объединенный
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ядерных
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B.M.Barbashov, A.M.Chervyakov, V.V.Nesterenko

FUNCTIONAL INTEGRAL FOR SYSTEMS WITH TIME-DEPENDENT CONSTRAINTS

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I. Introduction

The quantization of systems with singular Lagrangians by means of the functional integration in the phase-space was proposed by L.D.Faddeev in paper /1/. The proof of the formula which expresses the matrix element of the evolution operator by the functional integral over coordinates and momenta suggested there became the standard one and was accepted in subsequent publications /2–4/ without change. However, in the case when the canonical Hamiltonian is equal to zero identically (the Lagrangians homogeneous of the first-degree in the velocities: relativistic point particle, relativistic string, etc.) the proof in paper /1/, ought to be modified. In this case the gauge conditions, explicitly time-dependent, must be considered /5/. Usually one assumes that the gauge conditions have no such a dependence /1–4/. Besides, if the original Lagrangian has an explicit time dependence (e.g., a system coupled to an external nonstationary field), then the constraints in theory may turn out to be time-dependent as well. It may demand the choice of gauge conditions depending on time explicitly in turn.

In this paper we shall show that the rules for construction of the functional integral for the systems with degenerate Lagrangians in the phase space, proposed in /1/, remain valid when the time-dependent gauge conditions are used. The functional integral is written at first in terms of the physical canonical variables that are obtained by means of a canonical transformation adapted to the gauge conditions. In the case of the nonstationary gauge conditions the canonical transformation explicitly depends on time. In comparison with papers /1–4/ this leads to an additional term in the Hamiltonian which determines the dynamics on a physical submanifold of the phase space.

The organization of the paper is as follows. In Section 2 the equations of motion in the phase space for the systems with constraints and nonstationary gauge conditions are written in a generalized Hamiltonian form. In Section 3 using the canonical transformation we reduce the equations of motion to the Hamilton system which describes the dynamics only in terms of the physical variables. In Section 4 the functional integral for the matrix element of the evolution operator is constructed at first in terms of the physical variables. Then the functional integration is extended to the whole phase space by inserting into the integrand appropriate δ - functions. In conclusion we note basic distinctions between the given proof and that of papers [1-4].

For simplicity in this paper the mechanical system with a finite number of degrees of freedom is considered.

2. Generalized Hamiltonian dynamics of systems with time-dependent constraints

The Lagrangian $L(q,\dot q)$ of the system with a finite number of degrees of freedom is setted as a function of the generalized coordinates $q=(q_1,\dots,q_n)$ and velocities $\dot q=(\dot q_1,\dots,\dot q_n)$.

We are interested in the case when in the whole configuration space $(q,\dot q)$ the rank of the symmetric Hessian matrix with elements

$$\lambda_{ij}(q,\dot q)=\frac{\partial^2 L(q,\dot q)}{\partial \dot q_i\,\partial \dot q_j}, \qquad i,j=1,\dots,n \tag{2.1}$$

is less than the number of degrees of freedom n

$$\mathrm{rank}\|\lambda_{ij}(q,\dot q)\|<n. \tag{2.2}$$

In particular, condition (2.2) will always be fulfilled, if the Lagrangian $L(q,\dot q)$ is the function homogeneous of first-degree in the velocities [6]. Indeed, the Euler theorem for such Lagrangians leads to the relation

$$\dot q_i\,\frac{\partial L(q,\dot q)}{\partial \dot q_i}=L(q,\dot q). \tag{2.3}$$

The differentiation of (2.3) with respect to $\dot q_j$ allows one to conclude that the matrix (2.1) has at least one zero eigenvector

$$\dot q_i\,\lambda_{ij}(q,\dot q)=0. \tag{2.4}$$

Everywhere we have been assuming the summation over repeated indices.

By virtue of condition (2.2) the phase space of canonical coordinates q and momenta p defined as

$$p_i=\frac{\partial L(q,\dot q)}{\partial \dot q_i}, \qquad i=1,\dots,n \tag{2.5}$$

is restricted by constraints [6-7]. The primary constraints directly follow from the condition (2.2). The secondary constraints can be obtained from the Lagrange equations of motion independent of the accelerations [8].

Let

$$\varphi_\alpha(q,p)=0, \qquad \alpha=1,\dots,k \tag{2.6}$$

be a complete set of all (primary and secondary) constraints in the theory. We shall propose that $\varphi_\alpha(q,p)$, $\alpha=1,\dots,k$ are functionally independent, that is

$$\mathrm{rank}\left\|\frac{\partial \varphi_\alpha(\omega)}{\partial \omega^\mu}\right\|=k, \qquad \omega^\mu=(q,p),\quad 1\le\mu\le 2n. \tag{2.7}$$

In virtue of (2.7) eqs. (2.6) determine a $(2n-k)$- dimensional submanifold M of the phase space (q,p).

Having the complete set of constraints (2.6) that obey the condition (2.7), one can derive the canonical equations for the considered system [8]. By means of the Lagrange multiplier method we obtain

$$\frac{dq_i}{dt}=\frac{\partial H_c}{\partial p_i}+\lambda_\alpha(t)\frac{\partial\varphi_\alpha}{\partial p_i},$$
$$\frac{dp_i}{dt}=-\frac{\partial H_c}{\partial q_i}-\lambda_\alpha(t)\frac{\partial\varphi_\alpha}{\partial q_i}, \tag{2.8}$$
$$i=1,\dots,n,$$

where $H_c(q,p)$ is the canonical Hamiltonian, constructed in accordance with the usual rules, and $\lambda_\alpha(t)$, $\alpha = 1,\dots,k$ are Lagrange multipliers. The equation of motion for a general phase-space function $\Psi(q,p,t)$ in accordance with (2.8), has the form

$$\frac{d\Psi}{dt} = \frac{\partial\Psi}{\partial t} + (H_c, \Psi) + \lambda_\alpha(t)(\varphi_\alpha, \Psi). \tag{2.9}$$

Here the Poisson brackets of two functions $f(q,p)$ and $g(q,p)$ are defined by

$$(f,g) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right). \tag{2.10}$$

We shall restrict ourselves in what follows to the case when the relations (2.6) are first-class constraints, that is the Poisson brackets with each other and the canonical Hamiltonian H_c vanish on M

$$(\varphi_\alpha, \varphi_\beta) = C_{\alpha\beta\gamma}(q,p)\varphi_\gamma \approx 0, \tag{2.11}$$

$$(\varphi_\alpha, H_c) = C_{\alpha\beta}(q,p)\varphi_\beta \approx 0, \tag{2.12}$$

$$\alpha,\beta,\gamma = 1,\dots,k,$$

where $C_{\alpha\beta\gamma}(q,p)$ and $C_{\alpha\beta}(q,p)$ are certain functions of the canonical variables q, p.

It follows from (2.11) and (2.12) that the constraints (2.6) are invariant relations for the canonical system (2.8) with arbitrary functions $\lambda_\alpha(t)$. In other words, equations (2.6) will be satisfied by any solution of the canonical system (2.8) with any form of the functions $\lambda_\alpha(t)$ if these equations do for the initial data. Indeed, substituting (2.6) into equations of motion (2.9) and taking into account (2.11), (2.12), we obtain

$$\frac{d\varphi_\alpha}{dt} = (H_c, \varphi_\alpha) + \lambda_\beta(t)(\varphi_\beta, \varphi_\alpha) \approx 0, \tag{2.13}$$

$$\alpha,\beta = 1,\dots,k.$$

Thus, if the relations (2.6) are the first-class constraints, the solution of the Cauchy problem for the system of the canonical equations (2.8) involves k arbitrary functions of time $\lambda_\alpha(t)$.

In this case the infinitesimal changes of the canonical variables q and p generated by the constraints (2.6)

$$\delta q_i = \varepsilon_\alpha(t)\frac{\partial\varphi_\alpha}{\partial p_i}, \quad \delta p_i = -\varepsilon_\alpha(t)\frac{\partial\varphi_\alpha}{\partial q_i}, \tag{2.14}$$

$$i = 1,\dots,n$$

transform the classes of infinitely neighbouring trajectories into themselves, and functions $\lambda_\alpha(t)$ inside each class change as follows

$$\delta\lambda_\alpha(t) = \dot{\varepsilon}_\alpha(t) + \varepsilon_\beta(t)C_{\beta\alpha} + \lambda_\beta(t)\varepsilon_\gamma(t)C_{\beta\gamma\alpha}, \tag{2.15}$$

$$\alpha,\beta,\gamma = 1,\dots,k.$$

The functional arbitrariness in the solution of canonical equations (2.8) can be removed by the requirement that all the observables in the theory do not change under the transformations (2.14)

$$\delta\Psi = \varepsilon_\alpha(t)(\varphi_\alpha, \Psi) \approx 0.$$

Hence by virtue of the independence of parameters $\varepsilon_\alpha(t)$ we obtain a set of k differential equations of the first order for the observable functions Ψ

$$(\varphi_\alpha, \Psi) = d_{\alpha\beta}(q,p)\varphi_\beta \approx 0, \tag{2.16}$$

$$\alpha = 1,\dots,k.$$

The substitution of relations (2.11) into the Jacobi identity yields the integrability condition for (2.16)

$$(\varphi_\alpha, (\varphi_\beta, \Psi)) - (\varphi_\beta, (\varphi_\alpha, \Psi)) = ((\varphi_\alpha, \varphi_\beta), \Psi) \approx 0. \tag{2.17}$$

It follows from (2.17) that eqs. (2.16) have $(2n-k)$ solutions Ψ_τ and there are k constraints (2.6) among them. Therefore the independent observable variables Ψ will be unambiguously defined by their values on the submanifold of the phase space with dimension $(2n-k) - k = 2(n-k)$.

Such a submanifold can be realized as follows [7]. In addition to the k constraints (2.6) the canonical variables q and p will be submitted to k gauge conditions that will be assumed explicitly time-dependent

$$\chi_\alpha(q,p,t)=0\,, \qquad \alpha=1,\dots,k. \tag{2.18}$$

The relations (2.18) have to be taken not invariant under the transformation (2.14)

$$\delta\chi_\beta=\varepsilon_\alpha(t)(\varphi_\alpha,\chi_\beta)\neq 0\,. \tag{2.19}$$

This allows us to remove completely the functional arbitrariness in the considered system, i.e. to express the Lagrange multipliers $\lambda_\alpha(t)$ through canonical variables.

From (2.19) it follows that the choice of functions $\chi_\alpha(q,p,t)$ is restricted by the requirement

$$\det\|(\varphi_\alpha,\chi_\beta)\|\neq 0\,. \tag{2.20}$$

According to (2.20) the set of constraints (2.6) and (2.18) becomes the second-class one.

In addition to the condition (2.20) it is necessary to require that eqs. (2.18) be invariant relations for the canonical system (2.8). The substitution of (2.18) into (2.9) yields k equations linear with respect to the functions $\lambda_\alpha(t)$

$$\frac{d\chi_\beta}{dt}=\frac{\partial\chi_\beta}{\partial t}+(H_c,\chi_\beta)+\lambda_\alpha(t)(\varphi_\alpha,\chi_\beta)=0\,,$$

$$\alpha,\beta=1,\dots,k. \tag{2.21}$$

The solution of the system (2.21) has the form

$$\lambda_\alpha(t)=-\left[\frac{\partial\chi_\beta}{\partial t}+(H_c,\chi_\beta)\right]a_{\beta\alpha}\,, \qquad \alpha,\beta=1,\dots,k, \tag{2.22}$$

where the matrix $\|a_{\alpha\gamma}\|$ is inverse to that of the Poisson brackets $\|(\varphi_\gamma,\chi_\beta)\|$

$$a_{\alpha\gamma}(\varphi_\gamma,\chi_\beta)=\delta_{\alpha\beta}\,. \tag{2.23}$$

Thus, the dynamics of the considered system is completely determined by the set of second-class constraints (2.6), (2.18) and the canonical equations (2.8) which owing to (2.22) can be represented as follows

$$\frac{dq_i}{dt}=\frac{\partial H_c}{\partial p_i}-\left[\frac{\partial\chi_\alpha}{\partial t}+(H_c,\chi_\alpha)\right]a_{\alpha\beta}\frac{\partial\varphi_\beta}{\partial p_i}\,, \tag{2.24}$$

$$\frac{dp_i}{dt}=-\frac{\partial H_c}{\partial q_i}+\left[\frac{\partial\chi_\alpha}{\partial t}+(H_c,\chi_\alpha)\right]a_{\alpha\beta}\frac{\partial\varphi_\beta}{\partial q_i}\,,$$

$$i=1,\dots,n.$$

Substituting (2.22) into (2.9), we obtain the equation of motion for the function $\Psi(q,p,t)$

$$\frac{d\Psi}{dt}=\frac{\partial\Psi}{\partial t}-\frac{\partial\chi_\alpha}{\partial t}a_{\alpha\beta}(\varphi_\beta,\Psi)+(H_c,\Psi)^*\,, \tag{2.25}$$

where the Dirac bracket $(H_c,\Psi)^*$ is defined by the equality

$$(H_c,\Psi)^*=(H_c,\Psi)-(H_c,\chi_\alpha)a_{\alpha\beta}(\varphi_\beta,\Psi)\,. \tag{2.26}$$

Finally, in the case of time-dependent gauge conditions (2.18) the equations of the generalized Hamiltonian dynamics take the form (2.24). If $H_c=0$, then eqs. (2.24) admit solutions, different from the static ones, only for the gauge conditions (2.18) dependent on time t explicitly.

3. Reduction to the physical variables

To describe the dynamics of the considered system in terms of $(n-k)$ independent degrees of freedom, we make use of the invariant relations (2.6) and (2.18). In the classical mechanics it is well known that the existence of first integrals or invariant relations of the Hamiltonian system allows one to reduce their order (see, e.g., Levi-Civita and Amaldi [9]). Herewith the reduced system of equations has also the canonical form.

If m first integrals or invariant relations of a canonical system of $2n$ differential equations ($m \leq n$) are in involution, then one can reduce the number of these equations to 2(n-m). In general, when m first integral or invariant relations are not in involution, the order of the Hamiltonian system can be reduced only to 2n-m. It is necessary to emphasize that the reduction by using the first integrals leads to the canonical system whose general integral is that of the original canonical equations too. On the contrary, if the invariant relations replace the known first integrals, then the general solution of the reduced system is only a particular solution of the original Hamiltonian system. This solution describes only the trajectories which lie as a whole on the submanifold of the phase space defined by these invariant relations.

In the case under consideration just this submanifold is the physical phase space. To recover from the system (2.24) the equations that describe the dynamics on the physical phase space, we carry out the canonical transformation of variables q and p. Following to paper [1], we shall suppose that the left-hand sides of gauge condition (2.18) satisfy the relation

$$(\chi_\alpha, \chi_\beta) = 0, \quad \alpha, \beta = 1, \dots, k. \tag{3.1}$$

According to (3.1) the new canonical variables can be introduced as follows

$$Q_\alpha = Q_\alpha(q,p,t), \quad P_\alpha = \chi_\alpha(q,p,t), \quad \alpha = 1, \dots, k; \tag{3.2}$$

$$Q_a = Q_a(q,p,t), \quad P_a = P_a(q,p,t), \quad a = k+1, \dots, n; \tag{3.3}$$

$$(P_i, Q_j) = \delta_{ij}, \quad 1 \leq i, j \leq n. \tag{3.4}$$

We notice that the gauge conditions (2.18) lead to the appearance of the explicit time dependence in transformation (3.2)-(3.4).

To write the equations of motion (2.24) in new variables, it is necessary to substitute the new coordinates Q_i and momenta P_i from (3.2) and (3.3) instead of $\psi(q,p,t)$ into (2.25). As a result, we obtain

$$\dot{Q}_i = \frac{\partial K_c}{\partial P_i} + \frac{\partial K_c}{\partial Q_\alpha} A_{\alpha\beta} \frac{\partial \Phi_\beta}{\partial P_i}, \tag{3.5}$$

$$\dot{P}_i = -\frac{\partial K_c}{\partial Q_i} - \frac{\partial K_c}{\partial Q_\alpha} A_{\alpha\beta} \frac{\partial \Phi_\beta}{\partial Q_i},$$

$$i = 1, \dots, n.$$

The constraints (2.6) and (2.18) are rewritten now as

$$\varphi_\alpha(q(Q,P,t), p(Q,P,t)) \equiv \Phi_\alpha(Q,P,t) = 0, \tag{3.6}$$

$$P_\alpha = \chi_\alpha(q,p,t) = 0, \quad \alpha = 1, \dots, k. \tag{3.7}$$

In formula (3.5) $A_{\alpha\beta}(Q,P,t)$ denotes the matrix obtained from $a_{\alpha\beta}(q,p,t)$ by the transformations (3.2)-(3.4). The relation (2.23) becomes now

$$A_{\alpha\gamma} \frac{\partial \Phi_\gamma}{\partial Q_\beta} = -\delta_{\alpha\beta}. \tag{3.8}$$

The new canonical Hamiltonian $K_c(Q,P,t)$ is given by the expression

$$K_c(Q,P,t) = H_c(q(Q,P,t),p(Q,P,t)) + R(Q,P,t). \quad (3.9)$$

The second form of (3.9), caused by the explicit time dependence of the canonical transformation (3.2)-(3.4), is determined up to an arbitrary additive function of t as follows

$$\frac{\partial R(Q,P,t)}{\partial P_i} = \frac{\partial Q_i(q,p,t)}{\partial t}, \quad \frac{\partial R(Q,P,t)}{\partial Q_i} = -\frac{\partial P_i(q,p,t)}{\partial t}, \quad (3.10)$$

$$i = 1,\dots,n.$$

Owing to (3.8) the second half of the canonical equations (3.5) at $i = \alpha = 1,\dots,k$ turns into an identity which means that (3.7) are invariant relations with respect to (3.5)

$$\frac{dP_\alpha}{dt} = 0, \quad (3.11)$$

$$\alpha = 1,\dots,k.$$

That is k generalized momenta P_α take constant zero values along those phase trajectories of the system (3.5) which lie in the manifold defined by (3.7). The coordinates Q_α canonically conjugate to the P_α can be eliminated by using the constraints (3.6). Indeed, by virtue of condition (2.21) the appropriate Jacobian does not vanish

$$\det\|G_{\alpha\beta}\| \neq 0, \quad G_{\alpha\beta} = \frac{\partial \varphi_\alpha}{\partial Q_\beta} \quad (3.12)$$

and system of equations (3.6) can be solved for k coordinates Q_α

$$Q_\alpha = \Psi_\alpha(Q_a,P,t), \quad (3.13)$$

$$\alpha = 1,\dots,k; \quad a = k+1,\dots,n.$$

In this case it follows from (3.6) that the partial derivatives of functions $\Psi_\alpha(Q_a,P,t)$ with respect to Q_a and P_i are connected with those of original functions $\varphi_\alpha(Q,P,t)$ by the equalities

$$\frac{\partial \varphi_\alpha}{\partial P_a} + G_{\alpha\beta}\frac{\partial \Psi_\beta}{\partial P_a} = 0, \quad (3.14)$$

$$\frac{\partial \varphi_\alpha}{\partial Q_a} + G_{\alpha\beta}\frac{\partial \Psi_\beta}{\partial Q_a} = 0,$$

$$a = k+1,\dots,n,$$

$$\frac{\partial \varphi_\alpha}{\partial P_\gamma} + G_{\alpha\beta}\frac{\partial \Psi_\beta}{\partial P_\gamma} = 0, \quad (3.15)$$

$$\gamma = 1,\dots,k.$$

Taking into account the obvious identity

$$\frac{\partial \varphi_\alpha}{\partial Q_\gamma} = G_{\alpha\beta}\frac{\partial (Q_\beta - \Psi_\beta)}{\partial Q_\gamma}, \quad \gamma = 1,\dots,k$$

we rewrite the relations (3.14), (3.15) as follows

$$\frac{\partial \varphi_\alpha}{\partial P_i} = G_{\alpha\beta}\frac{\partial (Q_\beta - \Psi_\beta)}{\partial P_i}, \quad (3.16)$$

$$\frac{\partial \varphi_\alpha}{\partial Q_i} = G_{\alpha\beta}\frac{\partial (Q_\beta - \Psi_\beta)}{\partial Q_i},$$

$$i = 1,\dots,n.$$

Now the expressions (3.13) and (3.14)-(3.15) allow one to completely eliminate the dependent variables Q_α from the remaining equations of the system (3.5). As a result, these equiations at $i = a = k+1,\dots,n$ take the usual Hamiltonian form

$$\dot{Q}_a = \frac{\partial K_c}{\partial P_a} + \frac{\partial K_c}{\partial Q_\alpha}\frac{\partial \Psi_\alpha}{\partial P_a} = \frac{\partial \tilde{K}_c}{\partial P_a}, \quad (3.17)$$

$$\dot{P}_a = -\frac{\partial K_c}{\partial Q_a} - \frac{\partial K_c}{\partial Q_\alpha}\frac{\partial \Psi_\alpha}{\partial Q_a} = -\frac{\partial \tilde{K}_c}{\partial Q_a}, \quad a = k+1, \dots, n,$$

and with $i = \alpha = 1, \dots, k$ they are reduced to the condition that the constraints (3.13) be the invariant relations

$$\dot{\Psi}_\alpha = \frac{\partial K_c}{\partial P_\alpha} + \frac{\partial K_c}{\partial Q_\beta}\frac{\partial \Psi_\beta}{\partial P_\alpha} = \frac{\partial \tilde{K}_c}{\partial P_\alpha}, \tag{3.18}$$

$$\alpha = 1, \dots, k.$$

Here the canonical Hamiltonian $\tilde{K}_c(Q_a, P, t)$ is a result of substituting (3.13) into the function $K_c(Q, P, t)$

$$\tilde{K}_c(Q_a, P, t) = K_c(\Psi_\alpha(Q_a, P, t), Q_a, P, t). \tag{3.19}$$

In what follows the conditions (3.18) may be dropped out of consideration because they are identically satisfied in virtue of equations (3.11) and (3.17)

$$\frac{\partial \Psi_\alpha}{\partial t} + (\tilde{K}_c, \Psi_\alpha)_{P_a, Q_a} = \frac{\partial \tilde{K}_c}{\partial P_\alpha}, \tag{3.20}$$

$$\alpha = 1, \dots, k.$$

To prove the identities (3.20), one can use the relations (2.11)-(2.12) after they have been expressed in new canonical variables (3.2)-(3.4). Then, replacing the derivatives of functions $\varphi_\alpha(Q, P, t)$ by the expressions (3.16) and taking into account the condition (3.12), we obtain from (2.11) the following equalities

$$(Q_\alpha - \Psi_\alpha, Q_\beta - \Psi_\beta) = \tag{3.21}$$

$$= \frac{\partial \Psi_\beta}{\partial P_\alpha} - \frac{\partial \Psi_\alpha}{\partial P_\beta} + (\Psi_\alpha, \Psi_\beta)_{P_a, Q_a} = 0, \quad \alpha, \beta = 1, \dots, k.$$

We notice that the relations (3.13) cannot be used to fulfill (3.21) because the right-hand sides of these equalities do not depend on the coordinates Q_α. In other words, the involution of constraints (3.6) on the manifold M entails the involution of constraints (3.13) in the whole phase space [1].

By similar arguments as it has been done for (2.11), relations (2.12) are reduced to the equations

$$\frac{\partial H_c}{\partial P_\alpha} - (H_c, \Psi_\alpha)_{P_a, Q_a} + \frac{\partial H_c}{\partial Q_\beta}\frac{\partial \Psi_\alpha}{\partial P_\beta} = 0, \quad \alpha = 1, \dots, k. \tag{3.22}$$

Passing from H_c to the new canonical Hamiltonian K_c according to (3.9) and taking into account (3.10), we obtain from (3.22)

$$\frac{\partial \Psi_\alpha}{\partial t} + (K_c, \Psi_\alpha)_{P_a, Q_a} = \frac{\partial K_c}{\partial P_\alpha} + \frac{\partial K_c}{\partial Q_\beta}\frac{\partial \Psi_\alpha}{\partial P_\beta}, \tag{3.23}$$

$$\alpha = 1, \dots, k.$$

Finally, after the replacement of the coordinates Q_α by the expressions (3.13), we add these equations and the conditions (3.21) side by side and obtain directly the identities (3.20).

Now we pass to the consideration of the remaining equations (3.17). By substituting the relations (3.7) these equations are reduced to the Hamilton system of order $2(n-k)$ only with the independent canonical variables Q_a, P_a

$$\frac{dQ_a}{dt} = \frac{\partial K_c^*}{\partial P_a}, \quad \frac{dP_a}{dt} = -\frac{\partial K_c^*}{\partial Q_a}, \tag{3.24}$$

$$a = k+1, \dots, n,$$

where

[1] Using the obvious relations $(P_\alpha, Q_\beta - \Psi_\beta) = \delta_{\alpha\beta}$ in addition to (3.21), it is easy to see that the procedure of elimination of the dependent variables Q_α given here is completely equivalent to the canonical transformation $P_i, Q_i \to \tilde{P}_i, \tilde{Q}_i$, such that $\tilde{P}_\alpha = P_\alpha$, $\tilde{Q}_\alpha = Q_\alpha - \Psi_\alpha$, $\alpha = 1, \dots, k$.

$$K_c^*(Q_a, P_a, t) = \tilde{K}_c(Q_a, 0, P_a, t). \tag{3.25}$$

If the general integral of this system containing $2(n-k)$ arbitrary constants is known, then the dependence of coordinates Q_α on time can be obtained by the substitution of this integral and conditions (3.7) into the relation (3.13). Besides, taking into account that the canonically conjugate momenta $P_\alpha = 0$ we get such particular solutions of the original canonical system (3.5) that satisfy the invariant relations (3.6), (3.7). As eqs. (3.13) are also the invariant relations, the substitution of the solutions $Q_a(t), P_a(t)$ into (3.18) turns (3.18) into identities.

Thus, the independent canonical variables for the considered system are Q_a, P_a, $a = k+1, \dots, n$ which satisfy the Hamilton equations (3.24) with the effective Hamiltonian defined by formulae (3.25), (3.19), (3.10), and (3.9).

4. Construction of the functional integral

The physical canonical variables Q_a, P_a can be used now for the quantization of the original constrained system. At first, we represent in accordance with the standard rules [3] the matrix element of the evolution operator for the Hamiltonian system (3.24)-(3.25).

$$U(t'', t') = \exp\{-i(t''-t')K_c^*(Q_a, P_a, t)\} \tag{4.1}$$

by continual integral

$$I = \langle Q''_{k+1}, \dots, Q''_n | U(t'', t') | Q'_{k+1}, \dots, Q'_n \rangle =$$

$$= \int \exp\{i\int_{t'}^{t''} [P_a \dot{Q}_a - K_c^*(Q_a, P_a, t)]dt\} \prod_{t,a} \frac{dQ_a(t)dP_a(t)}{2\pi}, \tag{4.2}$$

$$a = k+1, \dots, n.$$

Now the functional integration in (4.2) can be extended to the whole phase space Q_i, P_i, $i = 1, \dots, n$ by the ansatz

$$\int \prod_{t,\alpha} \delta(P_\alpha(t)) \delta(Q_\alpha - \Psi_\alpha(Q_a, P, t)) \prod_{t,\alpha} \frac{dQ_\alpha(t) dP_\alpha(t)}{2\pi} =$$

$$= \int \prod_{t,\alpha} \delta(P_\alpha(t)) \delta(\Phi_\alpha(Q, P, t)) \det \| G_{\alpha\beta} \| \prod_{t,\alpha} \frac{dQ_\alpha(t) dP_\alpha(t)}{2\pi} = 1. \tag{4.3}$$

Here the δ - functions restrict the integration over Q_α and P_α, $\alpha = 1, \dots, k$ to the submanifold defined by constraints (3.6), (3.7), and we use the formula generalized to the case of arbitrary dimensions the well-known property of the usual δ - function

$$\delta(f(x)) = \frac{\delta(x-a)}{|f'(a)|}, \tag{4.4}$$

$$f(a) = 0.$$

Substitution of (4.3) into (4.2) gives for the functional integral I the following representation

$$I = \int \exp\{i\int_{t'}^{t''} [P_i \dot{Q}_i - K_c(Q, P, t)]dt\} \det \| G_{\alpha\beta} \| \times \tag{4.5}$$

$$\prod_{t,\alpha} \delta(P_\alpha(t)) \delta(\Phi_\alpha(Q, P, t)) \prod_{t,i} \frac{dQ_i(t) dP_i(t)}{2\pi}.$$

To deduce this formula, we extend the sum $\sum_{a=k+1}^{n} P_a \dot{Q}_a$ in the exponential to all variables $P_i, \dot{Q}_i$, $i = 1, \dots, n$ due to (3.7) and replace $K_c^*(Q_a, P_a, t)$ by $K_c(Q_i, P_i, t)$ according to (3.25) and (3.19).

Making use of the canonical transformation inverse to (3.2)-(3.4) we perform now the change of the functional variables in integral (4.5) [2]. As a result, we obtain the final expression for the matrix element of the evolution operator in the form of the continual integral over the whole phase space q, p

$$I = \int \exp\{i\int_{t'}^{t''} [p_i \dot{q}_i - H_c(q, p)]dt\} \det \| (\varphi_\alpha, \chi_\beta) \| \times \tag{4.6}$$

$$\prod_{t,\alpha} \delta(\varphi_\alpha(q, p)) \delta(\chi_\alpha(q, p, t)) \prod_{t,i} \frac{dq_i(t) dp_i(t)}{2\pi}.$$

This formula completely coincides with the appropriate expression in paper [1].

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2) We do not discuss here the possibility of performing such a replacement of the variables in the functional integral [10,11].

5. Conclusion

The basic result of this paper is the proof of the Hamiltonian structure of dynamics on the physical submanifold of phase space defined by the constraints and the gauge conditions. Just this problem takes the central place in the general investigation of the constrained dynamics in the phase space /12/. In paper /1/ and monographs /2-4/ the Hamiltonian structure of dynamics on the physical submanifold was assumed implicitly. In contrast to this we give the consistent derivation of the appropriate Hamiltonian equations (3.24) and (3.25).

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Барбашов Б.М., Нестеренко В.В., Червяков А.М. E2-84-521

Функциональный интеграл для систем со связями, явно зависящими от времени

Показано, что правила построения континуального интеграла в фазовом пространстве для систем с сингулярными лагранжианами, предложенные Л.Д.Фаддеевым, остаются справедливыми и при использовании калибровочных условий, явно зависящих от времени. Такие условия приходится рассматривать, например, в том случае, когда канонический гамильтониан в теории тождественно равен нулю /точечная релятивистская частица, релятивистская струна и т.д./. Функциональный интеграл вначале записывается в терминах физических переменных, для выделения которых используется каноническое преобразование, определяемое условиями калибровки. В случае нестационарных калибровочных условий каноническое преобразование оказывается явно зависящим от времени. Это приводит к дополнительному /по сравнению со случаем, рассмотренным Фаддеевым/ слагаемому в гамильтониане, задающем динамику на физическом подмногообразии фазового пространства.

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Barbashov B.M., Chervyakov A.M., Nesterenko V.V. E2-84-521

Functional Integral for Systems with Time-Dependent Constraints

It is shown that the method for constructing the functional integral in the phase space proposed by L.D.Faddeev remains correct when the gauge conditions are explicitly time-dependent. Such gauges should be used in that case when the canonical Hamiltonian vanishes identically (a point relativistic particle, the relativistic string, and other parametrization-invariant theories). At first the functional integral is written in terms of physical canonical variables which are singled out by the canonical transformation defined by constraints and gauge conditions. For time-dependent gauge conditions this transformation appears to be explicitly time-dependent. In comparison with the case considered by L.D.Faddeev this results in an additional term in the Hamiltonian determining the dynamics on a physical submanifold of the phase space.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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