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дубна

E2-84-569

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# ON SUBTRACTION OF A CLASS OF INFRARED SINGULARITIES (R*-OPERATION)

Submitted to "ТМФ"

1984

## INTRODUCTION

In some new powerful methods [1,2] of analytical evaluation of Feynman integrals there emerges the necessity to subtract infrared (IR) divergences that arise at intermediate steps of the calculation by means of counterterms, local in the momentum representation.

We stress that we discuss only Feynman integrals of Euclidean theories and the IR divergences resulting from putting some masses and external momenta to zero. A complicating factor is the presence of ultraviolet (UV) divergences.

Ref. [1] contains a formulation of the $R^*$ operation that allows one to subtract UV and IR divergences of the said type simultaneously. However, the definition of IR subgraphs given there is not quite correct. Our aim here is to improve this definition, and to show the validity for the $R^*$-operation of a generalization of the well-known Bogolubov-Parasiuk theorem [3].

We emphasize that the only point of [1] that needs to be corrected is the definition of the IR subgraphs $\tilde{\gamma}$ (and, respectively, the IR index of divergence $\tilde{\omega}(\tilde{\gamma})$ ) that appear in eq.(6) of [1]. The new definition is:

The IR divergent subgraph $\tilde{\gamma}$ is any set of lines of the graph that satisfy the following conditions:

1) When all the momenta flowing along the lines of $\tilde{\gamma}$ are put to zero, there are no other lines in G whose momenta are multified due to the momentum conservation.

2) The index of divergence defined as

$$\tilde{\omega}(\tilde{\gamma}) = (2\ell_{\tilde{\gamma}} - \sum_{\ell\in\tilde{\gamma}} a_\ell - \sum_{v\in\tilde{\gamma}} b_v) - 4m_{\tilde{\gamma}}$$

is $\tilde{\omega} \geq 0$. Here $\ell_{\tilde{\gamma}}$ is the number of lines of $\tilde{\gamma}$, $a_\ell$ and $b_v$ are the dimensionalities in the units of mass, respectively, of the numerator of the propagator corresponding to $\ell$-th line and of the factor (which is a polynomial of momenta) corresponding to the vertex $v\in\tilde{\gamma}$ (we say that $v\in\tilde{\gamma}$ if all the lines attached to $v$ belong to $\tilde{\gamma}$ ); $m_{\tilde{\gamma}}$ is the number showing how many loops the initial graph loses, plus how many connected components the

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* The present paper is based on the university degree by the author carried out at the Faculty of Physics of the Moscow State University in 1982.

initial graph acquires, when $\tilde{\gamma}$ (i.e., all the lines and vertices belonging to $\tilde{\gamma}$ ) is delected.

## 1. R* OPERATION IN THE $\alpha$-REPRESENTATION

For simplicity we limit our analysis to the case of massless scalar particles and interactions without derivatives. These limitations are by no means a matter of principle and are introduced only to avoid cumbersome formulae.

We begin with some definitions.

Consider a Feynman integral in p-space, corresponding to graph G. Euclidean IR divergences arise at those pointes of p-space, where some propagators

$$\Delta^c(p) = \frac{1}{p^2} \qquad (1)$$

of the integrand tend to $\infty$, i.e., the corresponding momenta vanish.

Definition 1: the complete IR subgraph is any set of lines of the graph that satisfy the "completeness condition": if all the momenta flowing along the lines of the subgraph vanish, there is no another line such that the corresponding propagator diverges due to the momentum conservation.

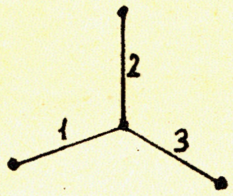


Fig.1

For example, consider the subgraph shown in fig.1. If one puts to zero the momenta flowing along lines 1,2, then the momentum flowing along line 3 also vanishes. So, the subgraph that contains only lines 1 and 2 is not complete, and to make it complete line 3 must be included into it.

Definition 2. The index of IR divergence of the subgraph $\tilde{\gamma}$ is

$$\tilde{\omega}_{\tilde{\gamma}} = 2\ell_{\tilde{\gamma}} - 4m_{\tilde{\gamma}}, \qquad (2)$$

where $m_{\tilde{\gamma}}$ is the number of loops that the graph G loses when all the lines of $\tilde{\gamma}$ are deleted, $\ell_{\tilde{\gamma}}$ is the number of lines of $\tilde{\gamma}$. This definition is easy to understand: $m_{\tilde{\gamma}}$ is the number of integrations over momenta flowing along the lines of $\tilde{\gamma}$, while $2\ell_{\tilde{\gamma}}$ characterizes the singularity resulting from putting to zero those momenta.

Definition 3. Now we can define the IR divergent subgraph as the complete IR subgraph $\tilde{\gamma}$ with a non-negative index of divergence (2): $\tilde{\omega}(\tilde{\gamma}) \geq 0$.



Indeed, the divergence corresponding to $\tilde{\gamma}$ is:

$$\frac{d^{4m_{\tilde{\gamma}}}p}{p^{2\ell_{\tilde{\gamma}}}} = \frac{d|p|}{|p|} \cdot |p|^{4m_{\tilde{\gamma}} - 2\ell_{\tilde{\gamma}}} .$$

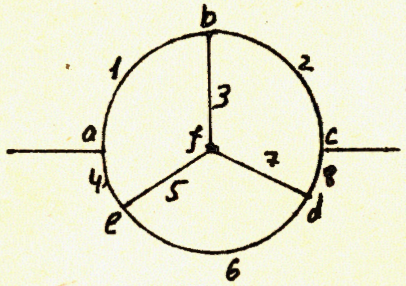


Fig.2

Let G be a Feynman graph, L be the set of its internal lines, v the set of its vertices. Let g be the set of vertices that are internal for G, i.e., no external momentum enters these vertices. For example, for the graph of Fig.2: v = {a, b, c, d, e, f}, g = {b,e,d,f}. The vertices from $v\setminus g$ will be called external.

According to Feynman rules, one can construct a function corresponding to G and defined formally:

$$\mathcal{F}(x_{v\setminus g}) = \int dx_g \prod_{\ell \in G} \Delta^c_\ell (x_{\ell_i} - x_{\ell_f}),$$

where $x_g$ are the internal vertices, $x_{v\setminus g}$ are the external vertices, $x_{\ell_i}$ and $x_{\ell_f}$ are the initial and final points for line (note, that the direction of line in the case of scalar propagator can be chosen at will). The formal Fourier transform of $\mathcal{F}(x_{v\setminus g})$

$$\tilde{\mathcal{F}}(p_{v\setminus g}) = \int \exp\cdot i\Big(\sum_{k\in v\setminus g} x_k p_k\Big)\, \mathcal{F}(x_{v\setminus g})\, dx_{v\setminus g}$$

is

$$\tilde{\mathcal{F}}(p_{v\setminus g}) = \int dp_g \prod_{\ell\in G} \Delta^c_\ell(p_\ell), \qquad (3)$$

where $p_g(p_{v\setminus g})$ are the internal (external) momenta of G and $p_\ell$ is the momentum flowing along line $\ell$ and $\Delta^c(p)$ is defined by (1).

However, the integral (3) is in general divergent, and $\tilde{\mathcal{F}}(p_{v\setminus g})$ does not exist due to singularities in the space of momenta.

It is convenient to introduce a parametric representation for the Feynman integral and subtracting operators.

Let $\alpha_\ell$ be a real parameter corresponding to the line $\ell$ of the graph G:

$$\Delta^c(p_\ell) = \int_0^\infty d\alpha_\ell \cdot \exp(-i)(\alpha_\ell\, p_\ell^2) . \qquad (4)$$

Substituting (4) into (3), we obtain the parametric representation for $\tilde{\mathcal{F}}(p_{v\setminus g})$

$$\tilde{\mathfrak{F}}(p_{v\setminus g}) = \int dp_g \prod_{\ell\in L} \left( \int_0^\infty da_\ell \right) \cdot \exp(-i) \left\{ \sum_{\ell\in L} a_\ell p_\ell^2 \right\}. \qquad (5)$$

In the present work we perform subtractions not only over internal but also over external momenta, i.e., we consider $\tilde{\mathfrak{F}}(p_{v\ g})$ as a functional on the class of test functions:

$$\phi(\underline{p}) = \int_0^\infty da_1 \ldots \int_0^\infty da_{L'} \phi'(\underline{a}) \cdot \exp(-i) \left[ \sum_{i=1}^{L'} a_i p_i^2 \right].$$

We impose the following conditions on $\phi$:

$$a)\ \int_0^\infty d\underline{a} \cdot \prod_{i=1}^{L'} a_i^{c_i} \phi'(\underline{a}) < \text{const}, \quad \text{at} \quad c_j \le R_j ,$$

where $R_j$ is a sufficiently large number to be fixed below. It follows that

$$\prod_i \left( \frac{d}{dp_i^2} \right)^{R_i} \phi(\underline{p}) \Big|_{\underline{p}=0} < \text{const},$$

i.e., $\phi(\underline{p})$ has a number of derivatives at $\underline{p}=0$

$$\beta)\ \int_0^\infty d\underline{a} \frac{\phi'(a)}{\prod_i a_i^{c_i}} < \text{const}, \quad c_i < \tilde{R}_i , \ \Rightarrow\ \phi(p)\big|_{p_i\to\infty} \le \prod_i \frac{\text{const}}{(P_i^2)^{R_i+1}}$$

(see, e.g., /5/), i.e., $\phi(\underline{p})$ decreases sufficiently fast as $p_i \to \infty$.

Now we can describe the functional $\tilde{\mathfrak{F}}$ that corresponds to a regularized but nonrenormalized Feynman integral. Its value on the test function $\phi(\underline{p})$ is

$$(\mathfrak{F}\circ\phi) = \prod_{\ell\in L\cup L'} \left( \int_0^\infty da_\ell \right) \phi'(a) \int dp \cdot \exp(-i) \left[ \sum_{\ell\in L\cup L'} a_\ell p_\ell^2 \right] \cdot \prod_{\ell\in L\cup L'} \chi_\ell(a_\ell, \hat{r}_\ell), \qquad (6)$$

where $\chi_\ell(a_\ell, \hat{r}_\ell)$ is a regularization (see, e.g., /6/). Now one can perform momentum integrations in (6). As a result one obtains:

$$(\mathfrak{F}\circ\phi) = \prod_{\ell\in L\cup L'} \left( \int_0^\infty da_\ell \right) \phi'(a) \prod_{\ell\in L\cup L'} \chi_\ell(a_\ell, \hat{r}_\ell) \frac{1}{(\det W_{\hat{G}})^2}, \qquad (7)$$

where $\det W_{\hat{G}} \equiv U(a) = \sum_{T_{\hat{G}}} \prod_{\ell\notin T_{\hat{G}}} a_\ell$, and $T_{\hat{G}}$ are the T-trees of the graph $\hat{G} = G + L'$.

Following /7/, we define the sector $\sigma_m$ in the integration region of (7) as $a_1 \le 1, \ldots a_m \le 1, a_{m+1} > 1, \ldots a_L > 1$ and perform the change of variables: $a_{m+1} \to 1/a_{m+1}, \ldots, a_L \to 1/a_L$.



Then the contribution to $\mathfrak{F}$ from $\sigma_m$ is:

$$(\mathfrak{F}\circ\phi)_{\sigma_m} = \prod_{\substack{\ell\in L \\ 1\le\ell\le m}} \left( \int_0^1 da_\ell \right) \prod_{\substack{\ell\in L \\ m<\ell\le L}} \left( \int_1^\infty da_\ell \right) \prod_{\ell\in L'} \left( \int da_\ell \right) \cdot \phi'(a) \times$$

$$\times \prod_{\ell\in L\cup L'} \chi_\ell(a_\ell, \hat{r}_\ell) \frac{1}{U^2(\underline{a})} = \prod_{\ell\in L} \left( \int_0^1 da_\ell \right) \cdot \prod_{\ell\in L'} \left( \int_0^\infty da_\ell \right) \cdot \phi'(a) \cdot \prod_{\ell\in L\cup L'} \chi_\ell(a_\ell, \hat{r}_\ell) \cdot \frac{1}{V^2(\underline{a})}, \qquad (8)$$

where $V(\underline{a}) = \sum_{T_{\hat{G}}} \prod_{\substack{1\le\ell\le m \\ \ell\notin T_{\hat{G}}}} a_\ell \prod_{\substack{m<\ell\le L \\ \ell\in T_{\hat{G}}}} a_\ell \prod_{\substack{\ell\in L' \\ \ell\notin T_{\hat{G}}}} a_\ell$.

Now we can introduce the $R^*$-operation.

Let $\{\tilde{\Gamma}_1 \ldots \tilde{\Gamma}_k\}$ be the set of all IR-divergent subgraphs, and $\{\Gamma_1 \ldots \Gamma_n\}$ be the set of all UV-divergent subgraphs which are defined as "generalized blocks" of Bogolubov and Parasiuk with the index of divergence $\omega(\gamma) = 4n_\gamma - 2\ell_\gamma$, where $n_\gamma$ is the number of loops of $\gamma$ and $\ell_\gamma$ is the number of lines of $\gamma$.

Furthermore, let $\tilde{M}(\tilde{\Gamma})$ be the operator that replaces the coefficient function of the complete IR subgraph $\tilde{\Gamma}$ in x-space by the sum of several first terms of its Taylor expansion. Besides, let $M(\Gamma)$ be the subtracting UV-operator defined in /6/.

Let us define "three-point products". We say that $\Gamma_i$ belongs to $\Gamma_j (\Gamma_i \subset \Gamma_j)$ if all lines of $\Gamma_i$ belong to $\Gamma_j$. We say that two subgraphs are partially intersecting, if they have at least one common line and non of them belongs to the other.

<u>Definition 4.</u> The three-point product $\vdots \tilde{M}(\tilde{\Gamma}_1) \ldots \tilde{M}(\tilde{\Gamma}_k) M(\Gamma_1) \ldots M(\Gamma_n) \vdots$ is the operator defined as follows:

1) if any two subgraphs $\Gamma_i, \Gamma_j$ (or $\tilde{\Gamma}_i, \tilde{\Gamma}_j$) are partially intersecting, then $\vdots \tilde{M}(\tilde{\Gamma}_1) \ldots M(\Gamma_n) \vdots \equiv 0$.

2) if two subgraphs $\Gamma_i, \tilde{\Gamma}_j$ have at least one common line, then $\vdots \tilde{M}(\tilde{\Gamma}_1) \ldots M(\Gamma_n) \vdots \equiv 0$.

3) Otherwise $\vdots \tilde{M}(\tilde{\Gamma}_1) \ldots M(\Gamma_n) \vdots \ne 0$ is equal to the usual product of $M, \tilde{M}$ so that if $\tilde{\Gamma}_j \supset \tilde{\Gamma}_i$ $(\Gamma_j \supset \Gamma_i)$, then $\tilde{M}(\tilde{\Gamma}_j)(M(\Gamma_j))$ stands to the left of $\tilde{M}(\tilde{\Gamma}_i)(M(\Gamma_i))$.

Now the $R^*$-operation is defined as

$$R^* = \vdots (1 - \tilde{M}(\tilde{\Gamma}_1)) \ldots (1 - \tilde{M}(\tilde{\Gamma}_k))(1 - M(\Gamma_1)) \ldots (1 - M(\Gamma_n)) \vdots \qquad (9)$$

It is easy to see that in the sector $\sigma_m$ (9) is equivalent to

$$R^*_{\sigma_m} = \vdots (1 - M(\Gamma_1)) \ldots (1 - M(\Gamma_p))(1 - \tilde{M}(\Gamma_{p+1})) (1 - \tilde{M}(\Gamma_n)) \vdots , \qquad (10)$$

where $\Gamma_1 \ldots \Gamma_p$ are UV divergent subgraphs containing lines $\ell \le m$, $\Gamma_{p+1} \ldots \Gamma_n$ are IR divergent subgraphs containing lines $\ell > m$.

$$\tilde{\mathcal{F}}(p_{v\setminus g}) = \int dp_g \prod_{\ell \in L} \Big( \int_0^\infty d\alpha_\ell \Big) \cdot \exp(-i) \Big\{ \sum_{\ell \in L} \alpha_\ell p_\ell^2 \Big\}. \tag{5}$$

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We impose the following conditions on $\phi$:

$\alpha$) $\int_0^\infty d\underline{\alpha} \cdot \prod_{i=1}^{L'} \alpha_i^{c_i} \phi'(\underline{\alpha}) < \text{const}, \quad \text{at} \quad c_j \le R_j,$

where $R_j$ is a sufficiently large number to be fixed below. It follows that

$$\prod_i \Big(\frac{d}{dp_i^2}\Big)^{R_i} \phi(\underline{p}) \Big|_{\underline{p}=0} < \text{const},$$

i.e., $\phi(\underline{p})$ has a number of derivatives at $\underline{p} = 0$

$\beta$) $\int_0^\infty d\underline{\alpha} \dfrac{\phi'(\alpha)}{\prod_i \alpha_i^{c_i}} < \text{const}, \quad c_i < \tilde{R}_i, \quad \Rightarrow \quad \phi(p)\big|_{p_i \to \infty} \le \prod_i \dfrac{\text{const}}{(P_i^2)^{R_i + 1}}$

(see, e.g., [5]), i.e., $\phi(\underline{p})$ decreases sufficiently fast as $p_i \to \infty$.

Now we can describe the functional $\tilde{\mathcal{F}}$ that corresponds to a regularized but nonrenormalized Feynman integral. Its value on the test function $\phi(\underline{p})$ is

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where $\chi_\ell(\alpha_\ell, \hat{r}_\ell)$ is a regularization (see, e.g., [6]). Now one can perform momentum integrations in (6). As a result one obtains:

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where $\det W_{\hat{G}} \equiv U(\alpha) = \sum_{T_{\hat{G}}} \prod_{\ell \notin T_{\hat{G}}} \alpha_\ell$, and $T_{\hat{G}}$ are the T-trees of the graph $\hat{G} = G + L'$.

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Then the contribution to $\mathcal{F}$ from $\sigma_m$ is:

$$(\mathcal{F} \circ \phi)_{\sigma_m} = \prod_{\substack{\ell \in L \\ 1 \le \ell \le m}} \Big( \int_0^1 d\alpha_\ell \Big) \prod_{\substack{\ell \in L \\ m < \ell \le L}} \Big( \int_1^\infty d\alpha_\ell \Big) \prod_{\ell \in L'} \Big( \int d\alpha_\ell \Big) \cdot \phi'(\alpha) \times$$

$$\times \prod_{\ell \in L \cup L'} \chi_\ell(\alpha_\ell, \hat{r}_\ell) \frac{1}{U^2(\underline{\alpha})} = \prod_{\ell \in L} \Big( \int_0^1 d\alpha_\ell \Big) \cdot \prod_{\ell \in L'} \Big( \int_0^\infty d\alpha_\ell \Big) \cdot \phi'(\alpha) \cdot \prod_{\ell \in L \cup L'} \chi_\ell(\alpha_\ell, \hat{r}_\ell) \cdot \frac{1}{V^2(\underline{\alpha})}, \tag{8}$$

where $V(\underline{\alpha}) = \sum_{T_{\hat{G}}} \prod_{\substack{1 \le \ell \le m \\ \ell \notin T_{\hat{G}}}} \alpha_\ell \prod_{\substack{m < \ell \le L \\ \ell \in T_{\hat{G}}}} \alpha_\ell \prod_{\substack{\ell \in L' \\ \ell \notin T_{\hat{G}}}} \alpha_\ell$.

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It is easy to see that in the sector $\sigma_m$ (9) is equivalent to

$$R^*_{\sigma_m} = \vdots (1 - M(\Gamma_1)) \ldots (1 - M(\Gamma_p))(1 - \tilde{M}(\Gamma_{p+1}))(1 - \tilde{M}(\Gamma_n)) \vdots, \tag{10}$$

where $\Gamma_1 \ldots \Gamma_p$ are UV divergent subgraphs containing lines $\ell \le m$, $\Gamma_{p+1} \ldots \Gamma_n$ are IR divergent subgraphs containing lines $\ell > m$.

Definition 5. The IR-(UV)-divergent subgraph $\tilde{\gamma}(\gamma)$ is an arbitrary set of lines with a non-negative index of divergence $\tilde{\omega}(\tilde{\gamma}) = 2\ell_{\tilde{\gamma}} - 4m_{\tilde{\gamma}}$ $(\omega(\gamma) = 4n_\gamma - 2\ell_\gamma)$.

Let us parametrize ${}^{\gamma}R^*_{\sigma_m}$. Let $\kappa_i$ be a parameter corresponding to $\Gamma_i$ and we introduce the variables:

$$\beta_\ell = \kappa_{i_1} \dots \kappa_{i_\ell} a_\ell\,, \quad i_j \le p\,, \quad \ell \le m, \quad \ell \in \Gamma_{i_j}\,,$$
$$\beta_\ell = \kappa_{j_1} \dots \kappa_{j_\ell} a_\ell\,, \quad j_k > p\,, \quad \ell > m, \quad \ell \in \Gamma_{j_k}\,. \tag{11}$$

In [6] a parametric representation of the UV-subtracting operators has been given

$$R^{uv}_\kappa = \int_0^1 d\kappa \frac{(1-\kappa)^{\omega/2}}{(\omega/2)!}\left(\frac{\partial}{\partial\kappa}\right)^{\omega/2+1} \kappa^{2n} = 1 - M(\Gamma) \tag{12}$$

based on the Schlomilch formula:

$$f(x) - \sum_{n_1+\dots+n_m \le k} \frac{x_j^{n_j}}{(n_j)!}\left(\frac{\partial^{n_j}}{\partial x_j^{n_j}} f(x)\right)\Big|_{x=0} = \frac{1}{k!}\int_0^1 d\kappa (1-\kappa)^k \frac{\partial^{k+1}}{\partial \kappa^{k+1}} f(\kappa x).$$

To obtain a similar parametric representation for $\tilde{M}(\tilde{\Gamma})$, we use the momentum representation and change the variables as $x \to x/\kappa$, $a \to 1/a$

$$(1 - \tilde{M}(\tilde{\Gamma})) = R^{IR}_\kappa = \int_0^1 d\kappa \frac{(1-\kappa)^{\tilde{\omega}/2}}{(\tilde{\omega}/2)!}\left(\frac{\partial}{\partial\kappa}\right)^{\tilde{\omega}/2+1} \kappa^{2\ell - 2m}\,. \tag{13}$$

So, using (12), (13) we obtain

$$R^*_{\sigma_m}(\mathcal{F}\circ\phi)_{\sigma_m} = \prod_{\ell\in L}\left(\int_0^1 da_\ell\right)\cdot \prod_{\ell\in L'}\left(\int_0^\infty da_\ell\right) \prod_{i=1}^{n}\left(\int_0^1 d\kappa_i\right) \prod_{j=1}^{n}\Bigg[\frac{(1-\kappa_j)^{\omega_j^*/2}}{(\omega_j^*/2)!}\times$$
$$\times\left(\frac{\partial}{\partial\kappa_j}\right)^{(\omega_j^*/2)+1} \cdot \kappa_j^{2r_j}\Bigg]\cdot \frac{\phi'(a)}{V^2(\beta)}\cdot \prod_{\ell\in L\cup L'} \chi_\ell(a_\ell, \hat{r}_\ell)\,, \tag{14}$$

where $\omega^*$ is the UV or IR index of divergence $r = n$ for UV-divergent subgraphs, $r = \ell - m$ for IR-divergent subgraphs.

The value of the functional $R^*\mathcal{F}$ on the test function is defined as $R^*(\mathcal{F}\circ\phi) = \sum_\sigma R_\sigma(\mathcal{F}\circ\phi)_\sigma$.

We stress that (14) is obvious if $\Gamma_1 \dots \Gamma_n$ and $\tilde{\Gamma}_1 \dots \tilde{\Gamma}_n$ are generalized blocks and complete IR subgraphs, respectively.



But (14) is equivalent to the $R^*$-operation also in the case when $\Gamma$ are understood in the sense of definition 5. The proof of this is based on the fact that in the presence of

$$R_\kappa = \int^1 d\kappa \frac{(1-\kappa)^{\omega/2}}{(\omega/2)!}\,\frac{\partial^{\omega/2+1}}{\partial\kappa^{\omega/2+1}}$$

additional subtractions that are absent in (10) are equivalent to an identity operation (cf. [6] and Appendix).

Our next aim is to prove the convergence of (14) in the limit when the IR and UV regularizations are removed (i.e., when $r \to 0$).

## 2. THE PROOF OF CONVERGENCE

Here we prove the convergence of (14) in the limit when IR and UV regularizations are removed.

Introduce an auxiliary regularization as:

$$R^*_{\sigma_m}(\mathcal{F}\circ\phi)^{\underline{\epsilon\delta}}_{\sigma_m} \equiv J'_{\underline{\epsilon\delta}} = \prod_{\ell\in L}\left(\int_{\epsilon_\ell}^1 da_\ell\right) \prod_{\ell\in L'}\left(\int_0^\infty da_\ell\right)\cdot \prod_{i=1}^{n}\left(\int_{\delta_i}^1 d\kappa_i\right)\cdot \Phi'(\underline{a},\underline{\kappa}) \tag{15}$$

$$\Phi'(\underline{a},\underline{\kappa}) = \prod_{j=1}^{n}\left(\frac{(1-\kappa_j)^{\omega_j^*/2}}{(\omega_j^*/2)!}\left(\frac{\partial}{\partial\kappa_j}\right)^{\omega_j^*/2+1} \kappa_j^{2r_j}\right)\cdot \frac{\phi'(a)}{V^2(\beta)}\cdot \prod_{\ell\in L\cup L'} \chi_\ell(a_\ell, \hat{r}_\ell)\,. \tag{16}$$

Then one only has to prove the existence of the limits: $J'_{00} = \lim_{\hat{r}_\ell\to 0}\lim_{\epsilon_\ell\to 0}\lim_{\delta_\ell\to 0} J'_{\underline{\epsilon\delta}}$.

Note that

$$\prod_{i=1}^{n} \frac{(1-\kappa_i)^{\omega_i^*/2}}{(\omega_i^*/2)!} \prod_{\ell\in L\cup L'} \chi_\ell(a_\ell, \hat{r}_\ell) < \text{const}\,.$$

Therefore it is sufficient to prove the convergence of the integral

$$J_{\underline{\epsilon\delta}} = \prod_{\ell\in L}\left(\int_{\epsilon_\ell}^1 da_\ell\right)\cdot \prod_{\ell\in L'}\left(\int_0^\infty da_\ell\right)\cdot \prod_{i=1}^{n}\left(\int_{\delta_i}^1 d\kappa_i\right)\Phi(\underline{a},\underline{\kappa})\,, \tag{17}$$

where

$$\Phi(\underline{a},\underline{\kappa}) = \prod_{i=1}^{n}\left\{\left(\frac{\partial}{\partial\kappa_j}\right)^{\omega_j^*/2+1}\right\} \prod_{j=1}^{p} \kappa^{2n_j} \cdot \prod_{k=p+1}^{n} \kappa^{2\ell_k - 2m_k} \frac{\phi'(a)}{V^2(\beta)}\,. \tag{18}$$

In (17) and (18) the natural variables are $\beta$ and $\kappa$ with $\beta$ de-

fined in (11). Therefore we change the variables: $\kappa_i = \kappa_i$, $a_\ell = a_\ell (\ell \in L')$, $\beta_\ell = \kappa_{i_\ell} \ldots \kappa_{j_\ell} a_\ell \equiv \pi_\ell a_\ell$.
The Jacobian is

$$I = \prod_{i=1}^{n} \kappa_i^{-L_i}, \tag{19}$$

where $L_i$ is the number of lines of the subgraph $\Gamma_i$. Eq.(18) takes the form:

$$\Phi(\underline{\kappa}, \underline{a}) = \prod_{j=1}^{n} \Big[ \Big\{ \frac{1}{\kappa_j} \Big( \kappa_j \frac{\partial}{\partial \kappa_j} + \sum_{i \in \Gamma_j} \beta_i \frac{\partial}{\partial \beta_i} \Big) \Big\}^{\omega_j^*/2+1} \kappa_j^{2r_j} \Big] \frac{\phi'(a)}{V^2(\beta)}, \tag{20}$$

where we have denoted: $r_j = n_j (j \le p)$, $r_j = L_j - m_j (j > p)$.
Performing differentiation with respect to $\kappa_j$ in (20) and introducing the operators

$$\mathfrak{L}_j = \prod_{s=0}^{\omega_j^*/2} \Big(2r_j - s + \sum_{i \in \Gamma_j} \beta_i \frac{\partial}{\partial \beta_i}\Big), \tag{21}$$

we obtain

$$\Phi(\underline{a}, \underline{\kappa}) = \prod_{j=1}^{n} \big(\kappa_j^{2r_j - (\omega_j^*/2+1)} \mathfrak{L}_j\big) \frac{\phi'(\underline{a})}{V^2(\underline{\beta})}. \tag{22}$$

Substituting (19) and (22) into (17), we get the following expression for $J_{\underline{\epsilon}\underline{\delta}}$

$$J_{\underline{\epsilon}\underline{\delta}} = \int_{\delta_1}^{1} \frac{d\kappa_1}{\kappa_1} \ldots \int_{\delta_n}^{1} \frac{d\kappa_n}{\kappa_n} \psi_{\underline{\epsilon}}(\kappa_1 \ldots \kappa_n), \tag{23}$$

where we have taken into account the definition of the divergence index: $\omega_j^*/2 = 2r_j - L_j$ and $\psi_\epsilon(\kappa_1 \ldots \kappa_n)$ in (23) has the form:

$$\psi_{\underline{\epsilon}}(\kappa_1 \ldots \kappa_n) = \prod_{\ell \in L} \Big( \int_{\pi_\ell \epsilon_\ell}^{\pi_\ell} d\beta_\ell \Big) \prod_{\ell \in L'} \Big( \int_0^\infty d a_\ell \Big) \cdot \prod_{j=1}^{n} \mathfrak{L}_j \frac{\phi'(\underline{a})}{V^2(\underline{\beta})}. \tag{24}$$

The pattern of singularities of $V^2(\underline{\beta})$ is somewhat complicated. Let us prove that they factorize in the sector $\beta_1 \le \beta_2 \le \ldots \le \beta_m$, $a_1 \le a_2 \le \ldots \le a_L$, $\beta_{m+} \le \ldots \le \beta_\ell$, and introduce the variables:

$$t_1 = \beta_1/\beta_2, \quad t_2 = \beta_2/\beta_3, \ldots, \quad t_m = \beta_m,$$

$$u_i = \beta_{m+1}/\beta_{m+2}, \ldots \quad u_k = \beta_L, \quad s_1 = a_1/a_2, \ldots, \quad s_{L'} = a_{L'},$$

recall that $V(\underline{\beta}) = \sum_{T_{\hat G}} \prod_{\substack{\ell \notin T_{\hat G} \\ \ell \le m}} \beta_\ell \prod_{\substack{\ell \in T_{\hat G} \\ \ell > m}} \beta_\ell \cdot \prod_{\substack{\ell \notin T_{\hat G} \\ \ell \in L'}} a_\ell$



Let us now construct a $T_{\hat G}$ tree such that the contribution corresponding to it factorizes the singularities:

$$V(\underline{\beta}) = t_1^{m_1'} \ldots t_m^{m_m'} \, u_1^{n_1'} \ldots u_k^{n_k'} \, s_1^{k_1'} \ldots s_{L'}^{k_{L'}'} \cdot (1 + D(s,t,u)), \tag{25}$$

where

$$D(s,t,u) = \sum_{\{\tilde m_i, \tilde n_j, \tilde k_\ell\}} \prod_{i,j,\ell} t_i^{\tilde m_i} u_j^{\tilde n_j} s_\ell^{\tilde k_\ell}, \quad \tilde m_j \ge 0, \; \tilde n_j \ge 0, \tag{26}$$

Let us construct the $T_{\hat G}$-tree as follows: include the line 1 into it and add to it a line with a minimal number such that if does not form a cycle with the line 1. Having tried all the lines $\ell \le m$, include into the $T_{\hat G}$ tree the maximal set of lines of $L'$ that do not form cycles with the lines already chosen. Then take the line L and add to it such a line with the maximal number which does not form cycles with the lines already chosen. In this way we try all the lines $\ell > m$. All the lines chosen should be included into the $T_{\hat G}$ -tree. So, we arrive at the representation (25), (26), and for $\Gamma_j$, $m_j'(n_j') = r_j$.
Note, that the derivatives with respect to $\underline{u}$ and $\underline{t}$ of $w(\underline{u},\underline{t})$

$$w(\underline{u},\underline{t}) = \prod_{\ell \in L'} \Big( \int_0^\infty ds_\ell \Big) \cdot \prod_i s_i^{-2k_i} \frac{\phi'(\underline{s})}{(1 + D(\underline{s},\underline{t},\underline{u}))^2}$$

have the form

$$\prod_{i,j} \frac{\partial}{\partial t_i^{\rho_i}} \frac{\partial}{\partial u_j^{\rho_j}} w(\underline{u},\underline{t}) = \sum_{\{\sigma_k, \tilde\sigma_i, \hat\sigma_j\}} \prod_{k,i,j} \cdot \prod_{\ell \in L'} \Big( \int_0^\infty ds_\ell \Big) \cdot s_k^{\sigma_k} \frac{t_i^{\tilde\sigma_i} u_j^{\hat\sigma_j}}{(1 + D(s,t,u))^a}. \tag{27}$$

Note that $\tilde\sigma_i > 0$, $\hat\sigma_j > 0$, $|\sigma_i| <$ const, for $\rho_i <$ const. In the variables $\underline{a}_L$, eq.(27) has the form

$$\int_0^\infty da_1 \ldots \int_0^\infty da_{L'} \prod_{i,j,k} a_k^{c_k} \frac{t_i^{\tilde\sigma_i} u_j^{\hat\sigma_j}}{(1 + D(s,t,u))^a}.$$

If $\phi_j \le \omega_{i/2}$, then $-R_i \le c_i \le \tilde R_i$, where $R_i$, $\tilde R_i$ enter into the conditions $a$ and $\beta$ on the test function $\phi'(a)$ and, consequently, $w(\underline{u},\underline{t})$ has $\omega_i$ derivatives with respect to $u_i$ (or $t_i$) at $u_i = 0$ ($t_i = 0$).
Changing the variables as $\underline{\beta}, \underline{a}_L \to \underline{u}, \underline{t}, \underline{s}$ and noting that the Jacobian is $I = \prod_i t_i^{L_i} \prod_j u_j^{L_j} \cdot \prod_k s_k^{L_k}$ we rewrite

$$\psi_{\underline{\epsilon}} = \int_{d_m}^{b_m} dt_m \ldots \int_{d_1}^{b_1} dt_1 \int_{\tilde{d}_k}^{\tilde{b}_k} du_k \ldots \int_{\tilde{d}_1}^{\tilde{b}_1} du_1\, \chi(\underline{u}, \underline{t}),\quad d_j(\tilde{d}_j) = \frac{\epsilon_j \pi_j}{t_{j+1} \ldots t_m} \left(\frac{\epsilon_j \pi_j}{u_{j+1} \ldots u_k}\right), \tag{28}$$

$$b_j(\tilde{b}_j) = \max\{d_j(\tilde{d}_j),\ \min[1, \frac{\pi_j}{t_{j+1} \ldots t_m} (\frac{\pi_j}{u_{j+1} \ldots u_k})]\},$$

where

$$\chi(\underline{u}, \underline{t}) = \prod_{i=1}^{m} t_i^{L_i - 1} \prod_{j=m+1}^{n} u_j^{L_j - 1}\, \mathcal{L}_1 \ldots \mathcal{L}_n \frac{w(\underline{u}, \underline{t})}{t_i^{2r_i} u_j^{2r_j}}. \tag{29}$$

The operators $\mathcal{L}_j$ in the variables $\underline{t}$, $\underline{u}$ are:

$$1 \le i \le p,\quad \mathcal{L}_i = \prod_{s=0}^{\omega_i^*/2} [2r_i - s + \sum_{j \in \Gamma_i} (t_j \frac{\partial}{\partial t_j} - t_{j-1} \frac{\partial}{\partial t_{j-1}})],$$

$$p < i < n,\quad \mathcal{L}_i = \prod_{s=0}^{\omega_i^*/2} [2r_i - s + \sum_{j \in \Gamma_i} (u_j \frac{\partial}{\partial u_j} - u_{j-1} \frac{\partial}{\partial u_{j-1}})].$$

Note that if the subgraph $\Gamma_i$ consists of the lines $\{1, 2, \ldots, i\}$, then $\mathcal{L}_i$ takes the form

$$\mathcal{L}_i = \prod_{s=i}^{2r_i} (s + t_i \frac{\partial}{\partial t_i}). \tag{30}$$

So, let $\Gamma_i$ be the subgraph which is divergent in the given sector. Expand $w(\underline{u}, \underline{t})$ in a series to order $\omega_i^*$:

$$\underline{1 < i < p},\quad \mathcal{L}_i \frac{w(\underline{u}, \underline{t})}{t_i^{2r_i}} = \mathcal{L}_i \frac{(c_0 + c_i t_i)}{t_i^{2r_i}} = \frac{c'}{t_i^{i-1}} + \frac{c''}{t_i^{i-2}} + \ldots. \tag{31}$$

Applying $\mathcal{L}_j$, $j > p$ in a similar manner and substituting (31) into (29), we obtain $\chi(\underline{u}, \underline{t}) = c_0 + \Sigma c_i t_i + \Sigma \tilde{c}_j u_j + \ldots$.

Now one can take the limit $\underline{\epsilon} \to 0$

$$\psi_0(\kappa_1 \ldots \kappa_n) < \text{const} \int_0^{b_m} dt_m \ldots \int_0^{b_1} dt_1 \int_0^{\tilde{b}_k} du_k \ldots \int^{\tilde{b}_1} du_1$$

or in the variables $\beta$

$$\psi_0(\kappa_1 \ldots \kappa_n) < \text{const} \cdot \int_0^{\pi_L} d\beta_L \ldots \int_0^{\min(\pi_1, \beta_2)} d\beta_1 \frac{1}{\beta_2 \ldots \beta_m \beta_{m+2} \ldots \beta_L}.$$



In the sector under consideration:

$$\frac{1}{\beta_2 \ldots \beta_m} \le \beta_1^{-1+1/m} \beta_2^{-1+1/m} \ldots \beta_m^{-1+1/m},\quad \frac{1}{\beta_{m+2} \ldots \beta_L} \le \beta_{m+1}^{-1+1/(L-m)} \ldots \beta_L^{-1+1/(L-m)}.$$

Therefore,

$$\psi_0 < \text{const} \cdot \kappa_1^{L_1/m} \ldots \kappa_p^{L_p/m} \kappa_{p+1}^{L_{p+1}/(L-m)} \ldots \kappa_n^{L_n/(L-m)}. \tag{32}$$

Substituting (32) into (23) we get an estimate on

$$\underline{J}_{0\delta} = \lim_{\underline{\epsilon} \to 0} \underline{J}_{\epsilon\delta} < \text{const} \cdot \prod_i \left(\int_{\delta_i}^1 d\kappa_i\right) \kappa_i^{L_1/m - 1} \ldots \kappa_p^{L_p/m - 1} \kappa_{p+1}^{L_{p+1}/(L-m) - 1} \ldots \times$$
$$\times \kappa_n^{L_n/(L-m) - 1}.$$

It is now obvious that the limit $\underline{J}_{00} = \lim_{\underline{\delta} \to 0} \underline{J}_{0\delta}$ exists. Finally, since $J_{00}$ does not depend on $\hat{r}_\ell$, one can take off the regularization, i.e., let $\hat{r} \to 0$. The proof is completed.

ACKNOWLEDGEMENTS

A am grateful to Professor V.A.Matveev and Professor A.N.Tavkhelidze for the support and to F.V.Tkachov (Institute for Nuclear Research, Moscow) for suggesting the problem and for helpful advices.

APPENDIX

The R*-operation can be written as: $R^* = (1 - \tilde{M}(\tilde{\Gamma}_1)) \ldots (1 - \tilde{M}(\tilde{\Gamma}_n))$. On the other hand, $R_\kappa^{IR} = 1 - \tilde{M}(\tilde{\Gamma})$. Let us prove that new subtractions resulting from partially intersecting divergences do not change $\mathcal{F}$. It is sufficient to check that

$$(1 - \tilde{M}(\Gamma_\cup)) \tilde{M}(\Gamma_1) \tilde{M}(\Gamma_2) (1 - \tilde{M}(\Gamma_\cap)) \mathcal{F}^{\kappa_1 \kappa_2 \kappa_\cap \kappa_\cup} \kappa_1^{2r_1} \kappa_2^{2r_2} \kappa_\cap^{2r} \kappa_\cup^{2r} = 0.$$

Here $\tilde{M}(\tilde{\Gamma}) \mathcal{F}^\kappa = \tilde{M}(\tilde{\Gamma}) \sum_{i=-k}^{\infty} c_i \kappa^i = \sum_{i=-k}^{-1} c_i \kappa^i = \mathcal{F}^\kappa - R_\kappa^{IR} \mathcal{F}^\kappa$, $\Gamma_\cup = \Gamma_1 \cap \Gamma_2$,

Then, using the fact that

$$\mathcal{F}^\kappa = \sum_{N_1, N_2, N_\cap} A_{N_1 N_2 N_\cap} (\kappa_1, \kappa_2 \kappa_\cup \kappa_\cap)^{N_\cap} (\kappa_1 \kappa_\cup)^{N_1} (\kappa_2 \kappa_\cup)^{N_2}$$

$$\Gamma_\cap = \Gamma_1 \cap \Gamma_2.$$

we get the conditions on $N_1$, $N_2$, $N_\cap$

$$N_\cap + N_1 + N_2 + 2r_\cup > 0\,, \quad N_\cap + 2r_\cap > 0\,, \quad -(N_1 + N_\cap + 2r_1) \geq 0\,, \quad -(N_2 + N_\cap + 2r_2) \geq 0.$$

Summing these inequalities and using the fact that $r_1 + r_2 = r_\cup + r_\cap$ we obtain $0 > 0$, i.e.,

$$(1 - \tilde{M}(\Gamma_\cup))\,\tilde{M}(\Gamma_1)\,\tilde{M}(\Gamma_2)\,(1 - \tilde{M}(\Gamma_\cap))\,\mathfrak{F}^{\kappa_1\kappa_2\kappa_\cap\kappa_\cup}\;\kappa_1^{2r_1}\,\kappa_2^{2r_2}\,\kappa_\cap^{2r_\cap}\,\kappa_\cup^{2r_\cup} = 0.$$

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Received by Publishing Department
on August 8, 1984.




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Об устранении одного класса инфракрасных расходимостей /R* -операция/


Исправлено определение инфракрасно-расходящегося подграфа, данное в работе [1]. Доказан для R*-операции аналог известной теоремы Боголюбова-Парасюка.



Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1984


Popov E.N. E2-84-569

On Subtraction of a Class of Infrared Singularities (R* -Operation)


The definition of the infrared-divergent subgraph given in ref.[1] is corrected. An analog of the Bogolubov-Parasiuk theorem is proved for the R*-operation.



The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1984