xi(t)\to e^{i\lambda(t)}\xi(t)$. It neutralizes (makes harmless) a superfluous degree of freedom of ξ as compared to $\vec{x}$. L can be written in terms of the covariant derivative: $L=2m(\xi^*\xi)((D\xi)^*D\xi)$, $D=\frac{d}{dt}+\frac{1}{2}(\xi^*\xi)^{-1}(\dot\xi^*\xi-\xi^*\dot\xi)$ (cf. CP^1-theories). Let us pass to the new Lagrangian

$$\tilde{L}=2m(\xi^*\xi)(\dot\xi^*\dot\xi)-V, \qquad (23)$$

where we omit the term $\frac{m}{2}(\dot\xi^*\xi-\xi^*\dot\xi)^2$ (like one usually does with the term $(\partial_\mu A_\mu)^2$ in electrodynamics). There remains only the symmetry under $U(1)_r$ transformations (7) with constant λ (r means rigid). It leads to the conservation law

$$(\xi^*\xi)(\dot\xi^*\xi-\xi^*\dot\xi)=\text{const}(t). \qquad (24)$$

If we put const=0 and thus assume the subsidiary condition (SC)

$$\dot{\xi}^*\xi - \xi^*\dot{\xi} = 0\,, \tag{25}$$

then the new theory becomes equivalent to the original one, i.e., it has equivalent equations, conservation laws, etc.). It is preferable for quantization, since it defines all four degrees of freedom. In the Coulomb case $\left(V = -\frac{e^2}{r} = -\frac{e^2}{\xi^*\xi}\right)$ the Lagrangian $\tilde{L}$ has SO(4)-symmetry, and equations of motion can be written as follows:

$$2m\overset{\circ\circ}{\xi} - H\xi = 0 \qquad \left(\overset{\circ}{\xi} = \frac{d\xi}{ds}\,,\ ds = \frac{dt}{r} = \frac{dt}{\xi^*\xi}\right), \tag{26}$$

where $H = 2m(\xi^*\xi)(\dot{\xi}^*\dot{\xi}) + V$ is Hamiltonian $(\dot{H}=0)$. From eq. (26) one can obtain (with the use of Fierz identities and SC) the Newton equations. When energy H is fixed, equation (26) is linear, and for $H<0$ it is an equation for 4-dimensional oscillator. In terms of the variables u this equation has been found by Kustaanheimo and Stiefel /22,23/. Symmetry properties of eq. (26) have been analysed in ref. /28/.

c) $$L = \frac{m}{2}\dot{\vec{x}}\dot{\vec{x}} - V = 2m(\psi^*\psi)(\dot{\psi}^*\dot{\psi}) + \frac{m}{2}(\dot{\psi}^*\psi - \psi^*\dot{\psi})^2 - -2m(\psi^*B\dot{\psi}^*)(\psi B\dot{\psi}) - V \tag{27}$$

$V = V(\vec{x}) = V(\psi^*\vec{\gamma}\psi)$ or $V(r) = V(\psi^*\psi)$. Lagrangian (27) has extra symmetry under gauge $SU(2)_1$ transformations (11), z_1 and z_2 being arbitrary functions of t. We omit the second and third terms from L, and assume the new Lagrangian

$$\tilde{L} = 2m(\psi^*\psi)(\dot{\psi}^*\dot{\psi}) - V. \tag{28}$$

It is invariant under $SU(2)_r$ transformations (11) only with independent of t parameters z_1 and z_2. This invariance leads to the conservation laws

$$\frac{d}{dt}\left[(\psi^*\psi)(\dot{\psi}^*\psi - \psi^*\dot{\psi})\right] = 0,\ \frac{d}{dt}(\psi^*\psi)(\psi B\dot{\psi}) = 0,\ \frac{d}{dt}(\psi^*\psi)(\psi^*B\dot{\psi}^*) = 0 \tag{29}$$

(three real conservation laws), that permits us to assume SC's

$$\dot{\psi}^*\psi - \psi^*\dot{\psi} = 0,\quad \psi B\dot{\psi} = 0,\quad \psi^*B\dot{\psi}^* = 0, \tag{30}$$

which ensure equivalence of the new theory with the original one. In the Coulomb case $\left(V = -\frac{e^2}{r} = -\frac{e^2}{\psi^*\psi}\right)$ Lagrangian (28) has SO(8)-symmetry and leads to the equation of motion

$$2m\overset{\circ\circ}{\psi} - H\psi = 0 \qquad \left(\overset{\circ}{\psi} = \frac{d\psi}{ds}\,,\ ds = \frac{dt}{r} = \frac{dt}{\psi^*\psi}\right), \tag{31}$$

where $H = 2m(\psi^*\psi)(\dot{\psi}^*\dot{\psi}) + V$ is Hamiltonian $(\dot{H}=0)$. As previously, this equation is linear, when energy H is fixed, and for $H<0$ it is an equation for a 8-dimensional oscillator. In terms of the variables u, Lagrangians (20), (23) and (28) and equations (21), (26) and (31) look alike, the only distinction is in their dimensionality.

4. Quantum mechanics. A few remarks concerning the Heisenberg picture. Quantization of theories with the Lagrangians L leads in the cases p=2 and 4 to degenerate canonical commutation relations (they cannot be solved, say, under ξ, ξ^*, $\dot{\xi}$ and $\dot{\xi}^*$). However, from the latter there follow usual commutation relations for $\vec{x}$ and $\vec{p} = m\dot{\vec{x}}$ (as it is natural to expect). If one performs the quantization of the theories with the Lagrangians $\tilde{L}$, then commutation relations, e.g.,

$$[\xi_\alpha, \xi_\beta] = [\xi_\alpha, \xi^*_\beta] = [\xi_\alpha, \pi^*_\beta] = [\xi^*_\alpha, \pi_\beta] = [\pi_\alpha, \pi_\beta] = [\pi^*_\alpha, \pi^*_\beta] = [\pi_\alpha, \pi^*_\beta] = 0$$

$$[\xi_\alpha, \pi_\beta] = i\hbar\delta_{\alpha\beta}\,,\quad [\xi^*_\alpha, \pi^*_\beta] = i\hbar\delta_{\alpha\beta}\,, \tag{32}$$

follow, where $\pi_\alpha = m((\xi^*\xi)\dot{\xi}^*_\alpha + \dot{\xi}^*_\alpha(\xi^*\xi))$ and $\pi^*_\alpha = m((\xi^*\xi)\dot{\xi}_\alpha + \dot{\xi}_\alpha(\xi^*\xi))$ are canonically conjugated momenta, are not degenerate and can be solved under ξ, ξ^*, $\dot{\xi}$ and $\dot{\xi}^*$ (see refs. /31/). However, from eqs. (32) there follow the commutation relations

$$[x_m, x_n] = 0\,,\quad [x_m, p_n] = i\hbar\delta_{mn}\,,$$

$$[p_m, p_n] = -\frac{i\hbar}{(\xi^*\xi)^3}\left(m(\xi^*\xi)(\dot{\xi}^*\xi - \xi^*\dot{\xi}) + i\hbar\right)\varepsilon_{mnl}\,\xi^*\sigma_l\xi, \tag{33}$$

the third one being distinct from the usual. This is payment for the modification of the Lagrangian. We can conclude that to reduce this theory to the original one, the following SC can be imposed on state vectors

$$(\xi\pi - \xi^*\pi^*)|\ \rangle = 0 \quad \text{or} \quad \left(m(\xi^*\xi)(\dot{\xi}^*\xi - \xi^*\dot{\xi}) + i\hbar\right)|\ \rangle = 0. \tag{34}$$

Note, that the operator $(\xi^*\xi)(\dot{\xi}^*\xi - \dot{\xi}^*\xi)$ commutes with the physical quantities $x_m = \xi^*\sigma_m\xi$, $r = \xi^*\xi$, $\dot{x}_m = (\xi^*\sigma_m\xi)^{\cdot}$ /31/, i.e., behaves like the quantity $\partial_\mu A_\mu$ in quantum electrodynamics /34/.

Now, we turn to the Schrödinger picture to find Green functions of the Schrödinger equations. To transform the Laplace operators, which enter into the Schrödinger equation, we need to solve in the cases p= 1,2 and 4 the following sets of equations

A) $$\frac{\partial}{\partial u_\alpha} = \frac{\partial x_m}{\partial u_\alpha}\frac{\partial}{\partial x_m}\,; \tag{35}$$

B) $$\frac{\partial}{\partial \xi_\alpha} = \frac{\partial x_m}{\partial \xi_\alpha}\frac{\partial}{\partial x_m}\,,\quad \frac{\partial}{\partial \xi^*_\alpha} = \frac{\partial x_m}{\partial \xi^*_\alpha}\frac{\partial}{\partial x_m}\,; \tag{36}$$

C) $$\frac{\partial}{\partial \psi_\alpha} = \frac{\partial x_m}{\partial \psi_\alpha}\frac{\partial}{\partial x_m}\,,\quad \frac{\partial}{\partial \psi^*_\alpha} = \frac{\partial x_m}{\partial \psi^*_\alpha}\frac{\partial}{\partial x_m} \tag{37}$$

with respect to $\partial/\partial x_m$. We find as solutions

$$\text{A)} \qquad 2r\frac{\partial}{\partial x_m} = u\sigma_m\frac{\partial}{\partial u}\,. \tag{38}$$

$$\text{B)} \qquad 2r\frac{\partial}{\partial x_m} = \xi\,\sigma_m^T\frac{\partial}{\partial \xi} + \xi^*\sigma_m\frac{\partial}{\partial \xi^*} \tag{39}$$

together with (since in this case set (36) is overdetermined) the relation

$$\xi\frac{\partial}{\partial\xi} - \xi^*\frac{\partial}{\partial\xi^*} = 0. \tag{40}$$

which does not contain $\partial/\partial x_m$ (a compatibility condition). We take it to be a condition on desired functions (to be SC):

$$\left(\xi\frac{\partial}{\partial\xi} - \xi^*\frac{\partial}{\partial\xi^*}\right)\langle\xi,\xi^*|\,\rangle = 0. \tag{41}$$

In fact, this is merely another form of SC (34), which can be also obtained by transforming eq. (34) into the ξ-representation, the canonical momenta being realized in the Schrödinger manner:

$$\pi_\alpha = -i\hbar\frac{\partial}{\partial\xi_\alpha}\,, \quad \pi_\alpha^* = -i\hbar\frac{\partial}{\partial\xi_\alpha^*}\,.$$

$$\text{C)} \qquad 2r\frac{\partial}{\partial x_m} = \psi\,\gamma_m^T\frac{\partial}{\partial\psi} + \psi^*\,\gamma_m\frac{\partial}{\partial\psi^*}\,, \tag{42}$$

and, since set (37) is overdetermined too, we find the relations

$$\psi\frac{\partial}{\partial\psi} - \psi^*\frac{\partial}{\partial\psi^*} = 0\,, \quad \psi B\frac{\partial}{\partial\psi^*} = 0, \quad \psi^* B\frac{\partial}{\partial\psi} = 0 \tag{43}$$

without $\partial/\partial x_m$. We interpret eqs. (43) also as SC's on state vectors

$$\left(\psi\frac{\partial}{\partial\psi} - \psi^*\frac{\partial}{\partial\psi^*}\right)\langle\psi,\psi^*|\,\rangle = 0,\ \psi B\frac{\partial}{\partial\psi^*}\langle\psi,\psi^*|\,\rangle = 0,\ \psi^* B\frac{\partial}{\partial\psi}\langle\psi,\psi^*|\,\rangle = 0. \tag{44}$$

Note, that the operators entering into eqs. (44) form a SU(2) algebra.

Using eqs. (38), (39) and (42), the Laplace operators can be transformed as follows:

$$\text{A)} \qquad \Delta_2^x \equiv \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_m} = \frac{1}{4r}\frac{\partial}{\partial u_\mu}\frac{\partial}{\partial u_\mu} \equiv \frac{1}{4r}\Delta_2^u\ ; \tag{45}$$

$$\text{B)} \qquad \Delta_3 \equiv \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_m} = \frac{1}{r}\frac{\partial}{\partial\xi_\alpha^*}\frac{\partial}{\partial\xi_\alpha} = \frac{1}{4r}\frac{\partial}{\partial u_\mu}\frac{\partial}{\partial u_\mu} \equiv \frac{1}{4r}\Delta_4\,; \tag{46}$$

$$\text{C)} \qquad \Delta_5 \equiv \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_m} = \frac{1}{r}\frac{\partial}{\partial\psi_\alpha^*}\frac{\partial}{\partial\psi_\alpha} = \frac{1}{4r}\frac{\partial}{\partial u_\mu}\frac{\partial}{\partial u_\mu} \equiv \frac{1}{4r}\Delta_8\,. \tag{47}$$

Equations (46) and (47) like eqs. (39) and (42) are valid only in application to functions, which satisfy SC's (41) or (44), respectively.

Define Green functions of the Schrödinger equation as follows:

$$G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,t) = \pm\theta(\pm t)\langle\vec{x}|e^{-i\hbar^{-1}\hat{H}t}|\vec{x}_0\rangle = \frac{1}{2\pi\hbar}\int dE\, e^{-i\hbar^{-1}Et}\,G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,E), \tag{48}$$

$$\left(i\hbar\frac{\partial}{\partial t} - H\right)G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,t) = i\hbar\,\delta(t)\,\delta^{p+1}(x-x_0), \tag{49}$$

$$(H-E)\,G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,E) = -i\hbar\,\delta^{p+1}(x-x_0), \tag{50}$$

where $H = -\frac{\hbar^2}{2m}\Delta_{(p+1)} + V(\vec{x}) \equiv -\frac{\hbar^2}{2m}\Delta_{(p+1)} - \frac{e^2}{r} + W(\vec{x})$, $\Delta_{(p+1)}$ is the Laplace operator in the $(p+1)$-dimensional space of the variables $\vec{x}$, $\delta^{p+1}(x-x_0)$ is the $(p+1)$-dimensional δ-function, $p=1,2,4$. Using relations (45)-(47), we can write the following Schrödinger equation in terms of the spinor variables (in spaces R^{2p}, $p=1,2,4$):

$$\left[-\frac{\hbar^2}{2m}\Delta_{(2p)} - 4e^2 + 4Wu^2 - 4Eu^2\right]\tilde{G}_{\substack{ret\\adv}}(u,u_0,e^2) = -i\hbar\,\delta^{2p}(u-u_0), \tag{51}$$

$$i\hbar\frac{\partial}{\partial s}\tilde{G}_{\substack{ret\\adv}}(u,u_0,s) = \left[-\frac{\hbar^2}{2m}\Delta_{(2p)} + 4Wu^2 - 4Eu^2\right]\tilde{G}_{\substack{ret\\adv}}(u,u_0,s) + i\hbar\,\delta(s)\,\delta^{2p}(u-u_0), \tag{52}$$

$$\tilde{G}_{\substack{ret\\adv}}(u,u_0,e^2) = \int ds\, e^{i\hbar^{-1}4e^2s}\,\tilde{G}_{\substack{ret\\adv}}(u,u_0,s). \tag{53}$$

Desired Green functions $G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,E)$ are expressed via these new Green functions as follows: in the 2-dimensional case (p=1) as the sum over fiber (for details see ref.[33])

$$G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,E) = \tilde{G}_{\substack{ret\\adv}}(u,u_0,e^2) + \tilde{G}_{\substack{ret\\adv}}(-u,u_0,e^2) = \int ds\, e^{i4\hbar^{-1}e^2s}\left[\tilde{G}_{\substack{ret\\adv}}(u,u_0,s) + \tilde{G}_{\substack{ret\\adv}}(-u,u_0,s)\right] \tag{54}$$

and in the 3- and 5-dimensional cases ($p=2$ and 4) as the integrals over fiber

$$G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,E) = \frac{\gamma}{\Omega_{p-1}}\int_{\text{fiber}} d\Omega_{p-1}\,\tilde{G}_{\substack{ret\\adv}}(u,u_0,e^2) = \frac{\gamma}{\Omega_{p-1}}\int_{\text{fiber}} d\Omega_{p-1}\int ds\, e^{i4\hbar^{-1}e^2s}\,\tilde{G}_{\substack{ret\\adv}}(u,u_0,s)\,, \tag{55}$$

where Ω_{p-1} is the surface of the unit sphere S^{p-1} , $d\Omega_{p-1}$ is the surface element of it, and a factor γ is defined below. If necessary, one can also integrate over the fiber corresponding to u_0. Integration over fiber takes into account relevant SC or SC's. Note a similarity to the Faddeev-Popov continual integral quantization of gauge theories [36]. When passing from the original Schrödinger equations to the new ones, we use the following relations between the δ-functions [31-33]

$$\delta^2(x-x_0)=\frac{1}{4r}\left[\delta^2(u-u_0)+\delta^2(u+u_0)\right] \quad \text{(sum over fiber)}, \qquad (56)$$

$$\delta^{p+1}(x-x_0)=\frac{1}{4r}\,\frac{\gamma}{\Omega_{p-1}}\int_{\text{fiber}} d\Omega_{p-1}\,\delta^{2p}(u-u_0), \qquad (57)$$

where $\gamma=\pi$ for $p=2$ and $\gamma=\pi^2$ for $p=4$. Note that E and e^2 exchange roles: E enters now into an oscillator-type potential (but with E of both signs), and e^2 serves as if it is a new energy variable. In the Coulomb case ($W=0$) the new potential is of a pure oscillator type, and $\widetilde{G}$ are of type of the Green functions for 2p-dimensional oscillator

$$\widetilde{G}_{\substack{ret\\adv}}(u,u_0,s)=\pm\theta(\pm s)\left(\frac{m\omega}{2\pi\hbar i\sin\omega s}\right)^p e^{\,i\frac{m\omega}{2\hbar\sin\omega s}\left[(u^2+u_0^2)\cos\omega s-2u\cdot u_0\right]}, \qquad (58)$$

where $\omega=2\sqrt{\frac{-2E}{m}}$. Putting these $\widetilde{G}$ into expressions (54) and (55), we obtain the Green functions $G(\vec{x},\vec{x}_0,E)$ explicitly (for details see in the case $p=2$ refs. /27,31/ and in the cases $p=1$ and $p=4$ ref. /33/). They can be represented in terms of the Pauli and Dirac spinors due to the relations

$$u^2=r=\xi^*\xi\,,\quad u_0^2=r_0=\xi_0^*\xi_0\,,\quad 2u\cdot u_0=\xi^*\xi_0+\xi_0^*\xi\,, \qquad \text{for } p=2 \quad (59)$$

$$u^2=r=\psi^*\psi\,,\quad u_0^2=r_0=\psi_0^*\psi_0\,,\quad 2u\cdot u_0=\psi^*\psi_0+\psi_0^*\psi\,. \qquad \text{for } p=4 \quad (60)$$

5. Hypercomplex numbers can serve as a natural language for defining the above fiber bundles (p=2,4,8). In these terms maps $R^{2p}\to R^{p+1}$ ($p=2^q=2,4,8,\ldots$) are given[*)]

$$\underset{\sim}{x}=2\,\underset{\sim}{\bar\alpha}\,\underset{\sim}{\alpha'}\,,\qquad x_p=\underset{\sim}{\bar\alpha}\,\underset{\sim}{\alpha}-\underset{\sim}{\bar\alpha'}\,\underset{\sim}{\alpha'}\,,\qquad \rho^2=\underset{\sim}{\bar\alpha}\,\underset{\sim}{\alpha}+\underset{\sim}{\bar\alpha'}\,\underset{\sim}{\alpha'}\,, \qquad (61)$$

where $\underset{\sim}{x}=x_\mu e_\mu$, $\underset{\sim}{\alpha}=\alpha_\mu e_\mu$, $\underset{\sim}{\alpha'}=\alpha'_\mu e_\mu$ ($\mu=0,1,\ldots,p-1$) are complex numbers (p=2), quaternions (p=4), octonions (p=8), etc. (we concern $p\geqslant 16$ too **)), $\underset{\sim}{\bar\alpha}=\alpha_0e_0-\alpha_me_m$, $\underset{\sim}{\bar\alpha'}=\alpha'_0e_0-\alpha'_me_m$ are conjugated quantities, and e_μ are hypercomplex unit elements with the multiplication tables

$$e_0^2=e_0\,,\qquad e_je_k=-\delta_{jk}e_0+\varepsilon_{jkl}e_l \qquad (j,k,l=1,2,\ldots,p-1) \qquad (62)$$

---

*) Map (1) can also be concisely written via complex numbers: $z=w^2$ ($z=x_0+ix_1$, $w=u_1+iu_2$), being the well-known transformation in the complex analysis.

**) For higher hypercomplex numbers see ref. /38/.

In eq. (62) ε_{jkl} is totally antisymmetric "tensor" with nonzero components ±1. From eq. (61) it is clear, that in the cases of complex numbers and quaternions each of the coordinates x_μ of a point of the base is invariant under the following transformations on fiber (U(1) and Sp(1), respectively)

$$\underset{\sim}{\alpha}\to\underset{\sim}{\widetilde{\alpha}}=\underset{\sim}{\bar z}\,\underset{\sim}{\alpha}\,,\qquad \underset{\sim}{\alpha'}\to\underset{\sim}{\widetilde{\alpha'}}=\underset{\sim}{\bar z}\,\underset{\sim}{\alpha'}\,,\qquad |z|^2=\underset{\sim}{\bar z}\,\underset{\sim}{z}=1\,, \qquad (63)$$

where $\underset{\sim}{z}=n_\mu e_\mu$ is a complex number or quaternion, respectively. For the octonions ($p=8$) and further algebras ($p\geqslant16$) there are no such invariances because of nonassociativity.

Let us give the maps $R^{2p}\to R^{p+1}$ in real terms

$$\begin{aligned} x_0&=2\alpha_\mu\alpha'_\mu=2\alpha\alpha',\\ x_j&=2(\alpha_0\alpha'_j-\alpha_j\alpha'_0\pm\varepsilon_{jkl}\alpha_k\alpha'_l) \quad (j,k,l=1,2,\ldots,p-1),\\ x_p&=\alpha_\mu\alpha_\mu-\alpha'_\mu\alpha'_\mu=\alpha\alpha-\alpha'\alpha',\\ \rho^2&=\alpha_\mu\alpha_\mu+\alpha'_\mu\alpha'_\mu=\alpha\alpha+\alpha'\alpha'. \end{aligned} \qquad (64)$$

The lower signs correspond to eq. (61), while the upper ones to the one, where the quaternions are multiplied in the opposite order as compared to eq. (61) ($\underset{\sim}{x}=\underset{\sim}{\alpha'}\,\underset{\sim}{\bar\alpha}$). Map (64) transforms the 2p-dimensional real vector (spinor) $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_{p-1},\alpha'_0,\alpha'_1,\ldots,\alpha'_{p-1})=(u_1,u_2,\ldots,u_{2p})$ into the real vector $x=(x_0,x_1,x_2,\ldots,x_p)$ (the coordinates of point of the "base") of smaller (when p>1) dimensionality p+1. The identities of interest with one vector (spinor) α and with two such vectors a and b are written as follows:

$$(|\underset{\sim}{\alpha}|^2+|\underset{\sim}{\alpha'}|^2)^2=\underset{\sim}{\bar x}\,\underset{\sim}{x}+x_p^2+X\,, \qquad (65a)$$

$$(\alpha^2+\alpha'^2)^2=(\alpha^2-\alpha'^2)^2+4(\alpha\alpha')^2+4\sum_{j=1}^{p-1}(\alpha_0\alpha'_j-\alpha_j\alpha'_0\pm\varepsilon_{jkl}\alpha_k\alpha'_l)^2+X\,, \qquad (65b)$$

$$(\underset{\sim}{\bar a}\,\underset{\sim}{a}+\underset{\sim}{\bar a'}\,\underset{\sim}{a'})(\underset{\sim}{\bar b}\,\underset{\sim}{b}+\underset{\sim}{\bar b'}\,\underset{\sim}{b'})=|\underset{\sim}{b}\,\underset{\sim}{\bar a}-\underset{\sim}{a'}\,\underset{\sim}{\bar b'}|^2+|\underset{\sim}{\bar a}\,\underset{\sim}{b'}+\underset{\sim}{\bar b}\,\underset{\sim}{a'}|^2+Y\,, \qquad (66a)$$

$$\begin{aligned}(a^2+a'^2)(b^2+b'^2)&=(ab-a'b')^2+\sum_{j=1}^{p-1}(a_0b_j-a_jb_0\mp\varepsilon_{jkl}a_kb_l+a'_0b'_j-a'_jb'_0\mp\varepsilon_{jkl}a'_kb'_l)^2+\\ &+(ab'+a'b)^2+\sum_{j=1}^{p-1}(a_0b'_j-a_jb'_0\pm\varepsilon_{jkl}a_kb'_l-a'_0b_j+a'_jb_0\mp\varepsilon_{jkl}a'_kb_l)^2+Y\,,\end{aligned} \qquad (66b)$$

where $\alpha^2=\alpha_\mu\alpha_\mu$, $\alpha\alpha'=\alpha_\mu\alpha'_\mu$. Again the lower signs correspond to eq. (66a), and the upper ones to eq. (66a) with the quaternions interchanged ($\underset{\sim}{b}\,\underset{\sim}{\bar a}\to\underset{\sim}{\bar a}\,\underset{\sim}{b}$, $\underset{\sim}{a'}\,\underset{\sim}{\bar b'}\to\underset{\sim}{\bar b'}\,\underset{\sim}{a'}$, $\underset{\sim}{\bar a}\,\underset{\sim}{b'}\to\underset{\sim}{b'}\,\underset{\sim}{\bar a}$, $\underset{\sim}{\bar b}\,\underset{\sim}{a'}\to\underset{\sim}{a'}\,\underset{\sim}{\bar b}$). $X=0$ for $p=1,2,4,8$, and identity (65) means that $r\equiv|\vec{x}|=\alpha^2+\alpha'^2\equiv\rho^2$ and that for fixed $r=\rho^2$ $S_\rho^{2p-1}\to S_r^p$. For $p\geqslant16$ $X\neq0$, and the connection with maps spheres onto spheres disappears. $Y=0$ only for $p=1,2,4$, and identity (66) is the form, we need, of the 2,4 and 8 square identities. From $p=8$ $Y\neq0$, and the identities of 16, 32, etc., squares are distorted by this "redundant" term Y (in accord with the Hurwitz theorem, that pure 16 and more square identities do not exist). One can derive that

$$X=-4\beta_{jklm}a_j a'_k a_l a'_m \qquad \text{for } p\geqslant 16 \tag{67}$$

$$Y=\begin{cases} -4\beta_{jklm}a_j b_k a'_l b'_m & \text{for } p=8 \\ -\beta_{jklm}\left[(a_j b_k+a'_j b'_k)(a_l b_m+a'_l b'_m)+(a_j b'_k-a'_j b_k)(a_l b'_m-a'_l b_m)\right] & \text{for } p\geqslant 16. \end{cases} \tag{68}$$

The "tensor" β_{jklm} appears as follows[38]

$$\varepsilon_{jkn}\varepsilon_{nlm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}+\beta_{jklm}\,,\quad \beta_{jklm}=\tfrac{1}{2}(d_{jklm}+d_{ljkm}-d_{kljm}), \tag{69}$$

where d corresponds to the associator $(e_j,e_k,e_l)=(e_j e_k)e_l-e_j(e_k e_l)=$ $=d_{jklm}e_m$. $\beta_{jklm}=0$ for p=2 and 4. For p=8 d_{jklm} is totally antisymmetric (alternativity), and $\beta_{jklm}=\frac{1}{2}d_{jklm}$.
When $p\geqslant 16$ only the properties

$$d_{jklm}=-d_{lkjm}=-d_{jmlk}=-d_{mjkl}=d_{lmjk}=-d_{klmj}\,,$$

$$d_{jjlm}=d_{jkll}=0 \qquad \text{(no summation)} \tag{70}$$

$$\beta_{jklm}=-\beta_{kjlm}=-\beta_{jkml}=\beta_{lmjk} \tag{71}$$

remain, and the algebras of hypercomplex numbers $\underline{a}=a_\mu e_\mu$ are noncommutative, nonassociative, nonalternative and nondivision algebras.

Note, that in terms of the next set of hypercomplex numbers e_J ($J=1,2,\dots,2p-1$) the identity (66) can be written as follows:

$$(\bar{\underline{a}}\,\underline{a})(\bar{\underline{b}}\,\underline{b})=(\bar{\underline{a}}\,\underline{b})(\bar{\underline{b}}\,\underline{a})-\beta_{JKLM}a_J b_K a_L b_M\,, \tag{72a}$$

$$(a_\mu a_\mu)(b_\nu b_\nu)=(a_\mu b_\mu)^2+\sum_{J=1}^{2p-1}(a_0 b_J-a_J b_0\pm\varepsilon_{JKL}a_K b_L)^2-\beta_{JKLM}a_J b_K a_L b_M, \tag{72b}$$

where $\mu,\nu=0,1,\dots,2p-1$; $J,K,L,M=1,2,\dots,2p-1$, $a_\mu b_\mu=a_0 b_0+a_M b_M$, $\underline{a}=a_0 e_0+a_M e_M$. For p=1,2, and 4 identity (72a) becomes $(\bar{\underline{a}}\,\underline{a})(\bar{\underline{b}}\,\underline{b})=(\bar{\underline{a}}\,\underline{b})(\bar{\underline{b}}\,\underline{a})$, and this is the well-known concise form of the 2,4 and 8 square identities. Two forms (66b) and (72b) of the same identity can be used to generate hypercomplex number algebras. Actually, starting with some known ε_{jkl} for an algebra A_q, we can write identity (66b), and then bring it by renumbering into form (72b), thus obtaining the next tensor ε_{JKL} for the next algebra A_{q+1} (and hence β_{JKLM}). Then, we can repeat the process, etc. This iterating process is equivalent to the Dickson one[37]. When transforming Lagrangians, Hamiltonians,etc., we need just identity of form (66b) (but not (72b)). Identities (16b), (17c) and (18c) are its particular cases, while identities (16a), (17a), (18a) and (19) are particular cases of identity (65).

Let us give an identity with four vectors:

$$(a_\mu b_\mu)(c_\nu d_\nu)=(a_\mu c_\mu)(b_\nu d_\nu)-\beta_{jklm}a_j d_k b_l c_m+$$
$$+\sum_{j=1}^{p-1}(a_0 d_j-a_j d_0\pm\varepsilon_{jkl}b_k c_l)(b_0 c_j-b_j c_0\pm\varepsilon_{jk'l'}a_{k'}d_{l'}), \tag{73}$$

where $a=(a_0,a_1,\dots,a_{p-1})$, b , c and d are vectors in R^p ($p=2^q$). The second term is absent for $p=2$ and 4, but is nonzero for $p\geqslant 8$.

Thus, the list of bilinear transformations of Sec. 2 was extended down, on p = 16,32,... (from p=16 connection with maps of spheres onto spheres desappears). Now, all this list can be continued to the right. Maps (2), (6), (10) and (15) are the first terms of the following series of maps:

A) $S^n\to RP^n$ (74)

B) $S^{2n+1}\to CP^n$ (75)

C) $S^{4n+3}\to QP^n$ (76)

D) $S^{8n+7}\to OP^n$ (77)

(n =1,2,3,...). We can associate with A) the transformations

$$x_0=u_1^2+u_2^2+\dots+u_n^2-u_0^2\,,\qquad x_1=2u_0u_1\,,\ \dots\,,\ x_n=2u_0u_n \tag{78}$$

satisfying the identity

$$(u_0^2+u_1^2+\dots+u_n^2)^2=(u_1^2+\dots+u_n^2-u_0^2)^2+4(u_0u_1)^2+\dots+4(u_0u_n)^2, \tag{79}$$

and with B), C), D) and with the cases $p\geqslant 16$ the transformations of the form

$$x_0=a_{1\mu}a_{1\mu}+\dots+a_{n\mu}a_{n\mu}-a_{0\mu}a_{0\mu}\,, \tag{80}$$
$$x_i=2a_{0\mu}a_{i\mu}\,,$$
$$x_{ij}=2(a_{00}a_{ij}-a_{0j}a_{i0}\pm\varepsilon_{jkl}a_{0k}a_{il})$$

($i=1,\dots,n$; $\mu=0,1,\dots,p-1$; $j,k,l=1,\dots,p-1$) satisfying the identity

$$(A_0+A_1+\dots+A_n)^2=(A_1+\dots+A_n-A_0)^2+4A_0(A_1+\dots+A_n)=$$
$$=(A_1+\dots+A_n-A_0)^2+4\sum_{i=1}^{n}\sum_{j=1}^{p-1}(a_{00}a_{ij}-a_{0j}a_{i0}\pm\varepsilon_{jkl}a_{0k}a_{jl})^2+$$
$$+4\sum_{i=1}^{n}(a_{0\mu}a_{i\mu})^2-4\sum_{i=1}^{n}\beta_{jklm}a_{0j}a_{ik}a_{0l}a_{im}\,, \tag{81}$$

where $A_i=a_{i\mu}a_{i\mu}$ (no summation over i). These identities correspond only to one of $n+1$ charts of the atlases for RP^n, CP^n, QP^n and OP^n , etc.

6. Let us transform, using terms of Sec. 5, known expressions for free Green functions in the cases p=2,4 and 8 into the form (55) with $\tilde G_{\substack{ret\\adv}}(u,u_0,s)$ (58) and demonstrate how to work with sphere S^{15}. Substitution $a\to a+b$, $b\to a-b$, $a'\to a'+b'$, $b'\to a'-b'$ into identity (66b) leads to the identity

$$2(rr_0+\vec x\vec x_0) = 4(ab+a'b')^2 + 4\sum_{j=1}^{p-1}(a_0b_j - b_0a_j \mp \varepsilon_{jkl}a_kb_l + a'_0b'_j - a'_jb'_0 \mp \varepsilon_{jkl}a'_kb'_l)^2 + \tilde Y, \qquad (82)$$

where the right-hand side contains p squares and for $p\geq 8$ term $\tilde Y$ of the form

$$\tilde Y = \begin{cases} 4Y = -16\beta_{jklm}a_jb_ka'_lb'_m & p=8 \\ -4\beta_{jklm}[(a_jb_k+a'_jb'_k)(a_lb_m+a'_lb'_m)+(a_ja'_k-b_jb'_k)(a_la'_m-b_lb'_m)] & p\geq 16. \end{cases} \qquad (83)$$

Now, we can prove that for p =2,4 and 8 the representation

$$\frac{I_{\frac{p-2}{2}}\left(w\sqrt{2(rr_0+\vec x\vec x_0)}\right)}{\left[w\sqrt{2(rr_0+\vec x\vec x_0)}\right]^{\frac{p-2}{2}}} = c_p\int d^pn\,\delta(n_\mu n_\mu-1)\,e^{w2u\cdot u_0} \qquad (84)$$

is valid, where $c_p = 2\left[2^{\frac{p-2}{2}}\left(\frac{p-2}{2}\right)!\,\Omega_{p-1}\right]^{-1}$, Ω_{p-1} is surface of unit sphere S^{p-1}, p=2,4,8. The same representation is valid for the Bessel function $J_{\frac{p-2}{2}}$, if one substitutes iw for w. To prove we put $u=(\tilde a,\tilde a')$, where $\tilde a$ and $\tilde a'$ are defined by eqs. (63), i.e.,

$$\begin{aligned} \tilde a_0 &= n_0a_0+n_ja_j, & \tilde a_j &= n_0a_j-n_ja_0\pm\varepsilon_{jkl}n_ka_l, \\ \tilde a'_0 &= n_0a'_0+n_ja'_j, & \tilde a'_j &= n_0a'_j-n_ja'_0\pm\varepsilon_{jkl}n_ka'_l, \end{aligned} \qquad (85)$$

where the vector (a,a') is somewhat fixed. Then,

$$\begin{gathered} 2u\cdot u_0 = 2(\tilde a b+\tilde a' b') = A_\mu n_\mu \qquad (u_0=(b,b'),\ \mu=0,1,\dots,p-1), \\ A_0 = 2(ab+a'b'),\ A_j = -2[a_0b_j-b_0a_j\mp\varepsilon_{jkl}a_kb_l+a'_0b'_j-a'_jb'_0\mp\varepsilon_{jkl}a'_kb'_l] \end{gathered} \qquad (86)$$

Performing integration in the r.h.s. of eq. (84) over n_μ in the spherical coordinates with the polar axis along A_μ, we obtain the standard representation for $[w\sqrt{A_\mu A_\mu}]^{-\frac{p-2}{2}}I_{\frac{p-2}{2}}(w\sqrt{A_\mu A_\mu})$. In the cases p =2 and 4 $\tilde Y=0$ in identity (82), and validity of eq. (84) is shown, (cf. similar calculations in ref.[33]). For p=8 term $\tilde Y\neq 0$ (owing to nonassociativity of octonions) hinders such a demonstration. However, we can make the term $\tilde Y$ to be zero, if we assume the special "gauge" $a'_j=0$ (or $a_j=0$) for the vector (a,a') (in the cases p =2 and 4 any gauge is acceptable and this one as

well). The remaining 9 components of $(a,a')=(a_0,a_1,\dots,a_7,a'_0,0,\dots,0)$ are sufficient to represent the 9-vector $\vec x$

$$x_0 = 2a_0a'_0,\quad x_j = -2a_ja'_0,\quad x_p = a_\mu a_\mu - a_0'^2. \qquad (87)$$

A fiber is introduced by eq. (85) as before and $u=(\tilde a,\tilde a')$ is a 16-vector (spinor) of a general form. Since now, in gauge adopted, $\tilde Y=0$ in identity (82), then representation (84) holds for p=8 too.

To transform free Green functions, use the relation [39]

$$\begin{aligned} &\left(\frac{\varkappa}{2\pi|\vec x-\vec x_0|}\right)^{\nu+\frac12}K_{\nu+\frac12}(\varkappa|\vec x-\vec x_0|) = \left(\frac{\varkappa}{2\pi\sqrt{a^2-b^2}}\right)^{\nu+\frac12}K_{\nu+\frac12}(\varkappa\sqrt{a^2-b^2}) = \\ &= \frac{1}{2(2\pi b)^\nu}\int_0^\infty dt\,e^{-at}(t^2-\varkappa^2)^{\frac{\nu}{2}}I_\nu(b\sqrt{t^2-\varkappa^2}) = \\ &= \frac{1}{2(2\pi)^\nu}\int_0^\infty\frac{w^{2\nu+1}dw}{\sqrt{w^2+\varkappa^2}}\,e^{-(r+r_0)\sqrt{w^2+\varkappa^2}}\,\frac{I_\nu\left(w\sqrt{2(rr_0+\vec x\vec x_0)}\right)}{\left[w\sqrt{2(rr_0+\vec x\vec x_0)}\right]^\nu}, \end{aligned} \qquad (88)$$

where $a=r+r_0$, $b=\sqrt{2(rr_0+\vec x\vec x_0)}$, $|\vec x-\vec x_0|=\sqrt{a^2-b^2}$, and the change of variable $t=\sqrt{w^2+\varkappa^2}$ is performed. Now, using eq. (84) (and the new variable σ, $w=\varkappa/sh\,\sigma$), we find the desired rapresentation for free Green functions of the Schrödinger equation (50) in (p+1)-dimensional space, p=2,4,8, for $E<0$

$$\begin{aligned} &G^{\circ}_{\substack{ret\\adv}}(\vec x,\vec x_0,E) = -\frac{im}{\pi\hbar}\left(\frac{\varkappa}{2\pi|\vec x-\vec x_0|}\right)^{\frac{p-1}{2}}K_{\frac{p-1}{2}}(\varkappa|\vec x-\vec x_0|) = \\ &= -\frac{im}{\pi\hbar}\frac{\varkappa^{p-1}c_p}{2(2\pi)^{\frac{p-2}{2}}}\int_{\text{fiber}} d^pn\,\delta(n_\mu n_\mu-1)\int_0^\infty\frac{d\sigma}{sh^p\sigma}\,e^{-\frac{\varkappa}{sh\sigma}[(u^2+u_0^2)ch\sigma-2u\cdot u_0]}, \end{aligned} \qquad (89)$$

where $\varkappa=\hbar^{-1}\sqrt{-2mE}$. Expression (89) can be written in terms of spinors according to eqs. (59) and (60) for p=2 and 4 and to

$$u^2 = r = \Psi^*\Psi,\quad u_0^2 = r_0 = \Psi_0^*\Psi_0,\quad 2u\cdot u_0 = \Psi^*\Psi_0+\Psi_0^*\Psi \quad \text{for p=8}. \qquad (90)$$

Representation (89) can be reduced to form (55) with $\tilde G_{\substack{ret\\adv}}(u,u_0,s)$ (58). Starting with eq. (89) one can obtain other representations and also pass to $E>0$ (for the corresponding expression in the case p=2, see refs.[31]). The Coulomb interaction leads merely to appearance of the factor $e^{2\nu\sigma}$ with $\nu=\frac{e^2m}{\hbar\sqrt{-2mE}}$ under the integral over σ. Examining analytical properties of this expression, one can find the known discrete spectra $\nu=\frac{p-2}{2}+n$, n=1,2,....

Let us note that the Coulomb and free cases can be easily expressed via coherent states in ξ (or u)-space with the "time" variable s (like the usual oscillator).

In field theory one can also pass to spinor variables (cf.refs.[35]), using the above fiber bundles. For example, representing a 3-dimensional vector field to be $A_m(x)=\xi^*(x)\sigma_m\xi(x)$ $(m=1,2,3;\ x=(x_1,x_2,x_3,ict))$ we can transform the simplest Lagrangian as follows

$$\mathcal{L}=-\frac{1}{2}\partial_\nu A_m(x)\partial_\nu A_m(x)=-2(\xi^*\xi)(\partial_\mu\xi^*\partial_\mu\xi)-\frac{1}{2}(\xi^*\partial_\mu\xi-\partial_\mu\xi^*\xi)(\xi^*\partial_\mu\xi-\partial_\mu\xi^*\xi).$$

(For the mass term: $-\frac{m^2}{2}A_mA_m=-\frac{m^2}{2}(\xi^*\xi)^2$.) Omitting the term $(\xi^*\partial_4\xi-\partial_4\xi^*\xi)^2$ we assume the Lagrangian in "temporal gauge"

$$\tilde{\mathcal{L}}=-2(\xi^*\xi)(\partial_\mu\xi^*\partial_\mu\xi)-\frac{1}{2}(\xi^*\partial_m\xi-\partial_m\xi^*\xi)(\xi^*\partial_m\xi-\partial_m\xi^*\xi).$$

The Lagrangian $\mathcal{L}$ is invariant under gauge transformations $\xi(x)\to e^{i\lambda(x)}\xi(x)$, $\lambda(x)$ being arbitrary functions of x. The Lagrangian $\tilde{\mathcal{L}}$ is invariant under similar gauge transformations, but with $\lambda(\vec{x})$ which are independent of t and are arbitrary functions of $\vec{x}=(x_1,x_2,x_3)$. The latter transformations lead to the conservation law

$$\frac{\partial}{\partial x_4}\left[(\xi^*\xi)(\xi^*\partial_4\xi-\partial_4\xi^*\xi)\right]=0 \qquad \text{for each } \vec{x}.$$

To ensure equivalence with the original theory we impose the following SC; in the classics

$$\xi^*(x)\partial_4\xi(x)-\partial_4\xi^*(x)\xi(x)=0,$$

and in quantum theory

$$(\xi(x)\pi(x)-\xi^*(x)\pi^*(x))|\ \rangle=0 \qquad \text{(Heisenberg picture)}$$

or

$$\left(\xi(\vec{x})\frac{\partial}{\partial\xi(\vec{x})}-\xi^*(\vec{x})\frac{\partial}{\partial\xi^*(\vec{x})}\right)\langle\xi,\xi^*|\ \rangle=0 \qquad \text{(Schrodinger picture).}$$

To take into account SC in quantum theory we integrate over fiber at each point $\vec{x}$ (like in ref.[8])

$$\langle A_m|\ \rangle=\prod_{\vec{x}}\int_0^{2\pi}d\lambda(\vec{x})\langle\xi,\xi^*|\ \rangle=\prod_{\vec{x}}\int d^2n(\vec{x})\,\delta(n_i(\vec{x})n_i(\vec{x})-1)\langle\xi,\xi^*|\ \rangle.$$

A 5-dimensional vector field (e.g., in de Sitter type theories) can be represented via spinors as follows

$A_\mu(x)=$ $\mu=1,2,3,4,5$	I: $\psi^*(x)\gamma_\mu\psi(x)$	II: $i\bar{\psi}(x)\gamma_\mu\psi(x)$	III: $i\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)$, $\mu=1,2,3,4$, $i\bar{\psi}(x)\gamma_5\psi(x)$ $\mu=5$
signature	$+++++$	$+++-+$	$+++-+$
$A_{\underline{\mu}}$ ($\underline{\mu}=1,2,3,4$)	Euclidean	time-like	space-like
Invariance group of A_μ, and $\psi^*\psi$ for I, and $\bar{\psi}\psi$ for II, III	$SU(2)$: $\tilde{\psi}=z_1^*\psi-z_2B\psi^*$, $\tilde{\psi}^*=z_1\psi^*+z_2^*\psi B$, $\lvert z_1\rvert^2+\lvert z_2\rvert^2=1$.	$SU(2)$: $\tilde{\psi}=z_1^*\psi+z_2B\bar{\psi}$, $\tilde{\bar{\psi}}=z_1\bar{\psi}-z_2^*\psi B$, $\lvert z_1\rvert^2+\lvert z_2\rvert^2=1$.	$SU(1,1)$: $\tilde{\psi}=z_1^*\psi+z_2C\psi$, $\tilde{\bar{\psi}}=z_1\bar{\psi}+z_2^*\psi C$, $\lvert z_1\rvert^2-\lvert z_2\rvert^2=1$.

Inclusion of the quantities $\partial_mA_m(x)$ or $\partial_\mu A_\mu(x)$ into a Lagrangian forces to introduce spinor variables for $x=(\vec{x},t)$ too.

Appendix A. Let us give one explicit representation of the matrices*)

$$\Gamma_1=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&-i&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&i&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&i&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-i\\ i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&-i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&-i&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&i&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ \Gamma_2=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ \Gamma_3=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&-i&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&-i&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&i\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&i&\cdot\\ \cdot&i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&-i&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&-i&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},$$

$$\Gamma_4=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\ \cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ \Gamma_5=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-i&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-i\\ \cdot&\cdot&\cdot&\cdot&-i&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&-i&\cdot&\cdot\\ \cdot&\cdot&i&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&i&\cdot&\cdot&\cdot&\cdot\\ i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ \Gamma_6=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\end{pmatrix},$$

$$\Gamma_7=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ \Gamma=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}. \tag{A.1}$$

This representation is so chosen that there is the correspondence

x of form (64)	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
notation of the same expressions according to eq. (12)	x_7	x_1	x_2	x_3	x_4	x_5	x_8	x_9	x_6

(A.2)

It is implied that in eq. (64) the lower sign is chosen and ε_{jkl} are assumed to be

$$\varepsilon_{123}=\varepsilon_{145}=-\varepsilon_{167}=\varepsilon_{246}=\varepsilon_{257}=\varepsilon_{347}=-\varepsilon_{356}=1, \tag{A.3}$$

which correspond to the Dickson multiplication table[37] with a natural correction $e_6\to -e_6$. The matrices I, Γ_m, $\Gamma_{mn}=\frac{i}{2}\Gamma_{[m}\Gamma_{n]}$, and $\Gamma_{mnp}=\frac{i}{6}\Gamma_{[m}\Gamma_n\Gamma_{p]}$, where [ ] means antisymmetrisation, form a complete set of 8x8 matrices. All they are Hermitian and, being squared, give unity. Their completeness relation can be written as follows

*) Dots replace zeros.

$$8\,\delta_{\alpha\beta}\delta_{\gamma\delta}=\sum_A(\Gamma^A)_{\alpha\delta}(\Gamma^A)_{\gamma\beta}\,, \tag{A.4}$$

where A runs values $0, \mathrm{I}, \mathrm{II}$ and III , and $\Gamma^A\ldots\Gamma^A$ means $I\ldots I$ for $A=0$, $\Gamma_m\cdots\Gamma_m$ for $A=\mathrm{I}$, $\frac{1}{2}\Gamma_{mn}\cdots\Gamma_{mn}$ for $A=\mathrm{II}$ and $\frac{1}{6}\Gamma_{mnp}\cdots\Gamma_{mnp}$ for $A=\mathrm{III}$. Using the relations

$$\begin{gathered}
\Gamma_j\Gamma_j=7, \quad \Gamma_j\Gamma_m\Gamma_j=-5\Gamma_m, \quad \Gamma_j\Gamma_{mn}\Gamma_j=3\Gamma_{mn}, \quad \Gamma_j\Gamma_{mnp}\Gamma_j=-\Gamma_{mnp},\\
\tfrac{1}{2}\Gamma_{jk}\Gamma_{jk}=21, \quad \tfrac{1}{2}\Gamma_{jk}\Gamma_m\Gamma_{jk}=9\Gamma_m, \quad \tfrac{1}{2}\Gamma_{jk}\Gamma_{mn}\Gamma_{jk}=\Gamma_{mn}, \quad \tfrac{1}{2}\Gamma_{jk}\Gamma_{mnp}\Gamma_{jk}=-3\Gamma_{mnp},\\
\tfrac{1}{6}\Gamma_{jkl}\Gamma_{jkl}=35, \quad \tfrac{1}{6}\Gamma_{jkl}\Gamma_m\Gamma_{jkl}=-5\Gamma_m, \quad \tfrac{1}{6}\Gamma_{jkl}\Gamma_{mn}\Gamma_{jkl}=-5\Gamma_{mn},\\
\tfrac{1}{6}\Gamma_{jkl}\Gamma_{mnp}\Gamma_{jkl}=3\Gamma_{mnp}, \quad \Gamma\Gamma^A\Gamma=\varkappa_A(\Gamma^A)^T, \quad \varkappa_A=\begin{cases}1 & \text{for } A=0,\mathrm{III}\\ -1 & \text{for } A=\mathrm{I},\mathrm{II}\end{cases}
\end{gathered} \tag{A.5}$$

we find the identities

$$\Gamma^B_{\alpha\beta}\Gamma^B_{\gamma\delta}=\frac{1}{8}\sum_A(\Gamma^B\Gamma^A\Gamma^B)_{\alpha\delta}\Gamma^A_{\gamma\beta}=\sum_A M_{BA}\Gamma^A_{\alpha\delta}\,\Gamma^A_{\gamma\beta}\,, \tag{A.6}$$

$$\begin{aligned}
\Gamma^B_{\alpha\beta}\Gamma^B_{\gamma\delta}&=\varkappa_B\,\Gamma^B_{\alpha\beta}(\Gamma\Gamma^B\Gamma)_{\delta\gamma}=\sum_A\varkappa_B M_{BA}\varkappa_A(\Gamma^A\Gamma)_{\alpha\gamma}(\Gamma\Gamma^A)_{\beta\delta}=\\
&=\sum_A N_{BA}(\Gamma^A\Gamma)_{\alpha\gamma}(\Gamma\Gamma^A)_{\beta\delta}\,.
\end{aligned} \tag{A.7}$$

The matrices M and N are of the form

$$M=\frac{1}{8}\begin{pmatrix}1 & 1 & 1 & 1\\ 7 & -5 & 3 & -1\\ 21 & 9 & 1 & -3\\ 35 & -5 & -5 & 3\end{pmatrix}, \quad N=\frac{1}{8}\begin{pmatrix}1 & -1 & -1 & 1\\ -7 & -5 & 3 & 1\\ -21 & 9 & 1 & 3\\ 35 & 5 & 5 & 3\end{pmatrix} \tag{A.8}$$

Hence for the combinations

$$L_A=(\Psi_1^*\Gamma^A\Psi_2)(\Psi_3^*\Gamma^A\Psi_4),\quad J_A=(\Psi_1^*\Gamma^A\Psi_4)(\Psi_3^*\Gamma^A\Psi_2),\quad K_A=(\Psi_1^*\Gamma^A\Gamma\Psi_3^*)(\Psi_2\Gamma\Gamma^A\Psi_4)$$

we obtain the identities of the Fierz type (A.9)

$$L_A=M_{AB}J_B\,, \qquad L_A=N_{AB}K_B\,, \tag{A.10}$$

where summation over B ($B=0,\mathrm{I},\mathrm{II},\mathrm{III}$) is implied. Combining these identities, one can find

$$\begin{array}{ll|ll}
2L_0= & J_0+J_{\mathrm{I}}+K_0-K_{\mathrm{I}}\,, & 2J_{\mathrm{I}}= & 3L_0-L_{\mathrm{I}}-3K_0+K_{\mathrm{I}}\,,\\
2L_{\mathrm{I}}= & 3J_0-J_{\mathrm{I}}-3K_0-K_{\mathrm{I}}\,, & 2K_0= & L_0-L_{\mathrm{I}}+J_0-J_{\mathrm{I}}\,,\\
2J_0= & L_0+L_{\mathrm{I}}+K_0+K_{\mathrm{I}}\,, & 2K_{\mathrm{I}}= & -3L_0-L_{\mathrm{I}}+3J_0+J_{\mathrm{I}}\,.
\end{array} \tag{A.11}$$

When $\Psi_1^*=\Psi^*$, $\Psi_2=\Psi$, $\Psi_3^*=\Psi^*$, $\Psi_4=\Psi$ the first of them gives identity (19). When $\Psi_1^*=\Psi^*$, $\Psi_2=\Psi$, $\Psi_3^*=\Psi_0^*$, $\Psi_4=\Psi_0$ it gives the identity relative to identities (16b), (17c), (18c) or (66b). However, it is more complicated, than eq. (66b), and the language of complex spinors seems in this case not quite adequate.

Note also that equations of the type of eq. (37) with Ψ substituted for ψ , have solutions of the form

$$\begin{gathered}
2r\frac{\partial}{\partial x_m}=\Psi\Gamma_m^T\frac{\partial}{\partial\Psi}+\Psi^*\Gamma_m\frac{\partial}{\partial\Psi^*}\,,\\
2r\frac{\partial}{\partial x_8}=\Psi^*\Gamma\frac{\partial}{\partial\Psi}+\Psi\Gamma\frac{\partial}{\partial\Psi^*}\,,\quad 2r\frac{\partial}{\partial x_9}=i\,\Psi^*\Gamma\frac{\partial}{\partial\Psi}-i\,\Psi\Gamma\frac{\partial}{\partial\Psi^*}\,.
\end{gathered} \tag{A.12}$$

<u>Appendix B</u>. Identities (17) follow (see refs./31/) from the completeness relation for the Pauli 2x2 matrices σ_μ ($\sigma_0=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$)

$$2\,\delta_{\alpha\beta}\delta_{\gamma\delta}=\sum_{\mu=0,1,2,3}(\sigma_\mu)_{\alpha\delta}(\sigma_\mu)_{\gamma\beta}\,. \tag{B.1}$$

Putting $(\sigma_0)_{\alpha\delta}(\sigma_0)_{\gamma\beta}=\delta_{\alpha\delta}\,\delta_{\gamma\beta}$ and

$$(\sigma_2)_{\alpha\delta}(\sigma_2)_{\gamma\beta}=-\varepsilon_{\alpha\delta}\varepsilon_{\gamma\beta}=-\delta_{\alpha\gamma}\delta_{\delta\beta}+\delta_{\alpha\beta}\delta_{\gamma\delta} \tag{B.2}$$

($\varepsilon_{11}=\varepsilon_{22}=0,\ \varepsilon_{12}=-\varepsilon_{21}=1$) , we obtain the relation

$$\sum_{m=1,3}(\sigma_m)_{\alpha\delta}(\sigma_m)_{\gamma\beta}=\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\gamma\beta}\,. \tag{B.3}$$

Hence there follow identities (16). The Fierz identities (18) are deduced in ref./33/. Let us give somewhat simplified derivation, similar to that in Appendix A. The set of 16 Hermitian 4x4 Dirac matrices I, γ_m and $\gamma_{mn}=\frac{i}{2}(\gamma_m\gamma_n-\gamma_n\gamma_m)$, $m,n=1,2,\ldots,5$ ($\{\gamma_m\gamma_n\}=2\,\delta_{mn}$) is complete with the completeness relation

$$4\,\delta_{\alpha\beta}\delta_{\gamma\delta}=\sum_A(\gamma^A)_{\alpha\delta}(\gamma^A)_{\gamma\beta}\,, \tag{B.4}$$

where A rans $0, \mathrm{I}, \mathrm{II}$, and $\gamma^A\ldots\gamma^A$ means $I\cdots I$ for $A=0$, $\gamma_m\cdots\gamma_m$ for $A=\mathrm{I}$ and $\frac{1}{2}\gamma_{mn}\cdots\gamma_{mn}$ for $A=\mathrm{II}$. We need the relations

$$\begin{gathered}
\gamma_j\gamma_j=5\,, \quad \gamma_j\gamma_m\gamma_j=-3\gamma_m\,, \quad \gamma_j\gamma_{mn}\gamma_j=\gamma_{mn}\,,\\
\tfrac{1}{2}\gamma_{jk}\gamma_{jk}=10, \quad \tfrac{1}{2}\gamma_{jk}\gamma_m\gamma_{jk}=2\gamma_m\,, \quad \tfrac{1}{2}\gamma_{jk}\gamma_{mn}\gamma_{jk}=-2\gamma_{mn}\,,\\
B^{-1}\gamma^A B=\varkappa_A(\gamma^A)^T, \qquad \varkappa_A=\begin{cases}1 & \text{for } A=0,\mathrm{I}\\ -1 & \text{for } A=\mathrm{II}\end{cases},\\
B^{-1}=B^T=-B, \qquad B\gamma_m B=-\gamma_m^T\,.
\end{gathered} \tag{B.5}$$

In the Pauli representation

$$\gamma_m=\begin{bmatrix}0 & -i\sigma_m\\ i\sigma_m & 0\end{bmatrix}\ (m=1,2,3),\quad \gamma_4=\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix},\quad \gamma_5=\begin{bmatrix}0 & -I\\ -I & 0\end{bmatrix},$$

$$B = C\gamma_5 = \gamma_5 C = \gamma_1\gamma_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{B.6}$$

($C = \gamma_2\gamma_4$ is the standard charge conjugation matrix). Using relation (B.5) we find the identities

$$\gamma^B_{\alpha\beta}\gamma^B_{\gamma\delta} = \frac{1}{4}\sum_A (\gamma^B\gamma^A\gamma^B)_{\alpha\delta}\,\gamma^A_{\gamma\beta} = \sum_A M_{BA}\gamma^A_{\alpha\delta}\gamma^A_{\gamma\beta}\,, \tag{B.7}$$

$$\gamma^B_{\alpha\beta}\gamma^B_{\gamma\delta} = \varkappa_B\gamma^B_{\alpha\beta}(B^{-1}\gamma^B B)_{\delta\gamma} = \sum_A \varkappa_B M_{BA}\varkappa_A(\gamma^A B)_{\alpha\gamma}(B\gamma^A)_{\beta\delta} =$$
$$= \sum_A N_{BA}(\gamma^A B)_{\alpha\gamma}(B\gamma^A)_{\beta\delta} \tag{B.8}$$

with the matrices

$$\|M_{BA}\| = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 \\ 5 & -3 & 1 \\ 10 & 2 & -2 \end{pmatrix}; \quad \|N_{BA}\| = \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 \\ 5 & -3 & -1 \\ -10 & -2 & -2 \end{pmatrix}. \tag{B.9}$$

Hence for the combinations

$$L_A = (\psi_1^*\gamma^A\psi_2)(\psi_3^*\gamma^A\psi_4),\ J_A = (\psi_1^*\gamma^A\psi_4)(\psi_3^*\gamma^A\psi_2),\ K_A = (\psi_1^*\gamma^A B\psi_3^*)(\psi_2 B\gamma^A\psi_4) \tag{B.10}$$

we get the following Fierz identities

$$\begin{array}{ll|l} 4L_0 = J_0 + J_I + J_{II}, & & 4L_0 = K_0 + K_I - K_{II}, \\ 4L_I = 5J_0 - 3J_I + J_{II}, & & 4L_I = 5K_0 - 3K_I - K_{II}, \\ 4L_{II} = 10J_0 + 2J_I - 2J_{II}, & & 4L_{II} = -10K_0 - 2K_I - 2K_{II}. \end{array} \tag{B.11}$$

Combining these identities we can obtain

$$4(\psi^*\psi)(\psi'^*\psi') = (\psi^*\psi')(\psi'^*\psi) + \sum_m (\psi^*\gamma_m\psi')(\psi'^*\gamma_m\psi) +$$
$$+ \frac{1}{2}\sum_{mn}(\psi^*\gamma_{mn}\psi')(\psi'^*\gamma_{mn}\psi) = L_0 + L_I + L_{II} = 2L_0 + 2L_I - 4K_0\,, \tag{B.12}$$

where $K_0 = -(\psi^* B\psi'^*)(\psi B\psi')$, and hence there follow the identities (18). There is the following correspondence

(B.13)

x of form (64) (p=4)	x_0	x_1	x_2	x_3	x_4
notation of the same expressions according to eq. (8)	x_5	x_3	$-x_2$	x_1	x_4

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Полубаринов И.В. E2-84-607

О применении расслоений Хопфа в квантовой теории

В связи с возрастающей ролью расслоений в квантовой теории обсуждаются двулистное покрытие окружности окружностью и расслоения Хопфа $S^3 \to S^2$, $S^7 \to S^4$ и $S^{15} \to S^8$ в терминах комплексных /или вещественных/ спиноров, а также в формализме гиперкомплексных чисел. Соответствующие тождества могут быть использованы для преобразования лагранжианов и гамильтонианов, например, в классической и квантовой механиках /переход от декартовых координат к спинорным/. Рассмотрены проблемы калибровочных инвариантностей, констрайнтов и квантования. С помощью указанных расслоений двухчастичные функции Грина в 2-, 3-, 5- и 9-мерных пространствах представлены как интегралы по слоям от некоторых других функций Грина соответственно в 2-, 4-, 8- и 16-мерных пространствах. Кулоновские и свободные функции Грина выражаются через известные функции Грина для многомерных гармонических осцилляторов.

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Polubarinov I.V. E2-84-607

On Application of Hopf Fiber Bundles in Quantum Theory

In connection with a growing role of fiber bundles in quantum theory a two-fold covering circle by circle and Hopf fiber bundles $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$ are discussed in terms of complex (or real) spinors, and also in the formalism of hypercomplex numbers. Relevant identities can be used to transform Lagrangians and Hamiltonians, e.g., in classical and quantum mechanics (passing from the Cartesian coordinates to spinor ones). Problems of gauge invariances, constraints, and quantization are considered. Using the above fiber bundles, two-particle Green functions of the Schrödinger equation in 2, 3, 5, and 9 dimensions are represented as integrals over fibers of some other Green functions in 2,4,8 and 16 dimensions, respectively. Coulomb and free Green functions are expressed via well-known Green functions for multidimensional harmonic oscillators.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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