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FLAT PROBLEM OF THE LINEAR TRANSPORT THEORY FOR UNIFORM SEMISPACE AND MNATZAKANJAN SEMIGROUP METHOD

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1. INTRODUCTION

We consider a uniform semispace $z > 0$, onto whose surface $z = 0$ there falls an independent of x, y flow of particles $u(p)\,v(p)$ (here $v(p)$ is the density of particles with momentum p in the semispace $z < 0$) from the region $z < 0$. Let $\rho(z, p)$ be the density of particles with momentum p on the depth z, $z \geq 0$.

Then we have to determine the function $\rho(z, p)$ from the equation

$$u_3(p)\frac{\partial}{\partial z}\rho(z, p) + \frac{\partial}{\partial p_\alpha}\left[u_\alpha(p)\frac{dp}{d\ell}(p)\,\rho(z, p)\right] = \int f(p, q)\,\rho(z, q)\,dq - f(p)\,\rho(z, p) \tag{1}$$

and the boundary condition

$$\rho^+(0, p) = v(p), \quad p_3 \geq 0. \tag{2}$$

Here functions $\rho^\pm(z,p)$ are defined by equations

$$\rho^+(z, p) = \begin{cases} \rho(z, p) & p_3 \geq 0 \\ 0 & p_3 < 0, \end{cases} \tag{3}$$

$$\rho^-(z, p) = \begin{cases} 0 & p_3 \geq 0 \\ \rho(z, p) & p_3 < 0 \end{cases} \tag{4}$$

so that

$$\rho(z, p) = \rho^+(z, p) + \rho^-(z, p). \tag{5}$$

In eq. (1) $\overrightarrow{u(p)}$ is the velocity

$$u_\alpha(p) = p_\alpha / \sqrt{p^2 + m^2}, \quad \alpha = 1, 2, 3, \tag{6}$$

where p is the momentum and m the mass of a particle (we have $h = c = 1$), u_3 - z-projection of the velocity. The quantity $\frac{dp}{d\ell}(p)$ in eq. (1) defines the momentum decrease per unit length due to the retardation in a medium.

One has also:

$$f(p, q) = \sigma(p, q)\,|u(q)|\,n_0, \tag{7}$$

where $\sigma(p, q)$ is the differential cross-section for the scattering of a particle with momentum q into a state with momentum p, n_0 is the density of the scatterers in semispace $z > 0$, $f(q) = \int f(p, q)\, d^3p$. One may find a derivation of eq. (1), e.g., in book /1/.

2. THE "CAUSALITY PRINCIPLE" IN THE LINEAR TRANSPORT THEORY

We state the existence of the following "causality principle" in the transport theory: if one knows the function $\rho^+(z_0, p) = \rho^+(z, p)|_{z=z_0}$ then eq. (1) in a semispace $z > z_0$ with the boundary condition

$$\rho^+(z, p) \to \rho^+(z_0, p), \quad z \to z_0, \quad z > z_0 \tag{8}$$

enables one to get the function $\rho(z,p)$ for all values of z, $z \geq z_0$*.

2.0. In other words, there exists a kernel $T(z_2, p; z_1, q)$ such that the functions $\rho(z_2, p)$ and $\rho^+(z_1, p)$, $z_2 \geq z_1$, are connected by a equation

$$\rho(z_2, p) = \int T(z_2, p; z_1, q)\, \rho^+(z_1, q)\, d^+q \tag{9}$$

($\int d^+q$ means the integration over semispace $q_3 \geq 0$).

2.1. Let us define in the region $z_2 \geq z_1$, $p_3 \geq 0$, $q_3 \geq 0$ the kernel $L(z_2, p; z_1, q)$ by the equation

$$L(z_2, p; z_1, q) = T(z_2, p; z_1, q). \tag{10}$$

Then eq. (9) gives

$$\rho^+(z_2, p) = \int L(z_2, p; z_1, q)\, \rho^+(z_1, q)\, d^+q, \tag{11}$$

whence there follows the composition rule

$$L(z_3, p; z_1, q) = \int L(z_3, p; z_2, s)\, d^+s\; L(z_2, s; z_1, q). \tag{12}$$

Here the depths z_3, z_2, z_1 may assume any values, satisfying the inequality

$$z_3 \geq z_2 \geq z_1 \geq 0. \tag{13}$$

*The authors are indebted to Professor D.V.Shirkov, who has communicated to them this principle.

One has, evidently

$$L(z_2, p; z_1, q) \to \delta(p - q) \quad \text{as} \quad z_2 \to z_1, \quad z_2 > z_1. \tag{14}$$

2.2. In our case of the <u>uniform</u> semispace the solution of the functional equation (12) with the boundary condition (14) is

$$L(z_2, p; z_1, q) = \sum_0^{\infty} \frac{(z_1 - z_2)^n}{n!} P_n(p, q). \tag{15}$$

where the functions $P_n(p, q)$, $n \geq 0$, are defined by the equations

$$P_0(p, q) = \delta(p - q), \quad P_1(p, q) = P(p, q),$$
$$P_{n+1}(p, q) = \int P_1(p, s)\, P_n(s, q)\, d^+s, \quad n = 0, 1, 2, \ldots . \tag{16}$$

Here $P(p, q)$ is an arbitrary function of its arguments; it is defined in the region $p_3 \geq 0$, $q_3 \geq 0$.

2.3. So, we may introduce the notation

$$K(z; p, q) = L(z_1 + z, p; z_1, q) \tag{17}$$

then eq. (15) gives

$$K(z; p, q) = \sum_0^{\infty} \frac{(-z)^n}{n!} P_n(p, q). \tag{18}$$

It follows from eqs. (11), (17)

$$\rho^+(z, p) = \int K(z; p, q)\, \rho^+(0, q)\, d^+q. \tag{19}$$

2.4. Let us consider now the function $\rho^-(z, p)$. According to the "causality principle" one may expect that there exists an "albedo kernel" $R(z; p, q)$ (in the region $p_3 < 0$, $q_3 \geq 0$) such that

$$\rho^-(z, p) = \int R(z; p, q)\, \rho^+(z, q)\, d^+q, \quad z \geq 0. \tag{20}$$

It is easy to show that the kernel R does not depend on z. Really let us represent the functions $\rho^{\pm}(z, p)$ as sums

$$\rho^+(z, p) = \sum_{n=0}^{\infty} \rho_n^+(z, p), \qquad \rho^-(z, p) = \sum_{n=1}^{\infty} \rho_n^-(z, p). \tag{21}$$

Here $\rho_n^{\pm}(z, p)$ denotes the density of the number of particles, which have visited the region $z' > z$ n times.

One has

One has, evidently

$$\rho^-_{n+1}(z, p) = \int R(0; p, q)\, \rho^+_n(z, q)\, d^+q. \tag{22}$$

Then eqs. (20)-(22) imply

$$R(z; p, q) = R(0; p, q) \equiv R(p, q). \tag{23}$$

2.5. Thus, we have got

$$\rho^-(z, p) = \int R(p, s)\, K(z; s, q)\, \rho^+(0, q)\, d^+s\; d^+q = \int R(p, s)\, \rho^+(z, s)\, d^+s. \tag{24}$$

Equations (19), (24), (18), (16) express the solution of our boundary problem in terms of two unknown kernels $P(p, q)$, $R(p, q)$.

3. THE SEMIGROUP

Equations (12), (15), (17) imply the operators (18) to form a semigroup: for any values of z_1, z_2, $z_1 \geq 0$, $z_2 \geq 0$, one has:

$$K(z_1 + z_2; p, q) = \int K(z_2; p, s)\, K(z_1; s, q)\, d^+s. \tag{25}$$

According to eq. (11) the kernel $K(q; p, q)$ allows one to express the function $\rho^+(z_1 + z, p)$ in terms of the function $\rho^+(z_1; p)$, $z_1 \geq 0$, $z \geq 0$.

3.0. We do not think, however, that the operator $K(z; \ldots)$ has an inverse, $\tilde{K}(z; \ldots)$. This inverse operator, were it exist would allow one to express the function $\rho^+(z_1, p)$ in terms of the function $\rho^+(z_1 + z, p)$.

Let us consider, e.g., the heat transport equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(t, x) = 0. \tag{26}$$

The kernel

$$F(t; x, y) = \exp\left[-\frac{(x-y)^2}{4t}\right] / (4\pi t)^{1/2} \tag{27}$$

allows one to express the function $U(t, x)$, $t > 0$, in terms of the function $U(0, x)$:

$$U(t, x) = \int F(t; x, y)\, U(0, y)\, dy. \tag{28}$$

The operator F has no inverse, $\tilde{F}$; it is impossible to express $U(0, x)$ in terms of $U(t, x)$. Were the equation

$$U(0, x) = \int \tilde{F}(t; x, y)\, U(t, y)\, dy \tag{29}$$

exist, one may substitute for $U(t, y)$ an arbitrary, e.g., discontinuous function of y; meanwhile this function, according to eqs. (27), (28), is an entire function of (the complex variable) y (if the function $U(0, x)$ is, e.g., bounded).

3.1. M.A.Mnatzakanjan was the first to apply the semigroup method for the solution of the transport equation (see, e.g., his work[2] and the references therein). His consideration, however, causes some doubt for he systematically uses the kernel $Y(z; p, q)$ that possesses the semigroup property (25) and he expresses the function $\rho^-(z_1, p)$ in terms of the function $\rho^-(z_1 + z, p)$, $z_1 \geq 0$, $z \geq 0$.

4. EQUATIONS DETERMINING THE KERNELS $P(p, q)$, $R(p, q)$

Let us substitute eqs. (19), (24) into the transport equation (1).

4.1. Consider at first the case

$$p_3 \geq 0. \tag{30}$$

Using the formula

$$\frac{\partial}{\partial z} K(z; p, q) = -\int P(p, q')\, d^+q'\, K(z; q', q), \tag{31}$$

which follows from eqs. (16), (18), one can get the expression

$$\frac{\partial}{\partial z} \rho^+(z, p) = -\int P(p, q')\, d^+q'\, \rho^+(z, q') \tag{32}$$

for the derivative that enters into the first term of the l.h.s. of eq. (1).

Then one can rewrite eq. (1) in the form

$$\int A(p, q')\, \rho^+(z, q')\, d^+q' = 0, \tag{33}$$

where

$$A(p, q') = -u_3(p)\, P(p, q') + \frac{\partial}{\partial p_\alpha}\left(u_\alpha(p) \frac{dp}{d\ell}(p)\, \delta(p - q')\right) - f(p, q') - \int d^-q''\, f(p, q'')\, R(q'', q') + f(p)\, \delta(p - q'), \quad p_3 \geq 0,\; q'_3 \geq 0. \tag{34}$$

The condition

$$A(p, q') = 0, \quad p_3 \geq 0, \quad q'_3 \geq 0 \tag{35}$$

which is sufficient for eqs. (1), (33) to be fulfilled in the region (30), allows one, evidently, to express the kernel P in terms of f and R.

4.2. Let us now consider eq. (1) in the case

$$p_3 < 0. \tag{36}$$

Equations (24) and (32) imply

$$\frac{\partial}{\partial z}\rho^-(z,p) = -\int R(p,q'')d^+q''\,P(q'',q')d^+q'\,\rho^+(z,q'). \tag{37}$$

Using eq. (37) one can rewrite eq. (1) in the form

$$\int B(p,q')\,\rho^+(z,q')d^+q' = 0, \tag{38}$$

where (c.f. eq. (34))

$$B(p,q') = -u_3(p)\int R(p,q'')P(q'',q')d^+q'' + \frac{\partial}{\partial p_\alpha}\left(u_\alpha(p)\frac{dp}{d\ell}(p)R(p,q')\right) - f(p,q') \tag{39}$$

$$- \int f(p,q'')\,d^-q''\,R(q'',q') + f(p)\,R(p,q'), \qquad q'_3 \geq 0, \quad p_3 < 0.$$

Analogously to eq. (35) one gets the equation

$$B(p,q') = 0, \quad p_3 < 0, \quad q'_3 \geq 0. \tag{40}$$

4.2.1. Equations (34), (35), (39), (40) allow one to calculate the kernels P, R.

4.3. The manifold of the eq. (1) solutions is determined by the Green function $\rho(z;p,q)$, $q_3 > 0$ which is defined as a solution of eq. (1) with the boundary condition $\rho^+(z;p,q) \to \delta(p-q)$ as $z \to 0$, $z > 0$. This Green function depends on seven independent variables. We have reduced the problem of solving eq. (1) to the nonlinear (quadratic) integral equation with respect to the albedo function $R(p,q)$ that depends only on six independent variables. This is all the progress we have achieved*.

5. CONCLUSION

We suspect the Ambartzumjan invariant embedding method (see, e.g. /1/) to be able to give our result. For the case of the single-velocity problem with isotropic scattering we have checked this statement (c.f. /1/).

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*As a matter of fact, the kernel $R(p,q)$ (and $P(p,q)$) does depend only on five independent variables (for this kernel is invariant under rotations around the z axis).

5.1. Our kernel $K(z;p,q)$ has the exponential construction (16), (18). One should not think, however, that series (18) in powers of z has a positive convergence radius.

The retardation in our semispace implies the Green function $\rho(z;p,q)$ to become zero in some region $z \geq z_0(p,q)$. This fact makes one to expect serious difficulties when attempting to use eq. (18) for the kernel $K(z;p,q)$ computation.

It is necessary to note also that the problem considered (if the retardation exists) is not completely stationary. The density $\rho(t,z,p)|_{p=0}$ of particles with zero momentum would constantly increase with time, even if the density $\rho(t,z,p)$, $|p| > 0$, does not not depend on time.

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Черненко Л.П., Заставенко Л.Г. E2-84-506

Плоская задача линейной теории переноса для однородного полупространства и метод полугрупп Мнацаканяна

Рассматривается стационарное кинетическое уравнение линейной теории переноса в однородном полупространстве. Целью работы является сведение задачи к решению уравнения относительно функции с меньшим числом независимых переменных. Эта цель достигается введением в рассмотрение полугруппы операторов, связанных с исходным кинетическим уравнением. Результатом работы является сведение исходной задачи к нелинейному интегральному уравнению относительно ядра альбедо $R(p,q)$ $p_3 < 0, q_3 \geq 0$. Здесь q - импульс частицы, входящей в полупространство, p -импульс частицы, покидающей полупространство, p_3 и q_3 - проекции импульсов на нормаль к границе полупространства. Наш результат, как кажется, может быть получен и методом инвариантного погружения Амбарцумяна.

Работа выполнена в Лаборатории ядерных проблем и Лаборатории теоретической физики ОИЯИ.

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Chernenko L.P., Zastavenko L.G. E2-84-506

Flat Problem of the Linear Transport Theory for Uniform Semispace and Mnatzakanjan Semigroup Method

We consider a stationary kinetic equation of the linear transport theory in a uniform semispace. Our aim is to reduce the problem to the solution of an equation with a diminished number of variables. We achieve this aim through the consideration of semi-group of operators, associated with our kinetic equation. Our result is the reduction of the problem to the solution of a nonlinear integral equation for the albedo kernel $R(p,q)$, $p_3 < 0, q_3 \geq 0$ determination (q is the momentum of a particleß which enters semispace; p the momentum of a particleß which leaves our semispace; q_3 and p_3 are the momentum projections on the normal to the semispace surface).

The investigation has been performed at the Laboratory of Theoretical Physics, and Laboratory of Nuclear Problems, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1984