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FORMULATION OF EUCLIDEAN QED MANIFESTLY OBEYING CONDITIONS OF GAUGE, TRANSLATIONAL, AND ROTATIONAL INVARIANCE

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At present no formulation is known of Quantum Electrodynamics (QED), which satisfies simultaneously conditions of the gauge, translational, and Lorentz invariance. The QED in Coulomb gauge, e.g., does manifestly satisfy the conditions of gauge and translational invariance, but does not manyfestly satisfy the condition of Lorentz invariance *.

The Lorentz-gauge QED,

$$\partial_\beta A_\beta = 0 \tag{1}$$

does manifestly satisfy the conditions of Lorentz and translational invariance, but does not satisfy (manifestly) the condition of gauge invariance, and so on.

1. Meanwhile, it seems to be very interesting to find a QED formulation that is manifestly invariant with respect to all the three groups of transformations.

1.1. In the present work we give the formulation of QED in Euclidean space of four dimensions, which does manifestly satisfy the conditions of gauge, translational, and rotational invariance.

One may hope to be able to obtain the Green functions of QED in pseudo-Euclidean space through an analytical continuation of the Euclidean QED Green functions.

One may hope, also that such pseudo-Euclidean Green functions would manifestly satisfy the condition of invariance under all the three groups of transformations in pseudo-Euclidean space.

2. It is to be noted that there exists a considerable difference between the QED-s in Euclidean and pseudo-Euclidean space. First of all, note that Lagrangians of these theories are different:

$$\mathcal{L}_{\mathrm{pseudo}} = \frac{1}{4}(\partial_\beta A_\delta - \partial_\delta A_\beta)^2 + \bar{\psi}[\gamma_\beta(\partial_\beta - ieA_\beta) + m + \delta m]\psi, \tag{2}$$

$$\bar{\psi} = \psi^*\gamma_4, \quad \mathrm{Re}A_4 = \mathrm{Im}A_j = 0, \quad j = 1, 2, 3; \tag{3}$$

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*This variant of QED is considered in detail in the textbook by Wentzel [1]. This book, however escapes discussing the zero-mode (of the potential A_μ) problem. This problem is discussed in work [2].

$$\mathcal{L}_{Eucl} = \frac{1}{4}(\partial_\beta A_\delta - \partial_\delta A_\beta)^2 + \psi^*[\gamma_\beta(\partial_\beta - ieA_\beta) + m + \delta m]\psi, \qquad (4)$$

$$\mathrm{Im} A_\beta = 0, \quad \beta = 1, 2, 3, 4. \qquad (5)$$

Also, the region

$$p^2 < \ell^2 \qquad (6)$$

in Euclidean space is finite unlike in pseudo-Euclidean space. For this reason one may regularize divergent integrals by the conditions that all integration momenta belong to the region (6) only in the case of Euclidean metrics. (But not in the case of pseudo-Euclidean metrics). Note also that the matrices

$$\bar{\gamma}_\beta \equiv \gamma_\beta - k_\beta (k_\delta \gamma_\delta)/k^2 \qquad (7)$$

which enter into vertices, are well defined (at $k \neq 0$) in Euclidean but not in pseudo-Euclidean metric (for the quantity k^2 may admit zero value at $k \neq 0$ in pseudo-Euclidean metric).

3. The gauge-transformation is

$$A_\beta(x) \xrightarrow{\Lambda(x)} A'_\beta(x) = A_\beta(x) + \partial\Lambda/\partial x_\beta, \quad \psi(x) \xrightarrow{\Lambda(x)} \psi'(x) = \psi(x)\exp[ie\Lambda(x)] \qquad (8)$$

here $\Lambda(x)$ is an arbitrary real function. Both the Lagrangians (2) and (4) are invariant under gauge transformations.

4. We shall find our QED formulation, transforming the functions A, ψ in the Lagrangian (4) according to the equations:

$$B_\beta = A_\beta - \partial_\beta \lambda, \quad \beta = 1, 2, 3, 4, \qquad (9a)$$

$$\eta(x) = \exp[-ie\lambda(x)]\,\psi(x), \qquad (9b)$$

$$\lambda(x) = \Box^{-1}\partial_\delta A_\delta; \qquad (9c)$$

here $\Box$ is a 4-dimensional Laplace operator. These eqs. define, evidently, the gauge-transformation (8) for a particular choice of the function Λ, $\Lambda(x) = -\lambda(x)$. Correspondingly, one has

$$\mathcal{L}_{Eucl}(A, \psi) = \mathcal{L}_{Eucl}(B, \eta). \qquad (10)$$

Let us show that the functions B_β do not change under gauge transformations (8). We begin with introducing, as usual, the periodicity cube in four-dimensional space, and considering the decompositions

$$A_\beta(x) = \Omega^{-1/2}\sum_p A_{p\beta} e^{ipx}$$

$$B_\beta(x) = \Omega^{-1/2}\sum_p B_{p\beta} e^{ipx}, \qquad (11a)$$

$$\Lambda(x) = \Omega^{-1/2}\sum_p \Lambda_p e^{ipx}, \qquad (11b)$$

$$\eta(x) = \Omega^{-1/2}\sum_p \eta_p e^{ipx}, \qquad (11c)$$

$$\Omega = L^4, \; -L/2 < x_\beta < L/2, \quad p = 2\pi n/L. \qquad (12)$$

Here n is a 4-dimensional vector, with integer components $n = (n_1, n_2, n_3, n_4)$; summation in eqs. (11) runs over all values of n_1, n_2, n_3, n_4.

Equations (8), (9a), (9c) give $B_\beta \xrightarrow{\Lambda} B'_\beta = B_\beta + \partial_\beta\Lambda - \partial_\beta\Box^{-1}\Box\Lambda$. According to eq. (11b) one has $\Box\Lambda(x) = -\Omega^{-1/2}\sum_{p^2\neq 0} p^2 \Lambda_p e^{ipx}$, so that

$$\Box^{-1}\Box\Lambda(x) = \Omega^{-1/2}\sum_{p^2\neq 0}\Lambda_p e^{ipx} = \Lambda(x) - \Omega^{-1/2}\sum_{p^2=0}\Lambda_p e^{ipx}$$

and

$$\partial_\beta(1 - \Box^{-1}\Box)\Lambda(x) = \Omega^{-1/2}\sum_{p^2=0} ip_\beta\Lambda_p e^{ipx};$$

for our case of the Euclidean metric the latter sum contains the only term $p = 0$ and is zero. Thus, the functions $B_\beta(x)$ really do not change under gauge transformation. (In the case of pseudo-Euclidean metric eq. $p^2 = 0$ has an infinite number of solutions so that the functions $B_\beta(x)$ are not invariant under gauge transformations). As far as we know, the transformation (9) has first been introduced in work /3/.

Further, equations (8) and (9) give

$$\eta(x) \xrightarrow{\Lambda} \eta'(x) = \exp[ie\sum_{p^2=0}\Lambda_p e^{ipx}\Omega^{-1/2}]\,\eta(x).$$

So, in the case of the Euclidean metric the function $\eta(x)$ under gauge transformation (8) gets only the phase factor $\exp[ie\Lambda_0\Omega^{-1/2}]$ that does not depend on coordinates x. (This factor corresponds to the charge conservation).

4.1. Thus, we have proved the functions B_β, η to be gauge invariant. This result gives a possibility of constructing the Euclidean QED formulation that is manifestly invariant under the gauge transformations, translations, and rotations. We will achieve this aim, defining, the electron G and photon D propagators by the equations:

$$G(p, \alpha, m + \delta m, \ell)_{\omega,\epsilon}\,\delta(0) = \int (\eta_p)_\omega (\eta^*_p)_\epsilon \, d\tau/I, \quad \alpha = e^2/(4\pi) \qquad (13)$$

$$D_{\beta\delta}(k, \alpha, m + \delta m, \ell)\,\delta(0) = \int B_{k\beta} B_{-k\delta}\, d\tau/I, \qquad (14)$$

$$I = \int d\tau, \quad S = \int \mathcal{L}_{Eucl}(B(x, \ell), \eta(x, \ell))\, d^4x, \qquad (15)$$

$$d\tau = e^{-S} \left(\prod_{q^2<\ell^2} \left(\prod_{\omega=1}^{3} dB(q)_\omega \right) d\eta_q d\eta_q^* \right). \qquad (16)$$

Here the Lagrangian $\mathcal{L}_{Eucl}$ is given by (4); the x-integration is carried over the whole 4-dimensional periodicity volume Ω; the functions $B(x,\ell)$ and $\eta(x,\ell)$ are obtained from functions (11) by introducing cut-off (6); functions $B(q)_\omega$ are defined by the equations

$$B_{p\beta} = \sum_{\delta=1}^{3} B(p)_\delta [e_\delta(p)]_\beta , \qquad (17)$$

where $e_\delta(p)$, $\delta = 1,2,3,4$ are the system of real unite vectors

$$e_4(p) = p/|p|, \quad [e_{\delta_1}(p)]_\beta [e_{\delta_2}(p)]_\beta = \delta_{\delta_1 \delta_2} . \qquad (18)$$

Equations (9a), (9c), and (11a) imply

$$\partial_\beta B_\beta(x) = 0 = p_\beta B_{p\beta} , \qquad (19)$$

eqs. (17), (18) give

$$B(p)_\delta = B_{p\beta} [e_\delta(p)]_\beta , \quad \delta = 1,2,3. \qquad (20)$$

Our definitions of propagators (13)-(15) follow those given in book /4/, chapter 4. The integrals over spinors $d\eta_q$ and $d\eta_q^*$ are to be understood in the Berjozin sense /5/. The indices ω, ϵ in eq. (13) define the Dirac bispinor. (We omit these indices in eq. (16)).

4.2. Gauge invariance of the functions B_β, η (item 4) implies gauge invariance of the propagators (13) and (14).

Let us show now that these propagators satisfy the translational-invariance condition. We will define

$$\tilde{G}(p, q, \dots) \equiv \int \eta_p \eta_q^* d\tau / I , \qquad (21)$$

and produce the transformation

$$\eta_q = e^{idq} \eta'_q , \quad \eta_q^* = e^{-idq} \eta'^*_q , \quad d \neq 0 \qquad B(q)_\delta = e^{idq} B'(q)_\delta ; \qquad (22)$$

the action S and "volume element" $d\tau$ are invariant under this transformation, and we get

$$\tilde{G}(p, q, \dots) = G(p, \dots)\, \delta(p - q) \qquad (23)$$

(c.f. /4,6/). Equation (23) is an adequate formulation of the propagator translational invariance. One may arrive at eq. (23) also by a straightforward consideration of the perturbation expansion of propagators.

Analogously, one may verify the rotational invariance of propagators (one has to take into account our rotationally invariant cut-off (6)).

4.3. So we have got the equations:

$$G(p, \dots)_{\omega\epsilon} = ip_\beta (\gamma_\beta)_{\omega\epsilon}\, a(p^2) + \delta_{\omega\epsilon}\, b(p^2, \dots), \qquad (24)$$

$$D_{\beta\delta}(k, \dots) = (\delta_{\beta\delta} - k_\beta k_\delta / k^2)\, d(k^2, \dots), \qquad (25)$$

in eq. (25) we have taken into account that eqs. (14) and (19) imply $D_{\beta\delta}(k, \dots) k_\beta = 0$.

4.3.1. The definitions (13)-(15) and eq. (25) allow us to obtain the equations

$$(\delta_{\beta\delta} - k_\beta k_\delta / k^2)\, d(k^2, \dots)^{-1} = k^2 \delta_{\beta\delta} - k_\beta k_\delta + \alpha \text{ ---}$$  $$\text{--- } + O(\alpha^2) \qquad (26)$$

$$G(p, \dots)^{-1} = ip_\beta \gamma_\beta + m + \delta m + \alpha$$  $$+ O(\alpha^2) \qquad (27)$$

from which one can construct propagators without computing integrals (13) and (14). In these equations to every internal solid line there corresponds an integral $\int G(p) dp$, $p^2 < \ell^2$, with the electron propagator; to every internal dotted line there corresponds an analogous integral with the photon propagator; to every vertex in a given diagram, except, one, there corresponds the function $\delta(p_1 + p_2 + k)$, where p_1, p_2 are the incoming electron momenta, k is the incoming photon momentum; to every vertex there also corresponds matrix (7), where k is the incoming photon momentum.

4.4. Knowledge of the propagators enables one to construct high-order Green functions which describe the processes of particle scattering and production; these functions in our QED also satisfy the conditions of gauge, translational, and rotational invariance.

5. Our Euclidean QED formulation is equivalent to its Lorentz Gauge formulation with the cut off (6).

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Эвклидова КЭД, явно удовлетворяющая условиям калибровочной, трансляционной и вращательной инвариантности

Рассматривается эвклидова КЭД с целью получения ее формулировки, которая одновременно и явно удовлетворяла бы условиям калибровочной, трансляционной и вращательной инвариантности. Главным моментом нашего рассмотрения является переход от обычных переменных A_μ, ψ к калибровочно-инвариантным переменным B_μ, η по формулам $B_\mu = A_\mu - \partial_\mu \lambda$, $\eta = \exp[-ie\lambda]\psi$, $\lambda = \Box^{-1}\partial_\mu B_\mu$. Пользуясь соответственно преобразованным лагранжианом, можно построить функции Грина, инвариантные относительно трех вышеуказанных групп преобразований. Можно надеяться получить функции Грина обычной псевдоэвклидовой КЭД путем аналитического продолжения наших функций Грина. Такие псевдоэвклидовы функции Грина будут, по-видимому, инвариантны относительно калибровочных преобразований, преобразований Лоренца и трансляций. Так, быть может, удастся получить формулировку псевдоэвклидовой КЭД, явно калибровочно- и Пуанкаре-инвариантную.

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Formulation of Euclidean QED Manifestly Obeying Conditions of Gauge, Translational, and Rotational Invariance

The Euclidean QED is considered with the aim to get its formulation which would be manifestly invariant under gauge transformations, translations, and rotations. The main point of our consideration is the introduction, instead of usual variables A_μ, ψ, of gauge-invariant veryables B_μ, η through the equations $B_\beta = A_\beta - \partial_\beta \lambda$, $\eta = \exp[-ie\lambda]\psi$, $\lambda = \Box^{-1}\partial_\beta A_\beta$. The thus transformed Lagrangian, enables one to construct Green functions which are invariant under all above-mentioned groups of transformations. One may hope to get the Green functions of the usual pseudo-Euclidean QED through an analytical continuation of our Green functions. So, one would get the formulation of the pseudo-Euclidean QED which is,probably,manifestly gauge and Poincare invariant.

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