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SOME PROPERTIES OF CONSTRAINTS IN THEORIES WITH DEGENERATE LAGRANGIANS

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1. INTRODUCTION

Most of the models interesting for the elementary particle physics are described by singular (or degenerate) Lagrangians [1-4]. The basic problem encountered here is the construction of a consistent Hamiltonian theory that should be used for the quantization.

The theory of degenerate Lagrangians has been worked out during the last few decades, but here there are still unsolved problems [5]. The search and classification of constraints between the canonical variables take a central place in the generalized Hamiltonian dynamics for the systems with degenerate Lagrangians [1]. However, in the literature there are no theorems that enable us to predict properties of the Poisson brackets of the constraints without their straightforward calculation. Obviously, the invariance of action under the transformations with arbitrary functions of time must play here an important role [6]. Just this invariance leads to the degeneracy of Lagrangian as it has been proved by using the second Noether theorem (Noether identities) [7,8]. In fact, this theorem allows us to prove something more. By means of its we shall be able to show that in a theory invariant under the transformations with arbitrary functions of time the primary constraints are in involution between themselves at least in a weak sense.

The paper is organized as follows. In Section 2 basic formulae and definitions are presented. The theorems on the properties of Lagrangian and Hamiltonian constraints will be proved in Section 3. We shall show here that the Lagrangian constraints are invariant relations for the equations of motion. In the framework of the Lagrangian formalism we propose a criterion of the appearance in the degenerate theory of the second-class constraints. The Poisson brackets of the primary constraints with the canonical Hamiltonian are determined. In Section 4 by means of the second Noether theorem we show that the action invariance under the transformations with arbitrary functions of time (see formula (4.2)) results in the primary constraints which are in involution between themselves at least in a weak sense. In conclusion (Section 5) we discuss other approaches to this problem.

2. BASIC FORMULAE AND DEFINITIONS

We shall consider for simplicity a mechanical system with a finite number of degrees of freedom. Let $q=(q_1,\dots q_n)$ are generalized coordinates of this system and $L(q,\dot q)$ is its Lagrange function depending on the generalized coordinates and velocities. The degeneracy of the Lagrangian $L(q,\dot q)$ means that the rank of a symmetric Hessian matrix with elements

$$\Lambda_{ij}(q,\dot q)=\frac{\partial^2 L(q,\dot q)}{\partial \dot q_i\,\partial \dot q_j},\quad i,j=1,\dots,n \tag{2.1}$$

is smaller than the number of degrees of freedom n

$$\mathrm{rank}\,\|\Lambda_{ij}\|=r=n-m,\quad m>0. \tag{2.2}$$

We shall suppose in what follows that $r>0$, that is, the case, when singular Lagrangians depend on the velocities $\dot q$ linearly, is not considered here.

By virtue of condition (2.2) the matrix $\|\Lambda_{ij}\|$ has eigenvectors with zero eigenvalues

$$\xi^s_i(q,\dot q)\,\Lambda_{ij}(q,\dot q)=\Lambda_{ji}\,\xi^s_i=0,\quad s=1,\dots,m,\quad i,j=1,\dots,n. \tag{2.3}$$

On account of the same reason the Euler equations

$$L_i(q,\dot q,\ddot q)=\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}-\frac{\partial L}{\partial q_i}=\Lambda_{ij}(q,\dot q)\ddot q_j-\ell_i(q,\dot q)=0,$$

$$\ell_i(q,\dot q)=-\frac{\partial^2 L}{\partial \dot q_i\,\partial q_j}\dot q_j+\frac{\partial L}{\partial q_i},\quad i,j=1,\dots,n \tag{2.4}$$

are reduced to r linearly independent second-order equations that can be represented in a normal form and $m=n-r$ first-order (or zero-order) equations. These last m equations independent of the accelerations $\ddot q$ are called the Lagrangian constraints. They are obtained by multiplying the Euler equations (2.4) by zero eigenvectors of the matrix Λ

$$B_s(q,\dot q)=\xi^s_i(q,\dot q)\,\ell_i(q,\dot q)=\xi^s_i\left(-\frac{\partial^2 L}{\partial \dot q_i\,\partial q_j}\dot q_j+\frac{\partial L}{\partial q_i}\right)=0,\quad s=1,\dots,m. \tag{2.5}$$

If eqs.(2.5) are fulfilled, then r accelerations $\ddot q_\alpha$, $\alpha=1,\dots,r$, can be found from system (2.4) as functions of q, $\dot q$ and the remaining $n-r$ accelerations

$$\ddot q_\alpha=Q_\alpha(q,\dot q;\ddot q_{r+1},\dots,\ddot q_n),\quad \alpha=1,\dots,r. \tag{2.6}$$

It is convenient to assume here that the first r rows of the matrix Λ are independent.

Condition (2.2) leads to m primary constraints between the coordinate q and the momenta p defined by

$$p_i = \frac{\partial L(q,\dot{q})}{\partial \dot{q}_i}, \quad i = 1, \dots, n. \tag{2.7}$$

These constraints are obtained by solving r equations (2.7) with respect to r velocities $\dot{q}_a$

$$\dot{q}_\alpha = h_\alpha(q, p_\beta, \dot{q}_{r+1}, \dots, \dot{q}_n), \quad \alpha, \beta = 1, \dots, r \tag{2.8}$$

and by substituting these expressions into the remaining $m = n - r$ equations (2.7)

$$p_{r+s} = g_{r+s}(q, p_\alpha), \quad s = 1, \dots, m, \quad \alpha = 1, \dots, r. \tag{2.9}$$

Instead of relations (2.9) usually an equivalent set of constraints involving the coordinates q and the momenta p in the symmetric form

$$\phi_s(q,p) = 0, \tag{2.10}$$

$$\mathrm{rank} \left\| \frac{\partial \phi_s(q,p)}{\partial p_i} \right\| = m, \quad s = 1, \dots, m, \quad i = 1, \dots, n. \tag{2.11}$$

is used. Constraints (2.9), and consequently (2.10), may contain no coordinates q, but they must involve the momenta p. After substituting expressions (2.7) into eqs.(2.10) we obtain the identities with respect to q and $\dot{q}$. The differentiation of these identities with respect to $\dot{q}_i$ yields

$$\frac{\partial \phi_s}{\partial p_j} \frac{\partial p_j}{\partial \dot{q}_i} = \frac{\partial \phi_s}{\partial p_j} \Lambda_{ij} = 0, \quad s = 1, \dots, m, \quad i, j = 1, \dots, n, \tag{2.12}$$

consequently, the quantities,

$$\xi_i^s(q,p) = \frac{\partial \phi_s(q,p)}{\partial p_i}, \quad s = 1, \dots, m, \quad i = 1, \dots, n \tag{2.13}$$

dependent on q and p only, can be used as zero eigenvectors of matrix $\|\Lambda_{ij}\|$. By virtue of condition (2.11) the vectors $\xi_i^s(q,p)$ are independent.

Owing to equations of motion (2.4) and equations of primary constraints (2.10) the equations

$$\frac{d}{dt}\phi_s(q,p) = \frac{d}{dt}\phi_s\left(q, \frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) = \frac{\partial \phi_s}{\partial q_i}\dot{q}_i + \frac{\partial \phi_s}{\partial p_i}\dot{p}_i = 0,$$
$$s = 1, \dots, m, \quad i = 1, \dots, n \tag{2.14}$$

are reduced to Lagrangian constraints (2.5) [9]. The differentiation of (2.10) with respect to q_i yields

$$\frac{\partial \phi_s}{\partial q_i} + \frac{\partial \phi_s}{\partial p_j}\frac{\partial p_j}{\partial q_i} = 0. \tag{2.15}$$

The substitution of (2.15) and $\dot{p}_i = \partial L/\partial q_i$ into (2.14) leads to the Lagrangian constraints (2.5)

$$\frac{\partial \phi_s}{\partial p_j}\left(\frac{\partial p_j}{\partial q_i}\dot{q}_i - \frac{\partial L}{\partial q_j}\right) = \xi_j^s\left(\frac{\partial^2 L}{\partial \dot{q}_j \partial q_i}\dot{q}_i - \frac{\partial L}{\partial q_j}\right) = 0. \tag{2.16}$$

In addition to the primary constraints (2.9) or (2.10) arising from condition (2.2), the secondary constraints on the canonical variables q and p, caused by the Lagrangian equation (2.5) and their derivatives with respect to time, may appear in the theory with degenerate Lagrangian. Substituting expressions (2.8) for r velocities into (2.5) we can obtain the equations containing q and p only [10]

$$B_s(q,\dot{q}) \to B_s(q,p), \quad s = 1, \dots, m. \tag{2.17}$$

In the same way we must deal with all the derivatives $d^k B_s(q,\dot{q})/dt^k$ as well. The differentiation of $B_s(q,\dot{q})$ and subsequent substitution of (2.8) have to be carried out until this procedure results in identities or expressions that contain the velocities explicitly. As a result, we shall obtain all secondary constraints

$$\omega_\sigma(q,p) = 0, \quad \sigma = 1, \dots, \mu,$$
$$\omega_\sigma(q,p) = \left\{B_s(q,p), \frac{d}{dt}B_s(q,p), \dots, \frac{d^k}{dt^k}B_s(q,p)\right\}. \tag{2.18}$$

Having the complete set of constraints (primary and secondary) in theory and proposing that these constraints are functionally independent we can derive the equations of motion in the phase space by the Lagrange multiplier method [10].

3. PROPERTIES OF LAGRANGIAN AND HAMILTONIAN CONSTRAINTS

In this Section we shall prove a series of theorems concerning Lagrangian and Hamiltonian constraints.

Theorem 1. The Lagrangian constraints form an invariant system with respect to the Euler equations.

Recall that equalities of the form

$$u_\mu (t, x, x^{(1)}, \dots, x^{(k-1)}) = 0, \qquad \mu = 1, 2, \dots \tag{3.1}$$

will be invariant relations [10,11] for the set of ordinary equations of the k-th order with unknown functions $x_i(t)$, $i = 1, \dots, n$ when the total derivative of $u_\mu(t, x, x^{(1)}, \dots, x^{(k-1)})$ with respect to t vanishes everywhere on the manifold defined by equalities (3.1) in virtue of the present set of equations.

Let us differentiate the left-hand sides of equations (2.5) with respect to t

$$\frac{dB_s(q,\dot q)}{dt} = \frac{d\xi_i^s}{dt}\ell_i + \xi_i^s \frac{d\ell_i}{dt}. \tag{3.2}$$

Taking into account the Euler equations (2.4) in the first term of (3.2) we obtain

$$\frac{dB_s}{dt} = \frac{d\xi_i^s}{dt}\Lambda_{ij}\ddot q_j + \xi_i^s \frac{d\ell_i}{dt}. \tag{3.3}$$

The differentiation of equalities (2.3) and equations of motion (2.4) gives

$$\frac{d\xi_i^s}{dt}\Lambda_{ij} + \xi_i^s \frac{d}{dt}\Lambda_{ij} = 0, \tag{3.4}$$

$$\frac{d\ell_i}{dt} = \frac{d}{dt}(\Lambda_{ik}\ddot q_k). \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3) we have now

$$\frac{dB_s}{dt} = -\xi_i^s \frac{d}{dt}(\Lambda_{ij})\ddot q_j + \xi_i^s \frac{d}{dt}(\Lambda_{ij}\ddot q_j) =$$

$$= -\xi_i^s \frac{d}{dt}(\Lambda_{ij})\ddot q_j + \xi_i^s \frac{d}{dt}(\Lambda_{ij})\ddot q_j + \xi_i^s \Lambda_{ij}\dddot q_j = 0, \quad s = 1, \dots, m, \quad i, j = 1, \dots, n. \tag{3.6}$$

Thus, the theorem 1 is proved.

The reason prompted us to prove this assertion is the conjecture made in some paper (see, e.g., [10]) that the Lagrangian constraints (2.5) must be invariant relations with respect to the Euler equations for consistency of the Lagrangian formalism. In fact, this assumption is not necessary because it is always fulfilled as is shown above.

We shall often use the following weak equality

$$\xi_i^s \xi_j^{s'} \left(\frac{\partial^2 L}{\partial q_j \partial \dot q_i} - \frac{\partial^2 L}{\partial \dot q_j \partial q_i}\right) = \xi_i^s \xi_j^{s'} \left(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}\right) \overset{\phi_{s''}}{\approx} \overset{\phi_{s''}}{\approx} (\phi_{s'}, \phi_s), \quad s, s', s'' = 1, \dots, m, \quad i, j = 1, \dots, n. \tag{3.7}$$

Here (f,g) is the Poisson brackets of two functions $f(q,p)$ and $g(q,p)$, defined by

$$(f,g) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \tag{3.8}$$

and the sign $\overset{\phi}{\approx}$ means equality on condition that $\phi = 0$. To prove (3.7), it is sufficient to substitute formulae (2.13) and (2.15) into the left-hand sides of these equalities.

Besides, we shall use linear differential operators (vector fields) of the type

$$X^s = \xi_j^s \frac{\partial}{\partial \dot q_j}, \quad s = 1, \dots, m. \tag{3.9}$$

They have the property

$$X^s p_i = X^s \frac{\partial L(q,\dot q)}{\partial \dot q_i} = \xi_j^s \frac{\partial^2 L(q,\dot q)}{\partial \dot q_i \partial \dot q_j} = 0, \quad i, j = 1, \dots, n, \quad s = 1, \dots, m. \tag{3.10}$$

What is more, the action of these operators on function $f(q,\dot q)$ that can be represented by virtue of (2.7) as some new function $\bar f(q,p)$ dependent on q and p only gives obviously zero

$$X^s f(q,\dot q) = \xi_j^s \frac{\partial}{\partial \dot q_j} \bar f\left(q, p = \frac{\partial L}{\partial \dot q}\right) = \frac{\partial \bar f}{\partial p_i}\frac{\partial p_i}{\partial \dot q_j}\xi_j^{(s)} = 0, \tag{3.11}$$

$i, j = 1, \dots, n, \quad s = 1, \dots, m.$

Theorem 2. The action of the linear differential operators (3.9) on the Lagrangian constraints (2.5) is expressed in terms of the Poisson brackets of the primary constraints (2.10)

$$X^s B_{s'} \overset{\phi}{\approx} (\phi_{s'}, \phi_s), \quad s, s' = 1, \dots, m. \tag{3.12}$$

We prove this theorem by direct calculation

$$X^s B_{s'} = \xi^s_i \frac{\partial}{\partial \dot q_i} \xi^{s'}_j \left(-\frac{\partial^2 L}{\partial \dot q_j \partial q_k}\dot q_k + \frac{\partial L}{\partial q_j}\right) =$$
$$= \xi^s_i \xi^{s'}_j \left(\frac{\partial^2 L}{\partial q_j \partial \dot q_i} - \frac{\partial^2 L}{\partial \dot q_j \partial q_i}\right) - \xi^s_i\xi^{s'}_j \dot q_k \frac{\partial^3 L}{\partial q_k \partial \dot q_i \partial \dot q_j} + \xi^s_i \ell_j \frac{\partial \xi^{s'}}{\partial \dot q_i}. \qquad (3.13)$$

The last term in (3.13) vanishes because the vectors ξ^s_j can be chosen always so that they will depend only on q and p (see eq.(2.13)). The vanishing of the second term in (3.13) becomes obvious after the following transformations

$$\xi^s_i\xi^{s'}_j \dot q_k \frac{\partial}{\partial q_k}\left(\frac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}\right) = \xi^s_i \dot q_k \frac{\partial}{\partial q_k}\left(\xi^{s'}_j \frac{\partial^2 L}{\partial \dot q_i\partial \dot q_j}\right) -$$
$$- \xi^s_i \frac{\partial^2 L}{\partial \dot q_i \partial \dot q_k}\dot q_k \frac{\partial \xi^{s'}_j}{\partial q_k} = 0. \qquad (3.14)$$

The first term in the right-hand side of (3.13) can be written by virtue of (2.13) and (2.15) in the form

$$\xi^s_i\xi^{s'}_j\left(\frac{\partial^2 L}{\partial q_j\partial \dot q_i} - \frac{\partial^2 L}{\partial \dot q_j \partial q_i}\right) = \xi^s_i\xi^{s'}_j\left(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}\right) \overset{\phi}{=}$$
$$\overset{\phi}{=} \frac{\partial\phi_s}{\partial p_i}\frac{\partial p_i}{\partial q_j}\frac{\partial \phi_{s'}}{\partial p_j} - \frac{\partial\phi_{s'}}{\partial p_j}\frac{\partial p_j}{\partial q_i}\frac{\partial\phi_s}{\partial p_i} = -\frac{\partial\phi_s}{\partial q_j}\frac{\partial\phi_{s'}}{\partial p_j} + \frac{\partial\phi_{s'}}{\partial q_i}\frac{\partial\phi_s}{\partial p_i} =$$
$$= (\phi_{s'},\phi_s).$$

Thus, formula (3.12) is proved.

Consequence of theorem 2. If the Lagrangian constraints (2.5) are fulfilled identically or they can be rewritten in the form containing only q and p, then in such a theory the primary constraints (2.10) are in involution in a weak sense at least.

Theorem 2 gives us in the framework of the Lagrangian formalism a sufficient criterion of the existence in the theory of the second-class constraints. If the Lagrangian constraints (2.5) cannot be reduced by virtue of the definition (2.7) to the equations containing only q and p (this can take place obviously when $m \geq 2$), then in such a theory there are the second-class constraints. This criterion according to its construction is sufficient but not necessary. It does not work in the case when the primary constraints are in involution only between themselves but not with the secondary constraints.

Let us consider now a few simple examples*

1. In the theory with the Lagrangian $L = \frac{1}{2}\dot x_1^2 - \dot x_2 x_3$

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* These examples of the degenerate Lagrangians were reported to the authors by L.V.Prokhorov.

there are one Euler equation of the second order $\ddot x_1 = 0$ and two Lagrangian constraints $\dot x_3 = 0$, $\dot x_2 = 0$. The canonical momenta are $p_1 = \dot x_1$, $p_2 = -x_3$, $p_3 = 0$. The velocities $\dot x_2$ and $\dot x_3$ cannot be eliminated from the Lagrangian constraints. Therefore the primary constraints $\phi_1 = p_2 + x_3$, $\phi_2 = p_3$ must be of the second class $(\phi_1, \phi_2) = 1$. In this theory there are no secondary constraints.

2. We consider now an example where there are the second-class constraints but the primary constraints are in involution between themselves $L = \frac{1}{2}\dot x_1^2 + (\dot x_2 + x_3)x_3 + (\dot x_3 + x_2)x_2$.

Here there are again one Euler equation of the second order $\ddot x_1 = 0$ and two Lagrangian constraints $x_2 = 0$, $x_3 = 0$. Making use of the expressions for the canonical momenta $p_1 = \dot x_1$, $p_2 = x_3$, $p_3 = x_2$, we obtain two primary constraints $\phi_1 = p_2 - x_3$, $\phi_2 = p_3 - x_2$. The primary constraints being in involution $(\phi_1, \phi_2) = 0$, the Lagrangian constraints do not depend on the velocities and give rise to the secondary constraints $\omega_1 = x_1 = 0$, $\omega_2 = x_3 = 0$. All the constraints ϕ_1, ϕ_2, ω_1, ω_2 are of the second class but the criterion proposed above does not work here because the nonvanishing Poisson brackets are between primary and secondary constraints $(\phi_1, \omega_1) = -1$, $(\phi_2, \omega_2) = -1$.

Theorem 3. The Poisson brackets of the primary constraints $\phi_s(q,p)$ with the canonical Hamiltonian $H = p_i \dot q_i - L$ are defined by

$$(\phi_s, H) \overset{\phi}{=} B_s(q,\dot q) + \dot q_{r+s}(\phi_s, \phi_{s'}). \qquad (3.15)$$

To prove this assertion, we have to calculate partial derivatives with respect to q_i and p_i of the canonical Hamiltonian. Taking into account (2.8) and (2.9) we obtain

$$\frac{\partial H}{\partial q_i} = p_\alpha \frac{\partial h_\alpha}{\partial q_i} + \frac{\partial g_{r+s}}{\partial q_i}\dot q_{r+s} - \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial \dot q_\alpha}\frac{\partial h_\alpha}{\partial q_i} = \frac{\partial g_{r+s}}{\partial q_i}\dot q_{r+s} - \frac{\partial L}{\partial q_i}, \qquad (3.16)$$

$$\frac{\partial H}{\partial p_\alpha} = \dot q_\alpha + p_\beta \frac{\partial h_\beta}{\partial p_\alpha} + \frac{\partial g_{r+s}}{\partial p_\alpha}\dot q_{r+s} - \frac{\partial L}{\partial \dot q_\beta}\frac{\partial h_\beta}{\partial p_\alpha} = \dot q_\alpha + \frac{\partial g_{r+s}}{\partial p_\alpha}\dot q_{r+s}, \qquad (3.17)$$

$$\frac{\partial H}{\partial p_{r+s}} = 0, \quad i = 1, \ldots, n, \quad \alpha, \beta = 1, \ldots, r = n - m, \quad s = 1, \ldots, m. \qquad (3.18)$$

Substitute now eqs.(3.16)-(3.18) into the Poisson brackets

$$(\phi_s, H) = \frac{\partial \phi_s}{\partial q_i}\cdot\frac{\partial H}{\partial p_i} - \frac{\partial \phi_s}{\partial p_i}\cdot\frac{\partial H}{\partial q_i} \overset{\phi}{=}$$

$$\stackrel{\phi}{=} -\xi_j^s \frac{\partial p_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \xi_i^s \frac{\partial H}{\partial q_i} = -\xi_j^s \left(\frac{\partial p_j}{\partial q_i}\dot q_i - \frac{\partial L}{\partial q_i}\right) -$$

$$- \dot q_{r+s'} \xi_j^s \left(\frac{\partial g_{r+s'}}{\partial q_i} + \frac{\partial p_j}{\partial q_\alpha}\frac{\partial g_{r+s'}}{\partial p_\alpha} - \frac{\partial p_j}{\partial q_{r+s'}}\right) = \tag{3.19}$$

$$= B_s(q,\dot q) + \dot q_{r+s'} \{\phi_s, \phi_{s'}\}, \qquad s,s' = 1,\dots,m.$$

Here we used eq.(2.5) and represented the primary constraint $\phi_{s'}(q,p)$ according to (2.9) in the form

$$\phi_{s'}(q,p) = g_{r+s'}(q,p_\alpha) - p_{r+s'}. \tag{3.20}$$

The left-hand side of eq.(3.19) is a function only of q and p. So, the right-hand side of this equality must depend also on q and p. If the primary constraints $\phi_s(q,p)$, $s = 1,\dots,m$ are involution at least in a weak sense, then the second term in the right-hand side of (3.15) considered as a function of q and $\dot q$ vanishes. In this case the Lagrangian constraints $B_s(q,\dot q)$ can be represented as functions of q and p, that follows in a straightforward way from eq.(3.12).

4. ACTION INVARIANCE AND PROPERTIES OF THE HAMILTONIAN CONSTRAINTS

If the action

$$S = \int L(q,\dot q)\,dt \tag{4.1}$$

is invariant under the transformations of coordinates q and time t which depend on m arbitrary functions of time $\epsilon_s(t)$, $s = 1,\dots,m$, then the Lagrangian $L(q,\dot q)$ is degenerate because the matrix Λ has m eigenvectors with zero eigenvalues in this case[6,3]. Using the second Noether theorem we shall show that the primary constraints among the canonical variables q and p corresponding to these eigenvectors are in involution with each other in a weak sense at least.

Consider the following form variations of functions $q(t)$ depending on m arbitrary functions of time $\epsilon_s(t)$, $s = 1,\dots,m$ and their derivatives with respect to t up to the σ order inclusive

$$\bar\delta q_i(t) = \bar q_i(t) - q_i(t) = \sum_{\rho=0}^{\sigma} \gamma_{i\rho}^s (t, q^{(0)}, q^{(1)},\dots, q^{(\sigma-\rho)})\, \epsilon_s^{(\rho)}(t),$$

$$f^{(\rho)} = \frac{d^\rho f}{dt^\rho}, \quad 0 \le \rho \le \sigma, \quad f^{(0)}(t) = f(t), \quad i = 1,\dots,n, \quad s = 1,\dots,m. \tag{4.2}$$

According to the second Noether theorem[7] the following identities

$$\sum_{\rho=0}^{\sigma} (-1)^\rho \frac{d^\rho}{dt^\rho}[\gamma_{i\rho}^s (t, q^{(0)}, q^{(1)},\dots, q^{(\sigma-\rho)})\, L_i(q^{(0)}, q^{(1)}, q^{(2)})] \equiv 0,$$
$$i = 1,\dots,n, \quad s = 1,\dots,m \tag{4.3}$$

take place, where $L_i(q^{(0)}, q^{(1)}, q^{(2)})$ mean the left-hand sides of the Euler equations (2.4) (the Lagrangian expressions). Equalities (4.3) are fulfilled identically with respect to the functions $q_i(t)$, $i = 1,\dots,n$. Consequently, in the left-hand side of formulae (4.3) the coefficients of each functions $q_i(t)$, $i = 1,\dots,n$ and of each their derivatives with respect to t, met in (4.3), must be equated to zero separately.

Equating to zero the coefficient of the $q_i^{(\sigma+2)}$, $i = 1,\dots,n$ in (4.3) one obtains

$$\gamma_{i\sigma}^s (t,q)\, \Lambda_{ij}(q,\dot q) = 0, \quad s = 1,\dots,m, \quad i,j = 1,\dots,n. \tag{4.4}$$

We shall suppose that in transformation (4.2) the coefficients $\gamma_{i\sigma}^s(t,q)$, $s = 1,\dots,m$, $i = 1,\dots,n$, considered as a set of m vectors with n components, are linear-independent. In this case from (4.4) it follows that the rank of matrix $\|\Lambda\|$ is equal to $(n-m)$ and, as has been shown in Section 2, m primary constraints of form (2.9) or (2.10) occur in the theory. Evidently, these constraints are linear-dependent on the momenta because $\gamma_{i\sigma}^s(t,q)$ are independent of p. The set of linear-independent vectors $\gamma_{i\sigma}^s(t,q)$ and that of the eigenvectors defined by formula (2.12), are connected by a nondegenerate transformation

$$\gamma_{i\sigma}^s (t,q) = \sum_{s'=1}^{m} C_{s'}^s(t,q)\, \xi_i^{s'}(q), \tag{4.5}$$

$$\mathrm{rank}\,\| C_{s'}^s(t,q)\| = m, \quad s,s' = 1,\dots,m, \quad i = 1,\dots,n. \tag{4.6}$$

In the case under consideration the primary constraints are in involution in the weak sense at least. To prove this, we write out in an explicit form the coefficients of $q_i^{(\sigma+1)}$, $i = 1,\dots,$ in the Noether identities (4.3) and equate them to zero

$$\gamma_{i\sigma-1}^s (t,q,\dot q)\, \Lambda_{ik}(q,\dot q) -$$

$$- \gamma_{i\sigma}^s (t,q) \left(\frac{\partial^3 L}{\partial q_j \partial \dot q_i \partial \dot q_k}\dot q_j + \frac{\partial^2 L}{\partial \dot q_i \partial q_k} - \frac{\partial^2 L}{\partial q_i \partial \dot q_k}\right) = 0, \tag{4.7}$$

$$i,j,k = 1,\dots,n, \quad s = 1,\dots,m.$$

Multiplying equalities (4.7) by $\xi_k^{s'}(q)$, one gets now

$$\gamma^{s}_{i\sigma}(t,q)\,\xi^{s'}_{k}(q)\,(\dot q_j \frac{\partial}{\partial q_j}\Lambda_{ik}(q,\dot q) + \frac{\partial^2 L}{\partial \dot q_i \partial q_k} - \frac{\partial^2 L}{\partial q_i \partial \dot q_k}) = 0 . \qquad (4.8)$$

The first term of (4.8) converts into zero

$$\gamma^{s}_{i\sigma}(t,q)\,\xi^{s'}_{k}(q)\,\dot q_j \frac{\partial}{\partial q_j}\Lambda_{ik}(q,\dot q) =$$

$$= \gamma^{s}_{i\sigma}(t,q)\,\dot q_j \frac{\partial}{\partial q_j}(\xi^{s'}_{k}(q)\,\Lambda_{ik}(q,\dot q)) - \gamma^{s}_{i\sigma}(t,q)\,\Lambda_{ik}(q,\dot q)\,\dot q_j \frac{\partial}{\partial q_j}\xi^{s'}_{k}(q) = 0. \qquad (4.9)$$

Taking into account (4.5) and (3.7), the expression remaining in (4.8) can be rewritten as follows

$$0 = \sum_{s''=1}^{m} C^{s}_{s''}(t,q)\,\xi^{s''}_{i}(q)\,\xi^{s'}_{k}(q)\,(\frac{\partial p_i}{\partial q_k} - \frac{\partial p_k}{\partial q_i}) \overset{\phi}{=}$$

$$\overset{\phi}{=} \sum_{s''=1}^{m} C^{s}_{s''}(t,q)\,(\phi_{s'},\phi_{s''}), \quad s,s',s''=1,\dots,m. \qquad (4.10)$$

Since the matrix $\| C^{s}_{s'}(t,q) \|$ has the rank m, we obtain from (4.10) that the matrix of the Poisson brackets $(\phi_s(q,p)$, $\phi_{s'}(q,p))$ has zero rank when the constraint equations $\phi_s = 0$, $s = 1,\dots,m$ are fulfilled. So

$$(\phi_s(q,p),\ \phi_{s'}(q,p)) \overset{\phi}{=} 0 . \qquad (4.11)$$

Therefore the primary constraints $\phi_s(q,p)$, $s = 1,\dots,m$ are in involution between themselves at least in a weak sense.

It should be noted that the conclusion about the singularity of Lagrangian may be done as well in the case when action (4.1) is invariant under the following transformations

$$\delta q_i(t) = \bar q_i(t) - q_i(t) = \sum_{\rho=0}^{\sigma} \gamma^{s}_{i\rho}(t,q^{(0)},q^{(1)},\dots,q^{(\sigma-\rho)},q^{(\sigma-\rho+1)})\,\epsilon^{(\rho)}_{s}(t). \qquad (4.12)$$

In contrast to transformations (4.2) the coefficients $\gamma^{s}_{i\rho}$ of (4.14) may possess derivatuves of $q_i(t)$ with respect to time the order of which is greater by unity than that of the derivatives in (4.2). Once again the Noether identities must be used here

$$\sum_{\rho=0}^{\sigma} (-1)^{\rho} \frac{d^{\rho}}{dt^{\rho}}[\gamma^{s}_{i\rho}(t,q^{(0)},q^{(1)},\dots,q^{(\sigma-\rho+1)})L_i(q,\dot q,\ddot q)] = 0 , \qquad (4.13)$$

$$s = 1,\dots,m, \quad i = 1,\dots,n.$$

Then equating to zero the coefficients of $q_i^{(\sigma+2)}$ one obtains

$$\gamma^{s}_{i\sigma}(t,q,\dot q)\,\Lambda_{ij}(q,\dot q) = 0 , \quad s = 1,\dots,m . \qquad (4.14)$$

The matrix $\Lambda_{ij}(q,\dot q)$ has zero eigenvectors depending on q and $\dot q$. Thus, it is degenerate. The primary constraints may depend nonlinearly on the momenta in this case. By arguments similar for transformations (4.2) one can establish that these constraints will be in involution when the coefficients $\gamma^{s}_{i\rho}$ of (4.12) satisfy the condition

$$\xi^{s'}_{k}(q,p)\frac{\partial \gamma^{s}_{i\rho}(t,q^{(0)},\dots,q^{(\sigma-\rho+1)})}{\partial q_k^{(\sigma-\rho+1)}} = 0, \quad s,s'=1,\dots,m,\ i,k=1,\dots,n,\ \rho=0,1,\dots,\sigma. \qquad (4.15)$$

The problem that would be interesting to consider further on the basis of the second Noether theorem is to prove that the primary constraints in the theory invariant under transformations with arbitrary functions of time should be first-class constraints. They must be in involution not only between themselves but also with the secondary constraints.

5. CONCLUSION

The assertion proved in the preceding Section is almost apparent because the invariance of theory under transformations of the configuration space with arbitrary functions of time results in the functional arbitrariness in the solution of the appropriate Euler equations. But in the generalized-Hamiltonian dynamics the functional arbitrariness may only appear in the case when the first-class constraints are present (see,e.g., /1/, p.423). However, such arguments cannot favour this statement because in the case of degenerate Lagrangians we are dealing with the singular Legendre mapping from the configuration space into the phase space. When attempting to prove this assertion by the consideration of dynamics in the phase space immediately without the second Noether theorem one is faced with the ambiguity in the correspondence between the given transformation of the configuration space and appropriate canonical transformation in the phase space /4/.

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Червяков А.М., Нестеренко В.В. E2-84-470
Некоторые свойства связей в теориях
с вырожденными лагранжианами

С помощью второй теоремы Нетера показано,что инвариантность действия по отношению к преобразованиям с произвольными функциями времени приводит к первичным связям, находящимся в инволюции по крайней мере в слабом смысле. Доказано, что лагранжевы связи являются инвариантными соотношениями для уравнений Эйлера. В рамках лагранжева формализма предложен критерий существования в теории связей второго рода. Вычислены скобки Пуассона первичных связей с каноническим гамильтонианом.

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Chervyakov A.M., Nesterenko V.V. E2-84-470
Some Properties of Constraints
in Theories with Degenerate Lagrangians

By means of the second Noether theorem it is shown that the action invariance under the transformations with arbitrary functions of time results in primary constraints, which are in involution between themselves at least in a weak sense. The Lagrangian constraints have been proved to be invariant relations for the Euler equations. In the framework of the Lagrangian formalism a criterion of the appearance in the degenerate theory of the second-class constraints is proposed. The Poisson brackets of the primary constraints with the canonical Hamiltonian are determined.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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