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# ON THE ELLIPTIC BASIS OF A CIRCULAR OSCILLATOR

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## 1. INTRODUCTION

In paper [1] we have found wave functions of the circular oscillator (c.o.) in elliptic coordinates (elliptic bases of c.o.). In our opinion, it is of interest to obtain a complete set of commuting operators determining the elliptic bases, and then, by using this set to express the elliptic bases by expansions in terms of polar and Cartesian bases of c.o.

## 2. THE ELLIPTIC INTEGRAL OF MOTION OF C.O.

We will take elliptic coordinates as follows: $x = \frac{R}{2}\,\mathrm{ch}\,\xi\cos\eta$, $y = \frac{R}{2}\,\mathrm{sh}\,\xi\sin\eta$, where the parameter $R$ and coordinates $\xi$ and $\eta$ change in the limits $0 \le R < \infty$, $0 \le \xi < \infty$, and $0 \le \eta \le 2\pi$. In the system $h = \mu = \omega = 1$ and variables $\xi$ and $\eta$ the Schrödinger equation for c.o. (with energy $E$)

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\Psi + \left[\frac{ER^2}{4}(\mathrm{ch}\,2\xi - \cos 2\eta) - \frac{R^4}{64}(\mathrm{ch}^2 2\xi - \cos^2 2\eta)\right]\Psi = 0$$

is splitted, and upon introducing the separation constant $Q$ and eliminating the energy $E$ from the obtained equations we arrive at the equation

$$\hat{Q}\Psi \equiv \left\{\frac{1}{\mathrm{ch}\,2\xi - \cos 2\eta}\left(\cos 2\eta\frac{\partial^2}{\partial\xi^2} + \mathrm{ch}\,2\xi\frac{\partial^2}{\partial\eta^2}\right) - \frac{R^4}{64}\,\mathrm{ch}\,2\xi\cos 2\eta\right\}\Psi = Q\Psi .$$

It is clear that operator $\hat{Q}$ commutes with the c.o. Hamiltonian and is therefore an integral of motion. It is easy to show that when $R \to 0$ and $R \to \infty$ there are valid the limiting relations

$$\lim_{R\to 0}\hat{Q} = -\hat{L}^2 \equiv -\left(-i\frac{\partial}{\partial\phi}\right)^2 , \quad \lim_{R\to\infty}\frac{2\hat{Q}}{R^2} = -\hat{\mathcal{P}} \equiv -\frac{1}{4}\left(x^2 - \frac{\partial^2}{\partial x^2} - y^2 + \frac{\partial^2}{\partial y^2}\right).$$

Operators $\hat{L}$ and $\hat{\mathcal{P}}$ commute with the c.o. Hamiltonian and are known [2] to have eigenfunctions in the form of solutions of the Schrödinger equation for c.o. in polar and Cartesian coordinates. Expressing operator $\hat{Q}$ in terms of the Cartesian coordinates and making some calculations we may prove the equality $\hat{Q} = -\hat{L}^2 - \frac{R^2}{2}\hat{\mathcal{P}} + \frac{R^4}{64}$. It is also convenient to employ the



operator $\hat{\Lambda} = \hat{Q} - \frac{R^4}{64} = -\hat{L}^2 - \frac{R^2}{2}\hat{\mathcal{P}}$. We shall call it the elliptic integral of motion of c.o. Eigenvalues of $\hat{\Lambda}$ will be denoted by $\lambda$. Obviously, $\lambda = Q - R^4/64$.

## 3. THE ELLIPTIC BASIS OF C.O.

Hamiltonian $\hat{H}$, elliptic integral of motion $\hat{\Lambda}$, and operators $\hat{P}_{xy}$ and $\hat{P}_y$ inverting coordinates $(x, y) \to (-x, -y)$ and $(x, y) \to (x, -y)$ form a complete set of commuting, and uniquely fixing the elliptic basis, operators. We present Table 1 of the classification of elliptic, polar, and Cartesian states over eigenvalues of operators $\hat{H}$, $\hat{P}_{xy}$, and $\hat{P}_y$.

*Table 1*

| Elliptic basis | Polar basis | Cartesian basis | $\varepsilon$ | $P_{xy}$ | $P_y$ | $\lambda$ |
|---|---|---|---|---|---|---|
| $\Psi^{(+,+)}_{E,\lambda}$ | $\Phi^{(+,+)}_{2n,2p}$ | $\Pi_{2k,2n-k}$ | $2n+1$ | + | + | $\lambda^{(+,+)}$ |
| $\Psi^{(-,+)}_{E,\lambda}$ | $\Phi^{(-+)}_{2n+1,2p+1}$ | $\Pi_{2k+1,2n-2k}$ | $(2n+1)+1$ | - | + | $\lambda^{(-,+)}$ |
| $\Psi^{(+,-)}_{E,\lambda}$ | $\Phi^{(+,-)}_{2n+2,2p+2}$ | $\Pi_{2k+1,2n-2k+1}$ | $(2n+2)+1$ | + | - | $\lambda^{(+,-)}$ |
| $\Psi^{(-,-)}_{E,\lambda}$ | $\Phi^{(--)}_{2n+1,2p+1}$ | $\Pi_{2k,2n-2k+1}$ | $(2n+1)+1$ | - | - | $\lambda^{(-,-)}$ |

The energy of c.o. is $E = N + 1$. In table 1 energy levels with $N = 2n$, $N = 2n+1$, $N = 2n+2$, and $N = 2n+1$ are placed in different cells, and the corresponding states differ, within a given basis (elliptic, polar, Cartesian), in $P_{xy}$ and $P_y$ parity denoted in the first two columns by superscripts of wave functions and in the third column by indices of Cartesian bases.

We will take the polar and Cartesian bases in the form

$$\Phi^{(+,+)}_{2n,2p} = R_{2n,2p}(r)\frac{1}{\sqrt{\pi}}\cos 2p\phi, \quad \Phi^{(-,+)}_{2n+1,2p+1} = R_{2n+1,2p+1}(r)\frac{1}{\sqrt{\pi}}\cos(2p+1)\phi,$$

$$\Phi^{(+,-)}_{2n+2,2p+2} = R_{2n+2,2p+2}(r)\frac{i}{\sqrt{\pi}}\sin(2p+2)\phi, \quad \Phi^{(-,-)}_{2n+1,2p+1} = R_{2n+1,2p+1}\frac{i}{\sqrt{\pi}}\sin(2p+1)\phi,$$

$$\Pi_{n_1 n_2} = \bar{H}_{n_1}(x)\,\bar{H}_{n_2}(y), \quad n_1 = 2k, \quad n_1 = 2k+1 .$$

Functions $R_{Nm}$ and $\bar{H}_n$ are normalized to unity, and their form is well known [3]. Integers p and k change within the limits $0 \le p \le n$ and $0 \le k \le n$.

## 4. THE CONNECTION OF THE C.O. ELLIPTIC BASIS WITH THE POLAR AND CARTESIAN ONE

The connection of the elliptic with the polar bases at a given energy and parities $P_{xy}$ and $P_y$ is given by the expansions

$$\Psi^{(+,+)}_{E\lambda} = \sum_{p=0}^{n} W^{(+,+)}_{2p}\,\Phi^{(+,+)}_{2n,2p}, \quad \Psi^{(-,+)}_{E\lambda} = \sum_{p=0}^{n} W^{(-,+)}_{2p+1}\,\Phi^{(-,+)}_{2n+1,2p+1},$$

$$\Psi^{(+,-)}_{E\lambda} = \sum_{p=0}^{n} W^{(+,-)}_{2p+2}\,\Phi^{(+,-)}_{2n+2,2p+2}, \quad \Psi^{(-,-)}_{E\lambda} = \sum_{p=0}^{n} W^{(-,-)}_{2p+1}\,\Phi^{(-,-)}_{2n+1,2p+1}.$$

With the help of these expansions the problem of eigenvalues and eigenfunctions of operator $\hat{\Lambda}$ can be reformulated in the form of systems of linear equations

$$\sum_{p=0}^{n} \{\mathcal{P}^{(++,++)}_{2p',2p} + \frac{2}{R^2}(\lambda^{(+,+)} + 4p^2)(1+\delta_{p0})\,\delta_{pp'}\}\,W^{(+,+)}_{2p} = 0,$$

$$\sum_{p=0}^{n} \{\mathcal{P}^{(-+,-+)}_{2p'+1,2p+1} + \frac{2}{R^2}[\lambda^{(-,+)} + (2p+1)^2]\,\delta_{pp'}\}\,W^{(-,+)}_{2p+1} = 0,$$

$$\sum_{p=0}^{n} \{\mathcal{P}^{(+-,+-)}_{2p'+2,2p+2} + \frac{2}{R^2}[\lambda^{(+,-)} + (2p+2)^2]\,\delta_{pp'}\}\,W^{(+,-)}_{2p+2} = 0,$$

$$\sum_{p=0}^{n} \{\mathcal{P}^{(--,--)}_{2p'+1,2p+1} + \frac{2}{R^2}[\lambda^{(-,-)} + (2p+1)^2]\,\delta_{pp'}\}\,W^{(-,-)}_{2p+1} = 0.$$

The matrix elements of operator $\hat{\mathcal{P}}$ over the polar bases are calculated with the use of the expansion of the c.o. polar basis found in [4]: $\Phi_{Nm} = R_{Nm}(r)\,\frac{e^{im\phi}}{\sqrt{2\pi}}$ having no definite $P_{xy}$ and $P_y$ parities over the Cartesian basis of this type

$$\Phi_{Nm} = \sum_{p=-N}^{N}{}' \, d^{N/2}_{\frac{m}{2},\frac{p}{2}}\left(\frac{\pi}{2}\right) \Pi_{\frac{N+p}{2},\frac{N-p}{2}} .$$

The prime of the sum means that summation rund only over the values of p with the parity of number N. The Wigner d-functions are taken from the monograph [5]. Using the recurrence relation

$$-M\,d^{J}_{M,M'}\left(\frac{\pi}{2}\right) = \frac{1}{2}\sqrt{(J+M')(J-M'+1)}\,d^{J}_{M,M'-1}\left(\frac{\pi}{2}\right) + \frac{1}{2}\sqrt{(J-M')(J+M'+1)}\,d^{J}_{M,M'+1}\left(\frac{\pi}{2}\right)$$



we may obtain the following formulae:

$$\mathcal{P}^{(++,++)}_{2p',2p} = \frac{1}{2}\sqrt{(n+p')(n-p'+1)}\,\delta_{p,p'-1} + \frac{1}{2}\sqrt{(n-p')(n+p'+1)}\,\delta_{p,p'+1} + \frac{1}{2}\sqrt{n(n+1)}\,(\delta_{p0}\,\delta_{p'1} + \delta_{p1}\,\delta_{p'0}),$$

$$\mathcal{P}^{(-+,-+)}_{2p'+1,2p+1} = \frac{1}{2}\sqrt{(n-p'+1)(n+p'+1)}\,\delta_{p,p'-1} + \frac{1}{2}\sqrt{(n-p')(n+p'+2)}\,\delta_{p,p'+1} + \frac{1}{2}(n+1)\,\delta_{p0}\,\delta_{p'0},$$

$$\mathcal{P}^{(--,--)}_{2p'+1,2p+1} = \frac{1}{2}\sqrt{(n-p'+1)(n+p'+1)}\,\delta_{p,p'-1} + \frac{1}{2}\sqrt{(n-p')(n+p'+2)}\,\delta_{p,p'+1} - \frac{1}{2}(n+1)\,\delta_{p0}\,\delta_{p'0},$$

$$\mathcal{P}^{(+-,+-)}_{2p'+2,2p+2} = \frac{1}{2}\sqrt{(n-p'+1)(n+p'+2)}\,\delta_{p,p'-1} + \frac{1}{2}\sqrt{(n-p')(n+p'+3)}\,\delta_{p,p'+1}.$$

Explicitly these matrices have the form:

$$\mathcal{P}^{(++,++)}_{2p',2p} = \begin{bmatrix} 0 & \sqrt{n(n+1)} & 0 & \dots & 0 & 0 \\ \sqrt{n(n+1)} & 0 & \frac{1}{2}\sqrt{(n-1)(n+2)} & \dots & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{(n-1)(n+2)} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \frac{\sqrt{2n}}{2} \\ 0 & 0 & 0 & \dots & \frac{\sqrt{2n}}{2} & 0 \end{bmatrix}$$

$$\mathcal{P}^{(-+,-+)}_{2p'+1,2p+1} = \begin{bmatrix} \frac{1}{2}(n+1) & \frac{1}{2}\sqrt{n(n+2)} & 0 & \dots & 0 & 0 \\ \frac{1}{2}\sqrt{n(n+2)} & 0 & \frac{1}{2}\sqrt{(n-1)(n+3)} & \dots & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{(n-1)(n+3)} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \frac{\sqrt{2n+1}}{2} \\ 0 & 0 & 0 & \dots & \frac{\sqrt{2n+1}}{2} & 0 \end{bmatrix}$$

$$
\mathcal{P}^{(--,--)}_{2p'+1,2p+1} = \begin{bmatrix}
-\frac{1}{2}(n+1) & \frac{1}{2}\sqrt{n(n+2)} & 0 & \dots & 0 & 0 \\
\frac{1}{2}\sqrt{n(n+2)} & 0 & \frac{1}{2}\sqrt{(n-1)(n+3)} & \dots & 0 & 0 \\
0 & \frac{1}{2}\sqrt{(n-1)(n+3)} & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & \frac{\sqrt{2n+1}}{2} \\
0 & 0 & 0 & \dots & \frac{\sqrt{2n+1}}{2} & 0
\end{bmatrix}
$$

$$
\mathcal{P}^{(+-,+-)}_{2p'+2,2p+2} = \begin{bmatrix}
0 & \frac{1}{2}\sqrt{n(n+3)} & 0 & \dots & 0 & 0 \\
\frac{1}{2}\sqrt{n(n+3)} & 0 & \frac{1}{2}\sqrt{(n-1)(n+4)} & \dots & 0 & 0 \\
0 & \frac{1}{2}\sqrt{(n-1)(n+4)} & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & \frac{\sqrt{2n+2}}{2} \\
0 & 0 & 0 & \dots & \frac{\sqrt{2n+2}}{2} & 0
\end{bmatrix}
$$

With the aid of the above results it is not difficult to establish the trinomial recurrence relations

$$
\sqrt{(n+p')(n-p'+1)}\,W^{(+,+)}_{2p'-2} + \sqrt{(n-p')(n+p'+1)}\,W^{(+,+)}_{2p'+2} +
$$

$$
+\sqrt{n(n+1)}\,(W^{(+,+)}_{0}\delta_{p'1} + W^{(+,+)}_{2}\delta_{p'0}) + \frac{4}{R^2}(\lambda^{(+,+)} + 4p'^2)(1+\delta_{p'0})\,W^{(+,+)}_{2p'} = 0,
$$

$$
\sqrt{(n-p'+1)(n+p'+1)}\,W^{(-,+)}_{2p'-1}(1-\delta_{p'0}) + \sqrt{(n-p')(n+p'+2)}\,W^{(-,+)}_{2p'+3} +
$$

$$
+ (n+1)\delta_{p'0}\,W^{(-,+)}_{1} + \frac{4}{R^2}[\lambda^{(-,+)} + (2p'+1)^2]\,W^{(-,+)}_{2p'+1} = 0,
$$

$$
\sqrt{(n-p'+1)(n+p'+1)}\,(1-\delta_{p'0})\,W^{(-,-)}_{2p'-1} + \sqrt{(n-p')(n+p'+2)}\,W^{(-,-)}_{2p'+3} -
$$

$$
-(n+1)\delta_{p'0}\,W^{(-,-)}_{1} + \frac{4}{R^2}[\lambda^{(-,-)} + (2p'+1)^2]\,W^{(-,-)}_{2p'+1} = 0,
$$



$$
\sqrt{(n-p'+1)(n+p'+2)}\,(1-\delta_{p'0})\,W^{(+,-)}_{2p'} + \sqrt{(n-p')(n+p'+3)}\,W^{(+,-)}_{2p'+4} +
$$

$$
+ \frac{4}{R^2}[\lambda^{(+,-)} + (2p'+2)^2]\,W^{(+,-)}_{2p'+2} = 0
$$

which make the basis for constructing the expansions of elliptic over polar bases. There also hold the conditions

$$
2|W^{(+,+)}_{0}|^2 + \sum_{p=1}^{n}|W^{(+,+)}_{2p}|^2 = 1,
$$

$$
\sum_{p=0}^{n}|W^{(-,+)}_{2p+1}|^2 = \sum_{p=0}^{n}|W^{(-,-)}_{2p+1}|^2 = \sum_{p=0}^{n}|W^{(+,-)}_{2p+2}|^2 = 1,
$$

representing the normalization of the bases participating in the expansion (all the elliptic bases are assumed to be normalized to unity). Then the method of calculation of the coefficients $W$ and eigenvalues $\lambda$ is obvious. Some results for small values of $n$ and $p$ we report in Tables 2-5.

*Table 2*

| n | p | $W^{(+,+)}_{2p}(R^2)$ | $\lambda^{(+,+)}$ |
|---|---|---|---|
| 0 | 0 | 1 | $\lambda^{(+,+)} = 0$ |
| 1 | 0 | $\left(\frac{\lambda^{(+,+)}+4}{3\lambda^{(+,+)}+4}\right)^{1/2}$ | $\lambda^{(+,+)}(\lambda^{(+,+)}+4) = \frac{R^4}{4}$ |
| 1 | 1 | $-\frac{2\sqrt{2}}{R^2}\lambda^{(+,+)}\left(\frac{\lambda^{(+,+)}+4}{3\lambda^{(+,+)}+4}\right)^{1/2}$ | |
| 2 | 0 | $\left\{1+\frac{8}{3}\left(\frac{\lambda^{(+,+)}}{R^2}\right)^2+\frac{2}{3}\left(\frac{\lambda^{(+,+)}}{\lambda^{(+,+)}+16}\right)^2\right\}^{-1/2}$ | $\lambda^{(+,+)}(\lambda^{(+,+)}+4)(\lambda^{(+,+)}+16) = -R^4(\lambda^{(+,+)}+12)$ |
| 2 | 1 | $-\frac{4}{\sqrt{6}}\frac{\lambda^{(+,+)}}{R^2}\left\{1+\frac{8}{3}\left(\frac{\lambda^{(+,+)}}{R^2}\right)^2+\frac{2}{3}\left(\frac{\lambda^{(+,+)}}{\lambda^{(+,+)}+16}\right)^2\right\}^{-1/2}$ | |
| 2 | 2 | $\frac{2}{\sqrt{6}}\frac{\lambda^{(+,+)}}{\lambda^{(+,+)}+16}\left\{1+\frac{8}{3}\left(\frac{\lambda^{(+,+)}}{R^2}\right)^2+\frac{2}{3}\left(\frac{\lambda^{(+,+)}}{\lambda^{(+,+)}+16}\right)\right\}^{-1/2}$ | |

The above method is, of course, applicable to finding expansions of the elliptic bases over the Cartesian ones. Here we will present only the final results. If the expansion is chosen in the form

$$
\Psi^{(+,+)}_{E\lambda} = \sum^{n} U^{(+,+)}_{2k}\,\Pi_{2k,2n-2k}\,, \quad \Psi^{(-,+)}_{E\lambda} = \sum^{n} U^{(-,+)}_{2k+1}\,\Pi_{2k+1,2n-2k}\,,
$$

*Table 3*

| n | p | $W_{2p+1}^{(-,+)}(R^2)$ | $\lambda^{(-,+)}$ |
|---|---|---|---|
| 0 | 0 | 1 | $\lambda^{(-,+)} = -1 - \frac{R^2}{4}$ |
| 1 | 0 | $\left(\frac{\lambda^{(-,+)}+9}{2\lambda^{(-,+)}+10+R^2/2}\right)^{1/2}$ | $(\lambda^{(-,+)}+1)(\lambda^{(-,+)}+9) = -\frac{R^2}{2}(\lambda^{(-,+)}+9)+\frac{3R^4}{16}$ |
| 1 | 1 | $-\frac{4}{\sqrt{3}R^2}(\lambda^{(-,+)}+1+R^2/2)\left(\frac{\lambda^{(-,+)}+9}{2\lambda^{(-,+)}+10+R^2/2}\right)^{1/2}$ | |
| 2 | 0 | $\left\{1+\frac{2}{R^4}(\lambda^{(-,+)}+1+3R^2/4)^2+\frac{5}{8}\left(\frac{\lambda^{(-,+)}+1+3R^2/4}{\lambda^{(-,+)}+25}\right)^2\right\}^{-1/2}$ | $(\lambda^{(-,+)}+1)(\lambda^{(-,+)}+9)(\lambda^{(-,+)}+25) = -\frac{3R^2}{4}(\lambda^{(-,+)}+9)(\lambda^{(-,+)}+25)+\frac{R^4}{16}(13\lambda^{(-,+)}+205)+\frac{15R^6}{64}$ |
| 2 | 1 | $-\frac{\sqrt{2}}{R^2}(\lambda^{(-,+)}+1+3R^2/4)\left\{1+\frac{2}{R^4}(\lambda^{(-,+)}+1+3R^2/4)^2+\frac{5}{8}\left(\frac{\lambda^{(-,+)}+1+3R^2/4}{\lambda^{(-,+)}+25}\right)^2\right\}^{-1/2}$ | |
| 2 | 2 | $\frac{\sqrt{10}}{4}\frac{\lambda^{(-,+)}+1+3R^2/4}{\lambda^{(-,+)}+25}\left\{1+\frac{2}{R^4}(\lambda^{(-,+)}+1+3R^2/4)^2+\frac{5}{8}\left(\frac{\lambda^{(-,+)}+1+3R^2/4}{\lambda^{(-,+)}+25}\right)^2\right\}^{-1/2}$ | |

*Table 4*

| n | p | $W_{2p+1}^{(-,-)}(R^2)$ | $\lambda^{(-,-)}$ |
|---|---|---|---|
| 0 | 0 | 1 | $\lambda^{(-,-)} = -1 + \frac{R^2}{4}$ |
| 1 | 0 | $\left(\frac{\lambda^{(-,-)}+9}{2\lambda^{(-,-)}+10-R^2/2}\right)^{1/2}$ | $(\lambda^{(-,-)}+1)(\lambda^{(-,-)}+9) = \frac{R^2}{2}(\lambda^{(-,-)}+9)+\frac{3R^4}{16}$ |
| 1 | 1 | $-\frac{4}{\sqrt{3}R}(\lambda^{(-,-)}+1-R^2/2)\left(\frac{\lambda^{(-,-)}+9}{2\lambda^{(-,-)}+10-R^2/2}\right)^{1/2}$ | |
| 2 | 0 | $\left\{1+\frac{2}{R^4}(\lambda^{(-,-)}+1-3R^2/4)^2+\frac{5}{8}\left(\frac{\lambda^{(-,-)}+1-3R^2/4}{\lambda^{(-,-)}+25}\right)^2\right\}^{-1/2}$ | $(\lambda^{(-,-)}+1)(\lambda^{(-,-)}+9)(\lambda^{(-,-)}+25) = \frac{3R^2}{4}(\lambda^{(-,-)}+9)(\lambda^{(-,-)}+25)+\frac{R^4}{16}(13\lambda^{(-,-)}+205)-\frac{15R^6}{64}$ |
| 2 | 1 | $-\frac{\sqrt{2}}{R^2}(\lambda^{(-,-)}+1-3R^2/4)\left\{1+\frac{2}{R^4}(\lambda^{(-,-)}+1-3R^2/4)^2+\frac{5}{8}\left(\frac{\lambda^{(-,-)}+1-3R^2/4}{\lambda^{(-,-)}+25}\right)^2\right\}^{-1/2}$ | |
| 2 | 2 | $\frac{\sqrt{10}}{4}\frac{\lambda^{(-,-)}+1-3R^2/4}{\lambda^{(-,-)}+25}\left\{1+\frac{2}{R^4}(\lambda^{(-,-)}+1-3R^2/4)^2+\frac{5}{8}\left(\frac{\lambda^{(-,-)}+1-3R^2/4}{\lambda^{(-,-)}+25}\right)^2\right\}^{-1/2}$ | |



*Table 5*

| n | p | $W_{2p+2}^{(+,-)}(R^2)$ | $\lambda^{(+,-)}$ |
|---|---|---|---|
| 0 | 0 | 1 | $\lambda^{(+,-)} = -4$ |
| 1 | 0 | $\frac{1}{\sqrt{2}}\left(\frac{\lambda^{(+,-)}+16}{\lambda^{(+,-)}+10}\right)^{1/2}$ | $(\lambda^{(+,-)}+4)(\lambda^{(+,-)}+16) = \frac{R^4}{4}$ |
| 1 | 1 | $-\frac{\sqrt{2}}{R^2}(\lambda^{(+,-)}+4)\left(\frac{\lambda^{(+,-)}+16}{\lambda^{(+,-)}+10}\right)^{1/2}$ | |
| 2 | 0 | $\left\{1+\frac{8}{5R^4}(\lambda^{(+,-)}+4)^2+\frac{3}{5}\left(\frac{\lambda^{(+,-)}+4}{\lambda^{(+,-)}+36}\right)^2\right\}^{-1/2}$ | $(\lambda^{(+,-)}+4)(\lambda^{(+,-)}+16)\times(\lambda^{(+,-)}+36) = R^4(\lambda^{(+,-)}+24)$ |
| 2 | 1 | $-\frac{4}{\sqrt{10}R^2}(\lambda^{(+,-)}+4)\left\{1+\frac{8}{5R^4}(\lambda^{(+,-)}+4)^2+\frac{3}{5}\left(\frac{\lambda^{(+,-)}+4}{\lambda^{(+,-)}+36}\right)^2\right\}^{-1/2}$ | |
| 2 | 2 | $\sqrt{\frac{3}{5}}\frac{\lambda^{(+,-)}+4}{\lambda^{(+,-)}+36}\left\{1+\frac{8}{5R^4}(\lambda^{(+,-)}+4)^2+\frac{3}{5}\left(\frac{\lambda^{(+,-)}+4}{\lambda^{(+,-)}+36}\right)^2\right\}^{-1/2}$ | |

$$\Psi_{E\lambda}^{(+,-)} = \sum^{n} U_{2k+1}^{(+,-)}\,\Pi_{2k+1,\,2n-2k+1},\quad \Psi_{E\lambda}^{(-,-)} = \sum^{n} U_{2k}^{(-,-)}\,\Pi_{2k,\,2n-2k+1}$$

the recurrence relations for coefficients U are of the form:

$$2\sqrt{(k+1)(2k+1)(n-k)(2n-2k-1)}\,U_{2k+2}^{(+,+)} + 2\sqrt{k(2k-1)(n-k+1)(2n-2k+1)}\,U_{2k-2}^{(+,+)} +$$

$$+\left[\lambda^{(+,+)} + 8k(n-k) + 2n + \frac{R^2}{2}(2k-n)\right]U_{2k}^{(+,+)} = 0,$$

$$2\sqrt{(k+1)(2k+3)(n-k)(2n-2k-1)}\,U_{2k+3}^{(-,+)} + 2\sqrt{k(2k+1)(n-k+1)(2n-2k+1)}\,U_{2k-1}^{(-,+)} +$$

$$+\left[\lambda^{(-,+)} + 4(2k+1)(n-k) + 2n+1 + \frac{R^2}{4}(4k-2n+1)\right]U_{2k+1}^{(-,+)} = 0,$$

$$2\sqrt{(k+1)(2k+3)(n-k)(2n-2k+1)}\,U_{2k+3}^{(+,-)} + 2\sqrt{k(2k+1)(n-k+1)(2n-2k+3)}\,U_{2k-1}^{(+,-)} +$$

$$+\left[\lambda^{(+,-)} + 2(2k+1)(2n-2k+1) + 2n + 2 - \frac{R^2}{2}(2k-n)\right]U_{2k+1}^{(+,-)} = 0,$$

$$2\sqrt{(k+1)(2k+1)(n-k)(2n-2k+1)}\,U_{2k+2}^{(-,-)} + 2\sqrt{k(2k-1)(n-k+1)(2n-2k+3)}\,U_{2k-2}^{(-,-)} +$$

$$+ [\lambda^{(-,-)} + 4k(2n-2k+1) + 2n+1 + \frac{R^2}{4}(4k-2n-1)]\,U_{2k}^{(-,-)} = 0$$

and the coefficients obey the normalization conditions

$$\sum_{k=0}^{n} |U_{2k}^{(+,+)}|^2 = \sum_{k=0}^{n} |U_{2k+1}^{(-,+)}|^2 = \sum_{k=0}^{n} |U_{2k+1}^{(+,-)}|^2 = \sum_{k=0}^{n} |U_{2k}^{(-,-)}|^2 = 1$$

and for small n and k are listed in Tables 6-9.

*Table 6*

| n | k | $U_{2k}^{(+,+)}(R^2)$ | $\lambda^{(+,+)}$ |
|---|---|---|---|
| 0 | 0 | 1 | $\lambda^{(+,+)} = 0$ |
| 1 | 0 | $\frac{1}{\sqrt{2}}\left(\frac{\lambda^{(+,+)}+2+R^2/2}{\lambda^{(+,+)}+2}\right)^{1/2}$ | $\lambda^{(+,+)}(\lambda^{(+,+)}+4)$ |
| 1 | 1 | $-\frac{1}{2\sqrt{2}}(\lambda^{(+,+)}+2-\frac{R^2}{2})\left(\frac{\lambda^{(+,+)}+2+R^2/2}{\lambda^{(+,+)}+2}\right)^{1/2}$ | $= \frac{R^4}{4}$ |
| 2 | 0 | $\left\{1+\frac{1}{24}(\lambda^{(+,+)}+4-R^2)^2+\left(\frac{\lambda^{(+,+)}+4-R^2}{\lambda^{(+,+)}+4+R^2}\right)^2\right\}^{-1/2}$ | $\lambda^{(+,+)}(\lambda^{(+,+)}+4)(\lambda^{(+,+)}+16)=$ |
| 2 | 1 | $-\frac{1}{2\sqrt{6}}(\lambda^{(+,+)}+4-R^2)\left\{1+\frac{1}{24}(\lambda^{(+,+)}+4-R^2)^2+\left(\frac{\lambda^{(+,+)}+4-R^2}{\lambda^{(+,+)}+4+R^2}\right)^2\right\}^{-1/2}$ | $= R^4(\lambda^{(+,+)}+12)$ |
| 2 | 2 | $\frac{\lambda^{(+,+)}+4-R^2}{\lambda^{(+,+)}+4+R^2}\left\{1+\frac{1}{24}(\lambda^{(+,+)}+4-R^2)^2+\left(\frac{\lambda^{(+,+)}+4-R^2}{\lambda^{(+,+)}+4+R^2}\right)^2\right\}^{-1/2}$ | |

## 5. PERTURBATION-THEORY METHOD

The disentagling of the above trinomial recurrence relations is connected with the solution of high-order algebraic equations and in the general case cannot be performed analytically. Nevertheless, when $R \ll 1$ and $R \gg 1$, properties of the elliptic integral of motion are determined by the term $\hat{L}^2$ or $\hat{\mathcal{P}}$, and perturbation theory may be used. When $R \ll 1$, the second term in operator $\hat{\Lambda}$ is taken as a perturbation. If we supply the eigenvalues of $\lambda$ with index q changing in the range $0 \le q \le n$, divide



*Table 7*

| n | k | $U_{2k+1}^{(-,+)}(R^2)$ | $\lambda^{(-,+)}$ |
|---|---|---|---|
| 0 | 0 | 1 | $\lambda^{(-,+)} = -1-\frac{R^2}{4}$ |
| 1 | 0 | $\left(\frac{\lambda^{(-,+)}+3+3R^2/4}{2\lambda^{(-,+)}+10+R^2/2}\right)^{1/2}$ | $(\lambda^{(-,+)}+1)(\lambda^{(-,+)}+9) =$ |
| 1 | 1 | $-\frac{1}{2\sqrt{3}}(\lambda^{(-,+)}+7-R^2/4)\left(\frac{\lambda^{(-,+)}+3+3R^2/4}{2\lambda^{(-,+)}+10+R^2/2}\right)^{1/2}$ | $= -\frac{R^2}{2}(\lambda^{(-,+)}+9)+\frac{3R^4}{16}$ |
| 2 | 0 | $\left\{1+\frac{1}{72}(\lambda^{(-,+)}+13-3R^2/4)^2+\frac{5}{9}\left(\frac{\lambda^{(-,+)}+13-3R^2/4}{\lambda^{(-,+)}+5+5R^2/4}\right)^2\right\}^{-1/2}$ | $(\lambda^{(-,+)}+1)(\lambda^{(-,+)}+9)(\lambda^{(-,+)}+25)=$ |
| 2 | 1 | $-\frac{\lambda^{(-,+)}+13-3R^2/4}{6\sqrt{2}}\left\{1+\frac{1}{72}(\lambda^{(-,+)}+13-3R^2/4)^2+\frac{5}{9}\left(\frac{\lambda^{(-,+)}+13-3R^2/4}{\lambda^{(-,+)}+5+5R^2/4}\right)^2\right\}^{1/2}$ | $= -\frac{3R^2}{4}(\lambda^{(-,+)}+9)(\lambda^{(-,+)}+25)+$ $+\frac{R^4}{16}(13\lambda^{(-,+)}+205)+\frac{15R^6}{64}$ |
| 2 | 2 | $\frac{\sqrt{5}}{3}\frac{\lambda^{(-,+)}+13-3R^2/4}{\lambda^{(-,+)}+5+5R^2/4}\left\{1+\frac{1}{72}(\lambda^{(-,+)}+13-3R^2/4)^2+\frac{5}{9}\left(\frac{\lambda^{(-,+)}+13-3R^2/4}{\lambda^{(-,+)}+5+5R^2/4}\right)^2\right\}^{1/2}$ | |

*Table 8*

| n | k | $U_{2k+1}^{(+,-)}(R^2)$ | $\lambda^{(+,-)}$ |
|---|---|---|---|
| 0 | 0 | 1 | $\lambda^{(+,-)} = -4$ |
| 1 | 0 | $\frac{1}{\sqrt{2}}\left(\frac{\lambda^{(+,-)}+10+R^2/2}{\lambda^{(+,-)}+10}\right)^{1/2}$ | $(\lambda^{(+,-)}+4)(\lambda^{(+,-)}+16) =$ |
| 1 | 1 | $-\frac{1}{6\sqrt{2}}(\lambda^{(+,-)}+10-R^2/2)\left(\frac{\lambda^{(+,-)}+10+R^2/2}{\lambda^{(+,-)}+10}\right)^{-1/2}$ | $= \frac{R^4}{4}$ |
| 2 | 0 | $\left\{1+\frac{1}{120}(\lambda^{(+,-)}+16-R^2)^2+\left(\frac{\lambda^{(+,-)}+16-R^2}{\lambda^{(+,-)}+16+R}\right)^2\right\}^{-1/2}$ | $(\lambda^{(+,-)}+4)(\lambda^{(+,-)}+16)\times$ |
| 2 | 1 | $-\frac{\lambda^{(+,-)}+16-R^2}{2\sqrt{30}}\left\{1+\frac{1}{120}(\lambda^{(+,-)}+16-R^2)^2+\left(\frac{\lambda^{(+,-)}+16-R^2}{\lambda^{(+,-)}+16+R}\right)^2\right\}^{-1/2}$ | $\times(\lambda^{(+,-)}+36) =$ $= R^4(\lambda^{(+,-)}+24)$ |
| 2 | 2 | $\frac{\lambda^{(+,-)}+16-R^2}{\lambda^{(+,-)}+16+R^2}\left\{1+\frac{1}{120}(\lambda^{(+,-)}+16-R^2)^2+\left(\frac{\lambda^{(+,-)}+16-R^2}{\lambda^{(+,-)}+16+R^2}\right)^2\right\}^{-1/2}$ | |

Table 9

| n | k | $U_{2k}^{(-,-)}(R^2)$ | $\lambda^{(-,-)}$ |
|---|---|---|---|
| 0 | 0 | $1$ | $\lambda^{(-,-)} = -1 + \frac{R^2}{4}$ |
| 1 | 0 | $\left(\frac{\lambda^{(-,-)}+7+R^2/4}{2\lambda^{(-,-)}+10-R^2/4}\right)^{1/2}$ | $(\lambda^{(-,-)}+1)(\lambda^{(-,-)}+9) = \frac{R^2}{2}(\lambda^{(-,-)}+9)+\frac{3R^4}{16}$ |
| 1 | 1 | $-\frac{1}{2\sqrt{3}}(\lambda^{(-,-)}+3-3R^2/4)\left(\frac{\lambda^{(-,-)}+7+R^2/4}{2\lambda^{(-,-)}+10-R^2/4}\right)^{1/2}$ | |
| 2 | 0 | $\left\{1+\frac{1}{40}(\lambda^{(-,-)}+5-5R^2/4)^2+\frac{3}{5}\left(\frac{\lambda^{(-,-)}+5-5R^2/4}{\lambda^{(-,-)}+13+3R^2/4}\right)^2\right\}^{-1/2}$ | $(\lambda^{(-,-)}+1)(\lambda^{(-,-)}+9)(\lambda^{(-,-)}+25) = \frac{3R^2}{4}(\lambda^{(-,-)}+9)(\lambda^{(-,-)}+25)+\frac{R^4}{16}(13\lambda^{(-,-)}+205)-\frac{15R^6}{64}$ |
| 2 | 1 | $-\frac{1}{2\sqrt{10}}(\lambda^{(-,-)}+5-5R^2/4)\left\{1+\frac{1}{40}(\lambda^{(-,-)}+5-5R^2/4)^2+\frac{3}{5}\left(\frac{\lambda^{(-,-)}+5-5R^2/4}{\lambda^{(-,-)}+13+3R^2/4}\right)^2\right\}^{-1/2}$ | |
| 2 | 2 | $\sqrt{\frac{3}{5}}\frac{\lambda^{(-,-)}+5-5R^2/4}{\lambda^{(-,-)}+13+3R^2/4}\left\{1+\frac{1}{40}(\lambda^{(-,-)}+5-5R^2/4)^2+\frac{3}{5}\left(\frac{\lambda^{(-,-)}+5-5R^2/4}{\lambda^{(-,-)}+13+3R^2/4}\right)^2\right\}^{-1/2}$ | |

them into four groups $\lambda_{2q}^{(+,+)}$, $\lambda_{2q+1}^{(-,+)}$, $\lambda_{2q+2}^{(+,-)}$ and $\lambda_{2q+1}^{(-,-)}$, and make use of the perturbative formulae obtained in /6/ and the expressions found in the previous section for matrix elements of operator $\hat{\mathcal{P}}$ over the polar bases with given $P_{xy}$ and $P_y$ parity, we arrive at the following results:

$$\lambda_{2q}^{(+,+)} = -(2q)^2 - \frac{R^4}{32}\,\frac{n^2+q^2+n}{4q^2-1} + \frac{R^4}{64}n(n+1)\delta_{q0} - \frac{R^4}{64}n(n+1)\delta_{q1},$$

$$\lambda_{2q+1}^{(-,+)} = -(2q+1)^2 - \frac{R^2}{4}(n+1)\delta_{q0} + \frac{R^4}{128}\left[\frac{(n-q)(n+q+2)}{q+1} - \frac{(n-q+1)(n+q+1)}{q}(1-\delta_{q0})\right],$$

$$\lambda_{2q+1}^{(-,-)} = -(2q+1) + \frac{R^2}{4}(n+1)\delta_{q0} + \frac{R^4}{128}\left[\frac{(n-q)(n+q+2)}{q+1} - \frac{(n-q+1)(n+q+1)}{q}(1-\delta_{q0})\right],$$

$$\lambda_{2q+2}^{(+,-)} = -(2q+2)^2 - \frac{R^4}{32}\,\frac{n^2+q^2+2q-3n+3}{(2q+1)(2q+3)},$$

$$\Psi_{E\lambda}^{(+,+)} = \Phi_{2n,2q}^{(+,+)} + \frac{R^2}{16}\Big\{-\frac{\sqrt{(n+q)(n-q+1)}}{2q-1}\Phi_{2n,2q-2}^{(+,+)} + \frac{\sqrt{(n-q)(n+q+1)}}{2q+1}\Phi_{2n,2q+2}^{(+,+)} +$$
$$+\sqrt{n(n+1)}\,\delta_{q0}\,\Phi_{2n,2}^{(+,+)} - \sqrt{n(n+1)}\,\delta_{q1}\,\Phi_{2n,0}^{(+,+)}\Big\},$$



$$\Psi_{E\lambda}^{(-,+)} = \Psi_{E\lambda}^{(-,-)} = \Phi_{2n+1,2q+1}^{(-,+ \atop -,-)} + \frac{R^2}{16}\Big\{-\frac{\sqrt{(n-q+1)(n+q+1)}}{q}(1-\delta_{q0})\Phi_{2n+1,2q-1}^{(-,+ \atop -,-)} +$$
$$+\frac{\sqrt{(n-q)(n+q+2)}}{q+1}\Phi_{2n+1,2q+3}^{(-,+ \atop -,-)}\Big\},$$

$$\Psi_{E\lambda}^{(+,-)} = \Phi_{2n+2,2q+2}^{(+,-)} + \frac{R^2}{16}\Big\{-\frac{\sqrt{(n-q+1)(n+q+2)}}{2q+1}\Phi_{2n+2,2q}^{(+,-)} + \frac{\sqrt{(n-q)(n+q+3)}}{2q+3}\Phi_{2n+2,2q+4}^{(+,-)}\Big\}.$$

In the above formulae factor $(1-\delta_{q0})/q$ is by definition put to zero at $q=0$.

For $R\gg1$, we divide both sides of the equation for eigenfunctions and eigenvalues of operator $\hat{\Lambda}$ by $R^2/2$ and consider the term $2\hat{L}^2/R^2$ to be perturbation and get:

$$\lambda_{2q}^{(+,+)} = -\frac{R^2}{2}(q-n) - 2q(2n-q) - 2n,$$

$$\lambda_{2q+1}^{(-,+)} = \lambda_{2q+1}^{(-,-)} = -\frac{R^2}{4}(2q-2n-1) - 2q(2n-q+1) - 2n-1,$$

$$\lambda_{2q+2}^{(+,-)} = -\frac{R^2}{2}(q-n-1) - 2q(2n-q+2) - 2n-2,$$

$$\Psi_{E\lambda}^{(+,+)} = \Pi_{2q,2n-2q} + \frac{2}{R^2}\Big\{\sqrt{q(2q-1)(n-q+1)(2n-2q+1)}\,\Pi_{2q-2,2n-2q+2} -$$
$$-\sqrt{(q+1)(2q+1)(n-q)(2n-2q-1)}\,\Pi_{2q+2,2n-2q-2}\Big\},$$

$$\Psi_{E\lambda}^{(-,+)} = \Pi_{2q+1,2n-2q} + \frac{2}{R^2}\Big\{\sqrt{q(2q+1)(n-q+1)(2q-2q+1)}\,\Pi_{2q-1,2n-2q+2} -$$
$$-\sqrt{(q+1)(2q+3)(n-q)(2n-2q-1)}\,\Pi_{2q+3,2n-2q-2}\Big\},$$

$$\Psi_{E\lambda}^{(-,-)} = \Pi_{2q,2n-2q+1} + \frac{2}{R^2}\Big\{\sqrt{q(2q-1)(n-q+1)(2n-2q+3)}\,\Pi_{2q-2,2n-2q+3} -$$
$$-\sqrt{(q+1)(2q+1)(n-q)(2n-2q+1)}\,\Pi_{2q+2,2n-2q-1}\Big\},$$

$$\Psi_{E\lambda}^{(+,-)} = \Pi_{2q+1,2n-2q+1} + \frac{2}{R^2}\Big\{\sqrt{q(2q+1)(n-q+1)(2n-2q+3)}\,\Pi_{2q-1,2n-2q+3} -$$
$$-\sqrt{(q+1)(2q+3)(n-q)(2n-2q+1)}\,\Pi_{2q+3,2n-2q-1}\Big\}.$$

We have derived these equations using the limiting relations [1]:

$$\lim_{R\to\infty} \frac{2\lambda^{(+,+)}_{2q}}{R^2} = -2(q-n), \quad \lim_{R\to\infty} \frac{2\lambda^{(+,-)}_{2q+2}}{R^2} = -2(q-n-1),$$

$$\lim_{R\to\infty} \frac{2\lambda^{(-,+)}_{2q+1}}{R^2} = \lim_{R\to\infty} \frac{2\lambda^{(-,-)}_{2q+1}}{R^2} = -(2q-2n-1).$$

Higher perturbation orders can be analogously calculated.

## 6. CONCLUSIONS

The method we have presented for the calculation of eigenvalues and common eigenfunctions of the elliptic integral of motion and Hamiltonian of c.o. is applicable to a larger class of operators. In particular, it is not difficult to write the recurrence relations determining the expansion of eigenfunctions of the operator $\hat{D} = g(R^2)\hat{L}^2 = f(R^2)\hat{\mathcal{P}}$ ($g(R^2)$ and $f(R^2)$ are given functions of $R^2$ ) and the corresponding eigenvalues. These eigenfunctions are obviously also eigenfunctions of the c.o. Hamiltonian, however now, as a rule, they are outside the class of solutions which can be obtained within the method of separation of variables. The exception is the cases: a) $g=0$; b) $f=0$, and c) $f=g(R^2/2)$ for which the separation of variables can be made in the c.o. Schrödinger equation in the Cartesian, polar, and elliptic coordinates.

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К эллиптическому базису кругового осциллятора

Получен полный набор коммутирующих между собой операторов, определяющих эллиптический базис квантового кругового осциллятора /к.о./. Введен эллиптический базис к.о. и найдены генерирующие его трехчленные рекуррентные соотношения. Вычислены эллиптические поправки к полярному и декартовому базисам к.о.

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