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THE YANG-FELDMAN EQUATION IN CONFORMALLY INVARIANT QED

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In papers [1,2] a new conformally invariant formulation of electromagnetic-field quantization was proposed. It was suggested by the aspiration of preserving the Maxwell equation symmetries in quantum theory too. An essential feature of this quantization scheme is the existence of conformal symmetry only in the physical state space $\mathcal{H}_{ph}$. As is known, the space $\mathcal{H}_{ph}$ is usually imbedded in a larger unphysical space $\mathcal{H}$ of the state-vectors of the vector field $A_\mu(x)$. Matrix elements of this field between the states from $\mathcal{H}_{ph}$ are electromagnetic potentials. That is why the conformal symmetry in $\mathcal{H}_{ph}$ is usually achieved setting it in $\mathcal{H} \supset \mathcal{H}_{ph}$. However, such a way of its introduction is not suggested by the physical problem properties. In fact, it is enough to attain an invariance under the conformal group action only in $\mathcal{H}_{ph}$ (as, for example, in [1]). The requirement for the larger space $\mathcal{H}$ invariance has the mathematical origin. As was shown in [3,4] such a requirement gives us a possibility of calculating exact (and unique) values of some quantities which might play the role of two- and three-point functions in the theory.

It seems that without conformal symmetry in the larger space $\mathcal{H}$ we shall loss this possibility. However, even if so, it is evident that such a refusal leads, on the other hand, to a more general solution of the problem of conformally invariant quantization of the electromagnetic field. Thus the class of functions which might play the role of the Green functions in the theory is essentially extended.

In the present paper we shall show that some unique results may be obtained with the requirement for conformal symmetry of the photon sector only in $\mathcal{H}_{ph}$. In particular, it is enough to fix the form of the integral equation for electromagnetic field which is of the type of the Yang-Feldman one. In the case of massless spinor quantum electrodynamics this equation allows us to formulate the theory so that Maxwell's equation is automatically fulfilled in the physical space $\mathcal{H}_{ph}$.

1. We shall begin with a brief review of some necessary information from paper [1]. In this paper a new gauge condition for the vector field $A_\mu(x)$ was proposed to construct the large space $\mathcal{H}$. This condition is a third-order equation and has the form (Greek indices take values from 0 to 3):

$$\Box \partial^\mu A_\mu(x) = f \partial^\mu S(x) j_\mu(x) , \qquad (1)$$

where $j_\mu(x)$ is the current of electric charges:

$$\partial^\mu j_\mu(x) = 0, \tag{2}$$

$S(x)$ is a scalar field with zero conformal dimension and with nonhomogeneous conformal transformations (in more detail this field is considered in papers /1,2,5,6/); f is a constant which depends on the $A_\mu(x)$ and $S(x)$ -fields normalization; $\Box$ is the D'Alembert operator $\Box = g^{\mu\nu}\partial_\mu \partial_\nu$; $\partial_\mu \equiv \partial/\partial x^\mu$; $g_{\mu\mu} \equiv (1,-1,-1,-1)$. In papers /7,8/ condition (1) for the free case (when $j_\mu(x) = 0$) is considered.

In paper /1/ Maxwell's equation was proposed to replace by equation (1) in quantum theory. Motivation was the following: Suppose that we are able to construct quantum fields $A_\mu(x)$, $S(x)$, and $j_\mu(x)$ which satisfy eq.(1). It turns out that among matrix elements of the operator $A_\mu(x)$ there are conformally invariant ones and they satisfy Maxwell's equation with the current $j_\mu(x)$. In this case the role of Maxwell's equations is an additional condition separating conformally-symmetric matrix elements of the operator $A_\mu(x)$ which obey eq.(1). (For more details see papers /1,2/). The state-vectors of operator $A_\mu(x)$ form the space $\mathcal{H}$. Its subspace $\mathcal{H}_{ph} \subset \mathcal{H}$ is spanned on those of them giving rise to conformally symmetric matrix elements. Let us denote by $A^{ph}_\mu(x)$ and $j^{ph}_\mu(x)$ matrix elements of operators $A_\mu(x)$ and $j_\mu(x)$ between physical states (i.e., states from $\mathcal{H}_{ph}$). Then:

i) $A^{ph}_\mu(x)$ satisfies Maxwell's equation

$$(\Box g^\mu{}_\nu - \partial^\mu \partial_\nu) A^{ph}_\mu(x) = j^{ph}_\nu(x). \tag{3}$$

ii) the set of functions $A^{ph}_\mu(x)$ is invariant under canonical special conformal transformations

$$A^{ph}_\mu(x) \to \frac{\partial x'^\nu}{\partial x^\mu} A^{ph}_\nu(x'); \quad x'_\mu = \frac{x_\mu + a_\mu x^2}{1 + 2(ax) + a^2 x^2}, \tag{4}$$

where a_μ are transformation parameters.

In fact, this means that in $\mathcal{H}^{ph}$ there acts a conformal-group representation which preserves this space itself and, of course, eq.(3), though at the same time the space $\mathcal{H}$ might not be conformally symmetric (at all).

In the case of a free electromagnetic field eq.(1) takes the form

$$\Box \partial^\mu A_\mu(x) = 0 \tag{5}$$

and field operators $A_\mu(x)$ and $S(x)$ obey the following commutation relations:

$$[A_\mu(x), A_\nu(y)] = -i g_{\mu\nu} D(x-y) + i\kappa \partial_\mu \partial_\nu E(x-y), \tag{6}$$

$$[A_\mu(x), S(y)] = i r \partial_\mu E(x-y), \tag{7}$$

$$[S(x), S(y)] = -i\lambda E(x-y). \tag{8}$$

Here r , κ and λ are constants, and commutation functions $D(x)$ and $E(x)$ have the form

$$D(x) = \Box E(x) = \frac{1}{2\pi} \epsilon(x_0)\, \delta(x^2), \tag{9}$$

$$E(x) = \frac{1}{8\pi} \epsilon(x_0)\, \theta(x^2). \tag{10}$$

The space $\mathcal{H}$ is the five-component field $(A_\mu(x), S(x))$ Fock space and for $\mathcal{H}_{ph}$ we shall remind only the structure of its one-photon state vectors, namely $|\Phi\rangle \in \mathcal{H}_{ph}$, if

$$|\Phi\rangle = \int \Phi^\mu(x)\, A^+_\mu(x)\, |0\rangle\, d^4x, \tag{11}$$

where functions $\Phi^\mu(x) \in \mathcal{S}(R_4)$ satisfy the following equation

$$\partial^\mu \Phi_\mu(x) = 0 \tag{12}$$

($\mathcal{S}(R_4)$ is the space of smooth functions of four arguments increasing at infinity faster than any negative degree of all these arguments).

2. Let us now turn back to equation (1). As a first step we shall try to find some analogy of Green's functions for this equation. The problem is that such a function (let us denote it by $G_{\mu\nu}(x-y)$) has to be a second-rank tensor due to the special structure of eq.(1) and its Lorentz invariance. We obtain a vector $\Box\partial^\mu G_{\mu\nu}$ when acting on $G_{\mu\nu}(x-y)$ with the operator $\Box\partial^\mu$. The simplest vector function with a point support is $\partial_\mu \delta^4(x-y)$ ($\delta(x-y)$ is the Dirac function). Thus we are led to the conclusion that this Green's function analogy must satisfy the following equation

$$\Box\partial^\mu G_{\mu\nu}(x-y) = \partial_\nu \delta^4(x-y). \tag{13}$$

A partial solution of this equation when scale dimension of $G_{\mu\nu}(x)$ is taken into account is

$$G_{\mu\nu}(x) = a\, g_{\mu\nu} D_g(x) - (1+a)\partial_\mu \partial_\nu E_g(x), \tag{14}$$

where a is an arbitrary constant, $D_g(x)$ is the Green function of D'Alembert equation

$$\Box D_g(x) = -\delta^4(x) \tag{15}$$

and $E_g(x)$ is Green's function of the equation

$$\Box^2 E_g(x) = -\delta^4(x), \quad (\Box E_g(x) = D_g(x)). \tag{16}$$

It follows from eq.(13) that $G_{\mu\nu}$ must have a scale dimension of $(\text{length})^{-2}$ that is in agreement with the dimension of physical matrix elements of the field $A_\mu(x)$ (dimension of the latter is $(\text{length})^{-1}$). Hence (14) is the only scale invariant solution of eq.(13) with such a dimension. We would like to point out that there is one uncertainty in $D_g(x)$ and $E_g(x)$ because of the boundary conditions which are not yet taken into account, and later on we shall return to this problem once more.

Consider the following expression

$$h_\mu(x) = A_\mu(x) - f\int D_{\mu\nu}(x-y)\,S(y)\,j^\nu(y)\,d^4y + c_1\int D_g(x-y)\,j_\mu(y)\,d^4y + c_2\partial_\mu\phi(x). \tag{17}$$

Here $A_\mu(x)$ is an arbitrary solution of eq.(1), c_1 and c_2 are arbitrary constants, and $\phi(x)$ is a new field which satisfies the equation

$$\Box^2\phi(x) = 0. \tag{18}$$

A straightforward calculation (where relations (13) and (1) are used) convinces us that $h_\mu(x)$ obeys the free equation (5):

$$\Box\partial^\mu h_\mu(x) = \Box\,\partial^\mu A_\mu(x) - f\,\Box\,\partial^\mu\int D_{\mu\nu}(x-y)\,S(y)\,j^\nu(y)\,d^4y = 0. \tag{19}$$

Thus, equation (17) gives us the correspondence between a solution $A_\mu(x)$ of eq.(1) and the free field $h_\mu(x)$ (in the sense of eqs.(5) or (19)).

3. Let us now consider equation (17) in the quantum-theory context. This equation connects solutions of equations (1) and (5). Then the free field $h_\mu(x)$ and the interacting one $A_\mu(x)$ have a common state space $\mathcal{H}$. The relation between those two fields is linear and canonical conformal transformations are homogeneous. Then it follows that physical matrix elements of the field $h_\mu(x)$ arise when in the right-hand-side of (17) physical matrix elements of the field $A_\mu(x)$ have arised. And this means that the physical space $\mathcal{H}_{ph}$ for operators $A_\mu(x)$ and $h_\mu(x)$ will be common too. A simple calculation of the matrix elements of operator equation (17) between physical states gives us

$$h^{ph}_\mu(x) = A^{ph}_\mu(x) - f\int D_{\mu\nu}(x-y)(S(y)\,j^\nu(y))^{ph}\,d^4y + \tag{20}$$
$$+\,c_1\int D_g(x-y)\,j^{ph}_\mu(y)\,d^4y + c_2\partial_\mu\phi^{ph}(x)$$

(the superscript ph is used here to denote the physical matrix elements of corresponding operators). As is known conserved current $j^{ph}_\mu(x)$ is transformed under the special conformal transformations as

$$j^{ph}_\mu(x) \to j^{ph\prime}_\mu(x) = \frac{1}{\rho^2(a,x)}\,\frac{\partial x'^\nu}{\partial x^\mu}\,j^{ph}_\nu(x'), \tag{21}$$

where $\rho(a,x) = 1 + 2(ax) + a^2x^2$. At the same time the field $S(x)$ obeys a nonhomogeneous transformation law

$$S(x) \to S'(x) = S(x') + \chi\,\ln|\rho(a,x)|, \tag{22}$$

where the constant χ depends on the field $S(x)$ normalization.

We would like to remind that $h^{ph}_\mu(x)$ and $A^{ph}_\mu(x)$ must be symmetric solutions of the corresponding equations. Hence the last three terms in the right-hand side of eq. (20) as a whole have to be transformed under the conformal group action according to eq.(4), when the matrix elements $j^{ph}_\mu(x)$ and $(S(x)\,j(x))^{ph}$ transformed under their own transformation laws, defined by eqs.(2) and (22). Owing to (18) we may consider for the field $\phi(x)$ the same transformation law as for the field $S(x)$. So, the last term in eq.(20) will have canonical transformation law (4). For the remaining two terms

$$L_\mu(x, S, j_\mu) = c_1\int D_g(x-y)\,j^{ph}_\mu(y)\,d^4y - f\int D_{\mu\nu}(x-y)(S(y)\,j^\nu(y))^{ph}\,d^4y$$

we must find conditions that lead to the identity

$$L_\mu(x, S'; j') = \frac{\partial x'^\nu}{\partial x^\mu}\,L_\nu(x', S; j) \tag{23}$$

(S' and j' are defined with (21), (22); the superscript ph is omitted for simplicity).

A tedious, but not difficult calculation leads us to the following conditions obtained from the infinitesimal form of eq. (22) that are in an identity:

i) $a = 0$,

ii) $c_1 = \frac{\chi}{2}f$ (see (14)).

As we have reduced condition (23) only to the determination of arbitrary constants, both eqs.(20) and (17) are fixed. The constants f and χ are related by /1/ $f = 2/\chi$.

Therefore instead of (17) we finally obtain

$$h_\mu(x) = A_\mu(x) - \frac{2}{\chi}\int\partial_\mu\partial_\nu E_g(x-y)\,S(y)\,j^\nu(y)\,d^4y + \tag{24}$$
$$+\int D_g(x-y)\,j_\mu(y)\,d^4y + c_2\partial_\mu\phi(x).$$

The constant c_2 is not fixed. The corresponding term represents the gauge freedom, each solution of eqs.(1) on (5) possesses.

4. Equation (24) is in fact some generalization of the Yang-Feldman equation for electromagnetic potentials where the physical space conformal invariance is taken into account. To get convinced of this, we shall write (24) as a solution for the vectors $A_\mu(x)$:

$$A_\mu(x) = h_\mu(x) - \int D_g(x-y)\, j_\mu(y)\, d^4y + \frac{2}{\chi}\int \partial_\mu \partial_\nu E_g(x-y)\, S(y)\, j^\nu(y)\, d^4y. \quad (25)$$

We have added the term $c_2 \partial_\mu \phi$ to the field $h_\mu(x)$ making use of the gauge freedom of the latter. Equation (24) is valid for all solutions of eq.(1). So, each of them may be represented in the form (25).

Equation (25) differs from the Yang-Feldman equation by the third, gradient term. This terms do not vanish because in this case there would not exist conformal symmetric matrix elements of the operator $A_\mu(x)$.

The field $h_\mu(x)$ coincides with the (in)-electromagnetic field if corresponding retarded Green's functions $D_{ret}(x)$ and $E_{ret}(x)$ ($\Box E_{ret}(x) = D_{ret}(x)$) are taken in eq.(25). So. we may denote this field as $A^{in}_\mu(x)$ and then eq.(25) will read

$$A_\mu(x) = A^{in}_\mu(x) - \int D_{ret}(x-y)\, j_\mu(y)\, d^4y + \frac{2}{\chi}\int \partial_\mu \partial_\nu E_{ret}(x-y)\, S(y)\, j^\nu(y)\, d^4y. \quad (26)$$

5. We have not yet fixed the structure of the current. It may be either an external classical current, or a quantum fermionic one, if only conserved. In the second case we have to consider massless charged fermions because of the conformal symmetry of the theory (though eq. (26) might be valid in the massive case too). These massless fermions satisfy the Dirac equation

$$i\gamma^\mu \partial_\mu \psi(x) = e\gamma^\mu A_\mu(x)\psi(x) \quad (27)$$

(γ^μ are the Dirac matrices). Then (26) includes the quantum spinor current

$$j^\mu = e\bar{\psi}(x)\gamma^\mu\psi(x), \quad \partial^\mu j_\mu = 0 \quad (28)$$

(defined in the usual way).

Thus, we have obtained a new fomulation of quantum electrodynamics given by eqs.(26), (27), (28), along with eq.(5) for the field $A^{in}_\mu(x)$.In this formulation Maxwell's equation is automatically fulfilled but only in the physical conformal-invariant

state space $\mathcal{H}_{ph}$. The field $A^{in}_\mu(x)$ must be quantized according to the scheme in paper /1/ (some properties of this field are given by relations (6)-(12)). As is known, an iteration method applied to eq.(26), (27) allows one to express approximately (in powers of e) the out-fields through the in-fields and to calculate the S-matrix elements. Obviously, in the perturbation theory thus constructed the gradient term in (26) will play an essential role in the diagram internal lines. A diagram technique based on the Yang-Feldman equation in QED was considered in paper /9/. However, an essential generalization of this techniques is necessary in our case because of the additional term in the integral equation (26).

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Стоянов Д.Ц. E2-84-465

Уравнение Янга-Фельдмана в конформно-инвариантной КЭД

Исследуется ранее предложенное новое калибровочное условие для электромагнитных потенциалов в квантовой электродинамике. Это условие дает возможность расширить гильбертово пространство состояний таким образом, чтобы физическое подпространство, во-первых, оказалось конформно-инвариантным и, во-вторых, в нем выполнялись автоматически уравнения Максвелла. Такая структура пространства состояния дает возможность построить интегральное уравнение типа Янга-Фельдмана, которое приводит к новой формулировке квантовой электродинамики.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1984

Stoyanov D.T. E2-84-465

The Yang-Feldman Equation in Conformally Invariant QED

A recently proposed new gauge condition for electromagnetic potential in QED is discussed. This condition allows one to extend the Hilbert state-space in such a way that the physical subspace is conformal-invariant and Maxwell equations in it are automatically fulfilled. This state-space structure gives a possibility of constructing the Yand-Feldman type integral equation, which leads to a new formulation of quantum electrodynamics.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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