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UNCONSTRAINED OFF-SHELL N=3 SUPERSYMMETRIC YANG-MILLS THEORY

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1. Introduction

Recently we have presented unconstrained superfield (SF) off-shell formulations of all the N=2 supersymmetric theories /1/. They are based on the new principle of "harmonization" of superspace (SS). The harmonic SS has a 6-dimensional even part,

$$M^4 \otimes \frac{SU(2)}{U(1)}$$

The two extra dimensions form a compact homogeneous space (the sphere S^2) where the SU(2) automorphism group of N=2 supersymmetry (SUSY) is realized. The SF's defined on the harmonic SS are nontrivial U(1) representations but SU(2) singlets[*)]. Ordinary 4-dimensional SU(2) covariant fields enter as components of the harmonic expansion of the SF's on the sphere S^2. The crucial feature of the harmonic SS is that it possesses an analytic subspace with half the number of odd coordinates (a generalization of the N=1 concept of chirality). The simplest analytic SF's defined there describe the hypermultiplet. The preservation of the concept of analyticity is the basis of the formulation of the N=2 supersymmetric Yang-Mills theory (SYM- and N=2 supergravity).

The principle of harmonization of the SS is applicable not only to the case N=2. In the present paper it helps us to solve the long-standing problem of finding an off-shell formulation of the N=3 SYM theory. (On-shell it coincides with N=4 SYM and describes a vector, four spinors, and six real scalar fields). Up to now there have been doubts whether this was possible (the so-called "no-go theorems", "N=3 barrier" /2/). We jump over this barrier by using infinite sets of auxiliary fields.

The standard (constrained) SS approach to N=3 SYM is based on the SS with coordinates

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*) In a fixed U(1)-gauge they transform according to the non-linear realization of SU(2): undergo U(1)-transformations depending on SU(2)/U(1)-coordinates.

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$$Z^M = \{ x^{\alpha\dot\alpha}, \theta^\alpha_i, \bar\theta^{\dot\alpha i} \}. \qquad (1.1)$$

where $i,j = 1,2,3$ are SU(3) indices. The spinor derivatives $\mathcal{D}^i_\alpha$, $\bar{\mathcal{D}}_{\dot\alpha i}$, covariant with respect to the Yang-Mills group with parameters $\tau(z)$ (τ- group below)

$$\mathcal{D}^i_\alpha = D^i_\alpha + i A^i_\alpha(z), \quad D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} + i \partial_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i}$$

$$\bar{\mathcal{D}}_{\dot\alpha i} = \bar D_{\dot\alpha i} + i \bar A_{\dot\alpha i}(z), \quad \bar D_{\dot\alpha i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - i\theta^\alpha_i \partial_{\alpha\dot\alpha}$$

must obey the constraints[3,4]

$$\{\mathcal{D}^{(i}_\alpha, \mathcal{D}^{j)}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha(i}, \bar{\mathcal{D}}_{\dot\beta j)}\} = 0$$
$$\{\mathcal{D}^i_\alpha, \bar{\mathcal{D}}_{\dot\beta j}\} - \frac{1}{3}\delta^i_j \{\mathcal{D}^k_\alpha, \bar{\mathcal{D}}_{\dot\beta k}\} = 0. \qquad (1.2)$$

These constraints are known to contain the equations of motion[3,4], so the theory is on-shell and no way to go off-shell has been found so far. There exist two geometric interpretations of (1.2). One of them was advocated by Rosly[5] while another (more closely related to the standard twistor approach) by Witten[4], Volovich [6], etc. Using some isospinor [5] or spinor [4,6] parameters they have formed certain subspaces of the tangent space spanned on $\mathcal{D}^i_\alpha$, $\bar{\mathcal{D}}_{\dot\alpha i}$ and considered (1.2) as the condition for these subspaces to be flat (as in the absence of YM fields). Our approach implies just the interpretation by Rosly [5]. At the same time it puts the main accent on an understanding of (1.2) as analytic-representation-preserving constraints (recall that the N=1 constraints preserve the chiral or N=1 analytic representations [7,8], the N=2 ones - the hypermultiplet[7] or rather the N=2 analytic representations [1]).

Our approach to N=3 SYM is based on the harmonized N=3 SS with 10-dimensional even part*)

$$M^4 \otimes \frac{SU(3)}{U(1)\otimes U(1)}.$$

It has an analytic subspace which plays a crucial role similar to the role of N=1 analytic (chiral) and N=2 analytic SS's in the corresponding gauge theories[8,1]. We interpret (1.2) as constraints for the preservation of N=3 analyticity and solve a part of them. The solution is of the "pure gauge" type familiar from the N=1 and,

*) There also exist other harmonic N=3 SS's, e.g.,with a purely harmonic part SU(3)/U(2). Manin /8/ has listed possible SS's of this kind for all N and has given some deep mathematical reasoning why one should consider such SS's.

by now, the N=2 cases. It allows us to define another gauge group with analytic parameters λ (λ -group). In the λ -representation the harmonic covariant derivatives acting on the space SU(3)/U(1)⊗U(1) acquire connection which are analytic SF's. They are the true potentials of the theory, and all the other geometric objects can be expressed in their terms. The equations of motion, initially contained in (1.2), now become constraints on the new harmonic connections (i.e.,in the purely internal, harmonic directions). The crucial next step is to find an action from which the above equations can be derived. The action is given by an integral over the analytic SS and is unexpectedly simple involving only bi- and tri-linear terms in the potentials. Remarkably, the Lagrange density is not a tensor, it is invariant under gauge transformations only up to total harmonic derivatives. Thus, we arrive at the off-shell N=3 SYM theory. It is described by three analytic harmonic SF's which contain infinite sets of ordinary (M^4) auxiliary and gauge degrees of freedom.

In sect. 2 we define properly the N=3 harmonic SS and its analytic subspace. In sect. 3 we rewrite the constraints in a form in which they can be solved and introduce the harmonic potentials. Sect. 4 contains the action and an overview of the components. Some concluding remarks concerning, in particular, off-shell N=3 Einstein supergravity, are given in sect. 5.

2. Harmonic N=3 Superspace

The main idea[1] of the harmonization of SS is to implement the group of automorphisms G of N-extended SUSY on the homogeneous space G/H (H is some subgroup of G). Then the Grassmann coordinates and the SF's become H -covariant objects. The lower symmetry exhibited allows us to find smaller (so-called analytic) subspaces of the harmonic SS where the N=3 SUSY is realized. Track of the full symmetry G is kept by letting the SF's depend harmonically on the coordinates of G/H.

In the case N=3 the appropriate subgroup of $G = SU(3)$ turns out to be U(1)⊗U(1). The corresponding harmonic variables (the basic harmonics) associated with the 6-dimensional space SU(3)/U(1)⊗U(1) are defined in the following way. They are 3x3 unitary unimodular matrices u

$$u\bar u = \bar u u = 1, \quad \det u = 1 \qquad (2.1)$$

on which the SU(3) and U(1)⊗U(1) groups act as follows:

$$u' = g\, u \exp(i a_1 H_1 + i a_2 H_2). \tag{2.2}$$

Here $g \in SU(3)$; a_1, a_2 are parameters with an arbitrary dependence on u and

$$H_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \tag{2.3}$$

are generators of the two U(1) subgroups. Notice that one can fix the U(1) gauges in (2.2) by eliminating 2 of the 8 components of u and making the parameters a_1, a_2 nonlinear functions of g and these particular u; this will correspond to a nonlinear realization of SU(3) with U(1)⊗U(1) as a stability subgroup.

According to (2.2) and (2.3) the elements of the matrix $u = \| u_i^{(a,b)} \|$ *) have an SU(3) index $i = 1,2,3$ and the following assignement with respect to U(1) charges a and b :

$$u = \begin{pmatrix} u_1^{(1,1)} & u_2^{(1,1)} & u_3^{(1,1)} \\ u_1^{(-1,1)} & u_2^{(-1,1)} & u_3^{(-1,1)} \\ u_1^{(0,-2)} & u_2^{(0,-2)} & u_3^{(0,-2)} \end{pmatrix}, \tag{2.4}$$

$$\bar{u} = \left(\bar{u}^{(-1,-1)i} \quad \bar{u}^{(1,-1)i} \quad \bar{u}^{(0,2)i} \right).$$

The unitarity and unimodularity conditions (2.1) lead to a number of relations between the harmonic variables which are listed in the Appendix. There exist 6 (as many as the dimension of G/H) covariant harmonic derivatives consistent with (2.1):

$$\begin{aligned} D^{(1,3)} &= -u_i^{(1,1)} \frac{\partial}{\partial u_i^{(0,-1)}} + \bar{u}^{(0,2)i} \frac{\partial}{\partial \bar{u}^{(-1,-1)i}}, \\ D^{(-1,3)} &= u_i^{(-1,1)} \frac{\partial}{\partial u_i^{(0,-2)}} - \bar{u}^{(0,2)i} \frac{\partial}{\partial \bar{u}^{(1,-1)i}}, \\ D^{(2,0)} &= u_i^{(1,1)} \frac{\partial}{\partial u_i^{(-1,1)}} - \bar{u}^{(1,-1)i} \frac{\partial}{\partial \bar{u}^{(-1,-1)i}} \end{aligned} \tag{2.5}$$

*) This notation for the U(1) charges was suggested by B.Zupnik.

and their complex conjugates. They form the algebra of SU(3) (together with the two U(1) generators; see the Appendix).

The N=3 SS (1.1) is harmonized by adding newly introduced variables u as independent coordinates of the harmonic N=3 SS:

$$\{ x^{\alpha\dot\alpha}, \theta_i^{\alpha}, \bar\theta^{\dot\alpha i}, u \}. \tag{2.6}$$

In this "central basis" (CB) N=3 SUSY acts as shown in (A.1). The most remarkable fact on SS (2.6) is that after making the change of variables to the so-called "analytic basis" (AB)

$$\begin{aligned} x_A^{\alpha\dot\alpha} &= x^{\alpha\dot\alpha} + 2i \left[\theta^{(0,2)\alpha} \bar\theta^{(0,-2)\dot\alpha} - \theta^{(-1,-1)\alpha} \bar\theta^{(1,1)\dot\alpha} \right] \\ \theta^{(a,b)\alpha} &= \bar{u}^{(a,b)i} \theta_{\alpha i}, \quad \bar\theta^{(a,b)\dot\alpha} = u_i^{(a,b)} \bar\theta^{i\dot\alpha} \end{aligned} \tag{2.7}$$

one can pick out an <u>analytic subspace</u>

$$\{ \zeta_A^M = (x_A^{\alpha\dot\alpha}, \theta^{(1,-1)\alpha}, \theta^{(0,2)\alpha}, \bar\theta^{(1,1)\dot\alpha}, \bar\theta^{(-1,1)\dot\alpha}), u \}. \tag{2.8}$$

It is closed under the N=3 SUSY transformations

$$\begin{aligned} \delta x_A^{\alpha\dot\alpha} &= 2i \left[\theta^{(1,-1)\alpha} u_i^{(-1,1)} - 2\theta^{(0,2)\alpha} u_i^{(0,-2)} \right] \bar\varepsilon^{\dot\alpha i} \\ &\quad - 2i \left[\bar\theta^{(-1,1)\dot\alpha} \bar{u}^{(1,-1)i} - 2\bar\theta^{(1,1)\dot\alpha} \bar{u}^{(-1,-1)i} \right] \varepsilon_i^{\alpha}, \\ \delta \theta^{(a,b)\alpha} &= \bar{u}^{(a,b)i} \varepsilon_i^{\alpha}, \quad \delta\bar\theta^{(a,b)\dot\alpha} = u_i^{(a,b)} \bar\varepsilon^{\dot\alpha i}, \\ \delta u_i^{(a,b)} &= 0, \quad \delta \bar{u}^{(a,b)i} = 0. \end{aligned} \tag{2.9}$$

Remarkably, that the AB and the analytic subspace (2.8) are real with respect to the following combined conjugation. There is a special involution $(*)$ which affects only the U(1) charges of the u's. Combined with the ordinary complex conjugation $(\bar{*})$ it gives

$$\begin{aligned} & u_i^{(1,1)} \overset{(\bar{*})}{\longleftrightarrow} u^{(0,2)i}, \quad \theta_\alpha^{(1,-1)} \overset{(\bar{*})}{\longleftrightarrow} -\bar\theta_{\dot\alpha}^{(-1,1)}, \quad x_A^{\alpha\dot\alpha} \overset{(\bar{*})}{\longleftrightarrow} x_A^{\alpha\dot\alpha} \\ & u_i^{(0,-2)} \longleftrightarrow u^{(-1,-1)i}, \quad \theta_\alpha^{(0,2)} \longleftrightarrow \bar\theta_{\dot\alpha}^{(1,1)} \\ & u_i^{(-1,-1)} \longleftrightarrow -u^{(1,-1)i}. \end{aligned} \tag{2.10}$$

This allows to define real objects in the analytic SS.

In AB (2.7) the spinor derivatives $D_\alpha^{(1,1)}$ and $\bar{D}_{\dot\alpha}^{(0,2)}$ become simply

$$\left(D_\alpha^{(1,1)} \right)_{AB} = \frac{\partial}{\partial \theta^{(-1,-1)\alpha}}, \quad \left(\bar{D}_{\dot\alpha}^{(0,2)} \right)_{AB} = -\frac{\partial}{\partial \bar\theta^{(0,-2)\dot\alpha}}. \tag{2.11}$$

This means that the analytic SF's defined by the constraints

$$D_{\alpha}^{(1,1)}\Phi = \bar{D}_{\dot\alpha}^{(0,2)}\Phi = 0 \Rightarrow \Phi = \Phi(\zeta_A, u) \qquad (2.12)$$

are general unconstrained objects living in the analytic SS (2.8). The component expansion of analytic SF's is rather cumbersome and the SF language is much more concise. Here we shall show just the first few components of an analytic SF which will be used in sect. 3. It has overall charges (-1,3), and so has each term in its θ-decomposition

$$\begin{aligned} V^{(-1,3)}&(x_A, \theta_{\alpha}^{(1,-1)}, \theta_{\alpha}^{(0,2)}, \bar\theta_{\dot\alpha}^{(1,1)}, \bar\theta_{\dot\alpha}^{(-1,1)}, u) = V^{(-1,3)}(x_A, u) + \\ &+ \theta^{(1,-1)\alpha}\psi_{\alpha}^{(-2,4)}(x_A, u) + \theta^{(0,2)\alpha}\psi_{\alpha}^{(-1,1)}(x_A, u) + \\ &+ \bar\theta_{\dot\alpha}^{(1,1)}\bar\psi^{\dot\alpha(-2,2)}(x_A, u) + \bar\theta_{\dot\alpha}^{(-1,1)}\psi^{\dot\alpha(0,2)}(x_A, u) + \dots \end{aligned} \qquad (2.13)$$

One has to make harmonic decomposition of each component harmonic field, e.g.,

$$\begin{aligned} V^{(-1,3)}(x_A, u) &= u_i^{(-1,1)}u^{(0,2)j}V_j^i(x_A) + u^{(-1,-1)i}u^{(0,2)j}u^{(0,2)k}V_{(ijk)}(x_A) + \\ &+ u_i^{(-1,1)}u_j^{(-1,1)}u_k^{(1,1)}F^{(ijk)}(x_A) + \\ &+ u_i^{(-1,1)}u_\ell^{(1,1)}u^{(0,2)j}u^{(-1,-1)k}A_{(jk)}^{(i\ell)}(x_A) + \\ &+ u_i^{(-1,1)}u^{(0,2)j}u_k^{(0,-2)}u^{(0,2)\ell}B_{(j\ell)}^{(ik)}(x_A) + \dots \end{aligned} \qquad (2.14)$$

The general rule is that the U(1) charges are conserved throughout the expansion and the ordinary fields are irreps of SU(3) with zero U(1) charges. (This rule originates from the nonlinear realization version of the transformation law (2.2)).

Note that the harmonic derivatives (2.5) acquire space-time derivative terms in AB, e.g.,

$$(D^{(1,3)})_{AB} = (D^{(1,3)})_{CB} - 4i\,\theta^{(0,2)\alpha}\bar\theta^{(1,1)\dot\alpha}\partial_{\alpha\dot\alpha}^{A}, \text{ etc.} \qquad (2.15)$$

Finally, the integration rule for the harmonic variables is simply

$$\int du\, f^{(a,\ell)}(u) = \delta^{a,0}\delta^{\ell,0} f^{(a,\ell)}(0). \qquad (2.16)$$

It allows to integrate by parts w.r.t. the harmonic derivatives.

3. Kinematical and Dynamical N=2 SYM Constraints. Harmonic Connections

Our aim in this section is to consider the N=3 SYM constraints in harmonic SS and rewrite them in an equivalent form in which their meaning will become clear. Then we shall solve part of them (the kinematical ones) thus introducing the harmonic connections. The remaining constraints will be equations of motion for these quantities.

In harmonic SS we shall keep the original gauge group of the constraints (1.2) with u -independent parameters $\tau(z)$. The reason is that all the on-shell gauge-covariant tensors are u -independent SF's and the τ -group is most natural for them. Then the τ -covariant harmonic derivatives $\mathcal{D}^{(a,\ell)}$ (see (2.5)) need no connections,

$$(\mathcal{D}^{(a,\ell)})_{\tau} = D^{(a,\ell)}. \qquad (3.1)$$

Consequently, the commutation relations among $\mathcal{D}_{\alpha}^{i}$, $\bar{\mathcal{D}}_{\dot\alpha i}$ and $\mathcal{D}^{(a,\ell)}$, as well as among $\mathcal{D}^{(a,\ell)}$ themselves remain the same as in the rigid case.

The next step is to convert all SU(3) indices into U(1)$\otimes$U(1) ones with the help of the harmonics $u_i^{(a,\ell)}$. The spinor covariant derivatives become

$$\begin{aligned} (\mathcal{D}_{\alpha}^{(a,\ell)})_{\tau} &= u_i^{(a,\ell)}(\mathcal{D}_{\alpha}^{i})_{\tau} = D_{\alpha}^{(a,\ell)} + iA_{\alpha}^{(a,\ell)}(z,u) \\ (\bar{\mathcal{D}}_{\dot\alpha}^{(a,\ell)})_{\tau} &= \bar{u}^{(a,\ell)i}(\bar{\mathcal{D}}_{\dot\alpha i})_{\tau} = \bar{D}_{\dot\alpha}^{(a,\ell)} + i\bar{A}_{\dot\alpha}^{(a,\ell)}(z,u) \\ A_{\alpha}^{(a,\ell)}(z,u) &= A_{\alpha}^{i}(z)u_i^{(a,\ell)}, \quad \bar{A}_{\dot\alpha}^{(a,\ell)}(z,u) = \bar{A}_{\dot\alpha i}(z)\bar{u}^{(a,\ell)i} \end{aligned} \qquad (3.2)$$

So, now the tangent space is spanned on $\mathcal{D}_{\alpha}^{(a,\ell)}$, $\bar{\mathcal{D}}_{\dot\alpha}^{(a,\ell)}$, $\mathcal{D}^{(a,\ell)}$ and its geometry is specified by a number of constraints following from (1.2), (3.1) and (3.2). However, we shall show that it is sufficient to consider only a subspace spanned on $\mathcal{D}_{\alpha}^{(1,1)}$, $\bar{\mathcal{D}}_{\dot\alpha}^{(0,2)}$, $\mathcal{D}^{(\pm1,3)}$, $\mathcal{D}^{(2,0)}$ subjected to the constraints

$$\{\mathcal{D}_{\alpha}^{(1,1)}, \mathcal{D}_{\beta}^{(1,1)}\} = \{\bar{\mathcal{D}}_{\dot\alpha}^{(0,2)}, \bar{\mathcal{D}}_{\dot\beta}^{(0,2)}\} = \{\mathcal{D}_{\alpha}^{(1,1)}, \bar{\mathcal{D}}_{\dot\beta}^{(0,2)}\} = 0, \qquad (3.3)$$

$$\begin{aligned} [\mathcal{D}^{(\pm1,3)}, \mathcal{D}_{\alpha}^{(1,1)}] &= [\mathcal{D}^{(\pm1,3)}, \bar{\mathcal{D}}_{\dot\alpha}^{(0,2)}] = 0, \\ [\mathcal{D}^{(2,0)}, \mathcal{D}_{\alpha}^{(1,1)}] &= [\mathcal{D}^{(2,0)}, \bar{\mathcal{D}}_{\dot\alpha}^{(0,2)}] = 0, \end{aligned} \qquad (3.4)$$

$$[\mathcal{D}^{(\pm1,3)},\mathcal{D}^{(\mp1,3)}]=[\mathcal{D}^{(1,3)},\mathcal{D}^{(2,0)}]=0\,,$$
$$[\mathcal{D}^{(-1,3)},\mathcal{D}^{(2,0)}]=\mathcal{D}^{(1,3)}. \qquad (3.5)$$

Indeed, eqs. (3.5) are automatically satisfied as a consequence of (3.1) (see (A.4)). Then eqs. (3.4) mean that

$$D^{(\pm1,3)}A^{(1,1)}_{\alpha}=D^{(2,0)}A^{(1,1)}_{\alpha}=D^{(\pm1,3)}\bar{A}^{(0,2)}_{\dot\alpha}=D^{(2,0)}\bar{A}^{(0,2)}_{\dot\alpha}=0. \qquad (3.6)$$

Here we assume for the moment that $A^{(1,1)}_{\alpha}$, $\bar{A}^{(0,2)}_{\dot\alpha}$ are general harmonic functions, e.g.,

$$A^{(1,1)}_{\alpha}(z,u)=A^{i}_{\alpha}(z)u^{(1,1)}_{i}+A_{\alpha ij}(z)\bar{u}^{(1,-1)i}\bar{u}^{(0,2)j}+$$
$$+A^{ij}_{\alpha k}(z)u^{(1,1)}_{i}u^{(1,1)}_{j}u^{(-1,-1)k}+B_{\alpha}{}^{ik}{}_{j}u^{(1,1)}_{i}\bar{u}^{(1,-1)j}u^{(-1,1)}_{k}+\ldots \qquad (3.7)$$

Using (A.3) it is easy to show that only the first term in (3.7) satisfies (3.6), so the spinor connections depend on u only linearly, as in (3.2). Finally, inserting this into (3.3) one finds, e.g.,

$$u^{(1,1)}_{i}u^{(1,1)}_{j}\{\mathcal{D}^{i}_{\alpha},\mathcal{D}^{j}_{\beta}\}=0$$

which leads to (1.2).

The new equivalent form (3.3)-(3.5) of the N=3 SYM constraints deserves two comments. First, eqs. (3.3) are just the integrability conditions for the existence of analytic SF's (2.12) in the case of local symmetry. This is the first indication for the importance of the notion of analyticity in the N=3 SYM theory (recall its similar role in the cases N=1,2). Second, note that eqs. (3.3)-(3.5) can be viewed as the conditions for flatness of the tangent subspace. Similar interpretations of the N=3 (as well as N=1,2) SYM constraints have been given before in /4-6/ under the names of "lightlike line" or "CR-structure" integrability *). Our crucial new step is to realize that the parameters used to define the subspace ($u^{(a,b)}$ in our case) should be considered as new coordinates of the base manifold (harmonic SS). This will enable us to solve part of the constraints and interpret the rest as equations of motion.

In the τ-representation which we are now considering the constraints (3.3) have the following "pure gauge" type solution:

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*) It is worth to emphasize that the interpretation of (3.3)-(3.5) as the representation preserving conditions seems to us more fruitful as it immediately indicates the fundamental role of analytic N=3 SS in the geometry of N=3 SYM.

$$(\mathcal{D}^{(1,1)}_{\alpha})_{\tau}=e^{-iv}D^{(1,1)}_{\alpha}e^{iv};\ (\bar{\mathcal{D}}^{(0,2)}_{\dot\alpha})_{\tau}=e^{-iv}\bar{D}^{(0,2)}_{\dot\alpha}e^{iv}. \qquad (3.8)$$

Here the prepotential $v=v(z,u)$ is a harmonic real (in the sense (2.10)) SF:

$$v=\overset{*}{\tilde{v}} \Leftrightarrow \bar{A}^{(0,2)}_{\dot\alpha}=\overline{\overset{*}{A^{(1,1)}_{\alpha}}} \Leftrightarrow \bar{A}_{\dot\alpha i}=\overline{A^{i}_{\alpha}} \qquad (3.9)$$

It should be stressed that v is subject to constraints following from (3.6). Before discussing this we shall examine the transformation properties of v:

$$e^{iv'}=e^{i\lambda}e^{iv}e^{-i\tau}. \qquad (3.10)$$

Here the factor $e^{-i\tau}$ produces the desired transformation laws for $(\mathcal{D})_{\tau}$ in (3.8), and λ is the analytic parameter of a new, pregauge group leaving (3.8) invariant,

$$D^{(1,1)}_{\alpha}\lambda=\bar{D}^{(0,2)}_{\dot\alpha}\lambda=0. \qquad (3.11)$$

With the help of e^{iv} one can convert any τ-covariant object into a λ-covariant one,

$$\Phi'=e^{i\tau}\Phi \Rightarrow \varphi=e^{iv}\Phi\,,\ \varphi'=e^{i\lambda}\varphi. \qquad (3.12)$$

In particular (see (3.8)),

$$(\mathcal{D}^{(1,1)}_{\alpha})_{\lambda}=e^{iv}(\mathcal{D}^{(1,1)}_{\alpha})_{\tau}e^{-iv}=D^{(1,1)}_{\alpha}\,,$$
$$(\bar{\mathcal{D}}^{(0,2)}_{\dot\alpha})_{\lambda}=e^{iv}(\bar{\mathcal{D}}^{(0,2)}_{\dot\alpha})_{\tau}e^{-iv}=\bar{D}^{(0,2)}_{\dot\alpha}. \qquad (3.13)$$

So, one sees that in the new λ-representation the derivatives $\mathcal{D}^{(1,1)}_{\alpha}$, $\bar{\mathcal{D}}^{(0,2)}_{\dot\alpha}$ have no connections and therefore the constraints (3.3) are automatically satisfied. However, now the harmonic derivatives acquire connections

$$(\mathcal{D}^{(\pm1,3)})_{\lambda}=D^{(\pm1,3)}+iV^{(\pm1,3)}$$
$$(\mathcal{D}^{(2,0)})_{\lambda}=D^{(2,0)}+iV^{(2,0)} \qquad (3.14)$$

which have the form (see (3.1) and (3.12))

$$V^{(\pm1,3)}=-ie^{iv}(D^{(\pm1,3)}e^{-iv})$$
$$V^{(2,0)}=-ie^{iv}(D^{(2,0)}e^{-iv}) \qquad (3.15)$$

and transform as follows from (3.10):

$$V^{(a,b)\prime} = -i e^{i\lambda}\left(D^{(a,b)} + i V^{(a,b)}\right) e^{-i\lambda}. \qquad (3.16)$$

As a consequence of (3.9) $V^{(a,b)}$ have the reality properties

$$\overline{V^{(1,3)}}^{*} = -V^{(1,3)}, \quad \overline{V^{(-1,3)}}^{*} = V^{(2,0)}, \quad \overline{V^{(2,0)}}^{*} = V^{(-1,3)} \qquad (3.17)$$

Now we turn to the remaining constraints (3.4) and (3.5). Eqs. (3.4) imply that $V^{(a,b)}$ must be <u>analytic</u>

$$D_{\alpha}^{(1,1)} V^{(a,b)} = \bar{D}_{\dot{\alpha}}^{(0,2)} V^{(a,b)} = 0 \qquad (3.18)$$

and eqs. (3.5) yield

$$D^{(1,3)} V^{(-1,3)} - D^{(-1,3)} V^{(1,3)} + i\left[V^{(1,3)}, V^{(-1,3)}\right] = 0 \qquad (3.19a)$$

$$D^{(1,3)} V^{(2,0)} - D^{(2,0)} V^{(1,3)} + i\left[V^{(1,3)}, V^{(2,0)}\right] = 0 \qquad (3.19b)$$

$$D^{(-1,3)} V^{(2,0)} - D^{(2,0)} V^{(-1,3)} + i\left[V^{(-1,3)}, V^{(2,0)}\right] - V^{(1,3)} = 0 \qquad (3.19c)$$

So far we have considered the prepotential v as given and derived the connections $V^{(a,b)}$ from v . Now we shall reverse the argument. Suppose we are given the connections $V^{(a,b)}$ satisfying (3.17), (3.18). One can prove by inspection of the harmonic expansions that eqs. (3.19) are the integrability conditions for representing $V^{(a,b)}$ in the pure gauge form (3.15). So, the analytic λ -group connections $V^{(a,b)}$ are the true potentials of the theory. One can go back to the τ -representations by reintroducing the bridge e^{iv} as a solution of eq. (3.15) provided eqs. (3.19) hold.

It is important to realize that eqs. (3.19) are in fact the equations of motion for the theory, since they are the only remaining constraints after solving (3.3), (3.4) by (3.13) and (3.18). This is a new feature of the N=3 theory. In N=2 SYM /1/ there is only one harmonic derivative D^{++} and connection V^{++}, and the pure gauge form (3.15) is always possible without any integrability conditions (i.e.,equations of motion) like (3.19).

Let us summarize the above discussion. The N=3 SYM constraints can be divided into kinematical (which can be solved) and dynamical (which remain as equations of motion). The picture depends on the frame chosen. In the τ -frame we have:

i) no harmonic connections ((3.1)). This solves automatically the kinematical constraint (3.5) which quarantees the existence of -independent SF's;

ii) the spinor connections $A_{\alpha}^{(1,1)}$, $\bar{A}_{\dot{\alpha}}^{(0,2)}$ are linear in $u^{(a,b)}$ ((3.2)). This solves the kinematical constraint (3.4);

iii) $A_{\alpha}^{(1,1)}$, $\bar{A}_{\dot{\alpha}}^{(0,2)}$ are pure gauge ((3.8)) with a restricted prepotential v. The corresponding constraint (3.3) preserves the notion of covariant analyticity and is dynamical.

In the λ -frame, on the contrary, we have:

i') no spinor connections $A_{\alpha}^{(1,1)}$, $\bar{A}_{\dot{\alpha}}^{(0,2)}$ ((3.13)). Here analyticity becomes manifest and eq. (3.3) is now kinematical;

ii') the harmonic connections are analytic (eq. (3.18)). Again eq. (3.4) is kinematical;

iii') the harmonic connections are pure gauge (eq.(3.15)). The corresponding integrability conditions (3.19) equivalent to the constraints (3.5) contain now the dynamics. The notion of u -independent SF's becomes covariant.

Of course, on-shell both pictures are equivalent because there exists the bridge e^{iv} between the τ- and λ -frames. Off-shell this is not true as will be explained in the next section.

4. <u>The N=3 SYM Action. An Overview of Components</u>

Going off-shell would mean trying to relax the dynamical constraints iii) (τ-frame) or iii') (λ -frame) and finding an action from which they will follow. In the τ -frame this seems hardly possible whereas in the λ -frame it is very easy. Let us assume that we are given three analytic harmonic connections $V^{(\pm1,3)}$, $V^{(2,0)}$ satisfying (3.17). In the analytic SS (2.8) they live as unconstrained SF's. Then the commutators (everything in λ -frame)

$$\begin{aligned} \left[\mathcal{D}^{(1,3)}, \mathcal{D}^{(-1,3)}\right] &= F^{(0,6)} \\ \left[\mathcal{D}^{(1,3)}, \mathcal{D}^{(2,0)}\right] &= G^{(3,3)} \\ \left[\mathcal{D}^{(-1,3)}, \mathcal{D}^{(2,0)}\right] &= \mathcal{D}^{(1,3)} + H^{(1,3)} \end{aligned} \qquad (4.1)$$

define three non-vanishing tensors F, G, H. The equations of motion (3.19) $(F = G = H = 0)$ should be obtained by variation of an action. The latter is given as an integral over the analytic SS (2.8):*)

*) Note that eq. (3.19c) (or $H^{(1,3)} = 0$ in (4.1)) is a conventional constraint. Using it to eliminate $V^{(1,3)}$ one is left with two conjugated (see (3.17)) connections $V^{(2,0)}$, $V^{(-1,3)}$ and the action (4.2) gets its "second order" form.

$$
\begin{aligned}
S = \int d\zeta_A^{(-2,-6)} du \, \mathrm{Tr} \{ & V^{(2,0)} (D^{(1,3)} V^{(-1,3)} - D^{(-1,3)} V^{(1,3)}) - \\
- & V^{(-1,3)} (D^{(1,3)} V^{(2,0)} - D^{(2,0)} V^{(1,3)}) + V^{(1,3)} (D^{(-1,3)} V^{(2,0)} - D^{(2,0)} V^{(-1,3)}) - \\
- & (V^{(1,3)})^2 + 2i \, V^{(1,3)} [V^{(-1,3)}, V^{(2,0)}] \} .
\end{aligned}
\tag{4.2}
$$

Surprisingly, the Lagrange density in (4.2) is not built out of the tensors (4.1) nor any other covariant objects. Under infinitesimal λ-gauge transformations (3.16) the density in (4.2) changes by a harmonic derivatives and one finds after integrating by parts

$$
\delta S = \int d\zeta_A^{(-2,-6)} du \, \mathrm{Tr} \{ \lambda [\mathcal{D}^{(2,0)} F^{(0,6)} - \mathcal{D}^{(-1,3)} G^{(3,3)} + \mathcal{D}^{(1,3)} H^{(1,3)}] \} .
$$

The expression in the brackets vanishes as a consequence of Bianchi identity following from (4.1). Let us emphasize once more that the Lagrange density in (4.2) is invariant up to total harmonic derivatives. Straightforward variation of S (4.2) leads to the equations of motion (3.19). Finally, using (3.17) it is easy to check that S is real with respect to the ($\ast$) conjugation (2.10).

We stress that off-shell only the λ-frame geometry exists. The τ-frame and the original form (1.2) of the constraints are recovered only on-shell where the existence of the λ-τ bridge e^{iv} is guaranteed by the equations of motion (3.19). In fact, now one may forget at all about the τ-representation and deal entirely with the λ-representation and analytic N=3 SS. It is intriguing that the equations of motion in this scheme turn out to be the integbrability conditions on the purely internal even subspace SU(3)/U(1)⊗U(1) of analytic SS. This might suggest a nontrivial extension of the concept of completely integrable systems. Another comment concerns the apparent similarity between the action (4.2) and the so-called Chern--Simon terms in N=0 YM in $d=3$ /10/. The latter have a topological nature and serve to introduce masses in an invariant way. Such terms are also encountered in $d=5$, N=1 YM theory interacting with N=1 supergravity /10/. A common feature of these terms is the absense of quantum corrections /11/. It is likely that an analogous property of the action (4.2) may be responsible for the N=3 SYM (or N=4) finiteness.

Before ending this section we shall give some idea of the component content of the theory. While the harmonic SF formalism is very simple and concise, the component language is rather awkward. The physical fields occur in the harmonic connection $V^{(-1,3)}$:

$$
\begin{aligned}
& V^{(-1,3)}(x_A, \theta_\alpha^{(1,-1)}, \theta_\alpha^{(0,2)}, \bar\theta_{\dot\alpha}^{(1,1)}, \bar\theta_{\dot\alpha}^{(-1,1)}, u) = \\
& = \ldots + i\, \theta_\alpha^{(0,2)} \bar\theta_{\dot\alpha}^{(-1,1)} A^{\alpha\dot\alpha}(x_A) + \bar\theta_{\dot\alpha}^{(-1,1)} \bar\theta^{(-1,1)\dot\alpha} u_i^{(1,1)} \phi^i(x_A) - \\
& - \bar\theta_{\dot\alpha}^{(1,1)} \bar\theta^{(-1,1)\dot\alpha} u_i^{(-1,1)} \phi^i(x_A) + \ldots + \\
& + \bar\theta_{\dot\alpha}^{(-1,1)} \bar\theta^{(1,1)\dot\alpha} \theta^{(0,2)\alpha} \bar u^{(-1,-1)i} \chi_{\alpha i}(x_A) + \\
& + \bar\theta_{\dot\alpha}^{(-1,1)} \bar\theta^{(-1,1)\dot\alpha} \bar\theta_{\dot\beta}^{(1,1)} \bar\psi^{\dot\beta}(x_A) + \ldots
\end{aligned}
\tag{4.3}
$$

The pure gauge and auxiliary fields are omitted in (4.3). The other example deals with the few first dimensionless fields in (2.14), (2.13). The fields V^i_j, $V_{(ijk)}$; $B^{(ik)}_{(jl)}$ and the imaginary part of $A^{(il)}_{(jk)}$ are gauge degrees of freedom. The fields $F^{(ijk)}$ and the real part of $A^{(il)}_{(jk)}$ are auxiliary. Note that the physical fields in (4.3) coincide with those of N=4 SYM. At the same time the absence of auxiliary fields in lower SU(3) representations in (2.14) does not allow to form SU(4) multiplets. Hence, the off-shell formulation of N=3 SYM obtained is not an off-shell formulation of N=4 SYM.

5. Concluding Remarks

So, the "N=3 barrier" has been breached. The unconstrained N=3 SYM theory exists in manifestly supersymmetric and Lorentz invariant form. Moreover, in some respects it is simpler than the N=1 and N=2 theories. This has become possible in the context of harmonic SS with 10-dimensional even part M^4 ⊗ (SU(3)/U(1)⊗U(1)). Recall that the 6-dimensional space M^4 ⊗ (SU(2)/U(1)) proved relevant to the N=2 theory. However, N=2 SYM has also a formulation in M^4 only, in terms of a finite number of fields. From the 4-dimensional point of view N=3 SYM is a highly non-orthodoxial theory, whereas in 10-dimensional harmonic space it has a simple and natural formulation. This might be another argument in favour of the concept of higher dimensions. Currently this is a widely discussed subject in the context of Kaluza--Klein theories /12/. We should note that in the KK approach one starts with a general higher dimensional manifold and later on compactifies the extra dimensions with the help of the equations of motion. As a result an infinite tower of heavy massive excitation arises. In the harmonic approach, from the very beginning the extra dimensions lie in a compact manifold and one gets an infinite tower of auxiliary degrees of freedom. A crucial difference is also in the fact that the extra dimensions in our scheme are not introduced "by hand", the harmonic and analytic SS's naturally appear as certain homogeneous spaces of original extended SUSY in four dimensions.

The next comment concerns N=4 SYM. As explained at the end of sect. 4, we anticipate essential differences between the N=3 and N=4 cases off-shell, although they are the same on-shell. An additional argument is the existence of new type of constraint in the case N=4 /3/

$$\Phi^{ij} = \varepsilon^{ijkl}\bar{\Phi}_{kl}\,, \quad \Phi^{ij} = \{\mathcal{D}^{\alpha[i}, \mathcal{D}^{j]}_{\alpha}\}\,.$$

It is not clear how it fits in the picture described above.

It should be pointed out that the knowledge of the N=3 SYM off-shell theory helps to find out the field content of N=3 off-shell Einstein supergravity. Indeed, following /13/ one can show that coupling N=3 conformal supergravity/14/ to three off-shell N=3 Maxwell multiplets produces off-shell N=3 Einstein supergravity. To this end one has to decompose all SU(3) representations into SO(3) ones. Three N=3 Maxwell multiplets contain the following physical fields

$$\dim 1 \qquad A^{i}_{\mu}(\underline{3}),\ \phi^{ij} - \text{complex}\ (\underline{5}+\underline{3}+\underline{1})^2$$

$$\dim 3/2 \qquad \psi^{i}_{\alpha}(\underline{3}),\ \chi^{ij}_{\alpha}\ (\underline{5}+\underline{3}+\underline{1})\,.$$

Here i, j are indices of the adjoint representation of SO(3). The vectors become physical fields of Einstein supergravity (the only ones not already contained in the Weyl multiplet); the scalars $(\underline{1})^2$, $\underline{3}$, $\underline{5}$ and the spinors $\underline{3}$ can be used to compensate the Weyl, U(1), SU(3) and conformal SUSY gauge transformations. The remaining physical fields have the right dimensions to be Lagrange multipliers for the higher dimension auxiliary fields of the Weyl multiplet. The auxiliary fields of the Maxwell multiplets serve as auxiliary fields for Einstein supergravity.

Our last remark is that the new formulation of N=3 SYM may provide a better framework for an easier and more convincing proof of the famous N=4 (or N=3) finiteness. Recently new doubts have been cast on the legitimacy of the renormalization procedure in supersymmetric gauge theories/15/. Direct quantum computations in terms of N=3 harmonic SF's may not require any regularization to show the absence of divergencies.

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Appendix

N=3 SUSY transformations in central basis

$$\delta x^{\alpha\dot\alpha} = 2i(\theta^{\alpha}{}_{i}\bar{\varepsilon}^{\dot\alpha i} - \varepsilon^{\alpha}{}_{i}\bar{\theta}^{\dot\alpha i})$$
$$\delta\theta_{\alpha i} = \varepsilon_{\alpha i}\,,\ \delta\bar{\theta}^{i}_{\dot\alpha} = \bar{\varepsilon}^{i}_{\dot\alpha}\,,\quad \delta u = 0. \qquad \text{(A.1)}$$

Some properties of SU(/)/U(1)⊗U(1) harmonics

$$\overline{u^{(a,b)}_{i}} = u^{(-a,-b)i}\,, \quad (a,b) = (1,1), (-1,1), (0,-2)$$
$$u^{(a,b)}_{i}u^{(c,d)i} = \delta_{a,-c}\,\delta_{b,-d}$$
$$u^{(1,1)}_{i}u^{(-1,-1)j} + u^{(-1,1)}_{i}u^{(1,-1)j} + u^{(0,-2)}_{i}u^{(0,2)j} = \delta^{j}_{i} \qquad \text{(A.2)}$$
$$\varepsilon^{ijk}u^{(1,1)}_{i}u^{(-1,1)}_{j}u^{(0,-2)}_{k} = 1$$
$$\varepsilon_{ijk}u^{(-1,-1)i}u^{(1,-1)j}u^{(0,2)k} = 1\,,\quad \varepsilon_{ijk} = \varepsilon^{ijk}$$

Table. Harmonic differentiation rules

	$u^{(1,1)}_{i}$	$u^{(-1,1)}_{i}$	$u^{(0,-2)}_{i}$	$u^{(-1,-1)i}$	$u^{(1,-1)i}$	$u^{(0,2)i}$
$D^{(1,3)}$	0	0	$-u^{(1,1)}_{i}$	$u^{(0,2)i}$	0	0
$D^{(-1,3)}$	0	0	$u^{(-1,1)}_{i}$	0	$-u^{(0,2)i}$	0
$D^{(2,0)}$	0	$u^{(1,1)}_{i}$	0	$-u^{(1,-1)i}$	0	0

(A.3)

Algebra of flat harmonic derivatives:

$$[D^{(1,3)}, D^{(-1,3)}] = [D^{(1,3)}, D^{(2,0)}] = 0\,,$$
$$[D^{(-1,3)}, D^{(2,0)}] = D^{(1,3)}. \qquad \text{(A.4)}$$

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Расширенная $N=3$ суперсимметричная теория Янга-Миллса

С помощью гармонического суперпространства сформулирована $N=3$ суперсимметричная теория Янга-Миллса вне массовой оболочки и без сторонних связей. Теория определяется в аналитическом $N=3$ суперпространстве с четной частью $M^4 \times \frac{SU(3)}{U(1) \circ U(1)}$. Основными объектами служат аналитические потенциалы - связности для гармонических производных. Действие является интегралом по аналитическому суперпространству. Лагранжева плотность удивительно проста и калибровочно инвариантна с точностью до гармонической производной. Уравнения движения имеют вид условий интегрируемости на внутреннем пространстве $SU(3)/U(1)\times U(1)$. Бесконечный набор вспомогательных полей позволяет обойти "$N=3$ барьер".

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Unconstrained Off-Shell $N=3$ Supersymmetric Yang-Mills Theory

The harmonic superspace is used to build up an unconstrained off-shell formulation of $N=3$ supersymmetric Yang-Mills theory. The theory is defined in an analytic $N=3$ superspace having $M^4 \times \frac{SU(3)}{U(1)\times U(1)}$ as an even part. The basic objects are the analytic potentials which serve as gauge connections entering harmonic derivatives. The action is an integral over analytic superspace. The Lagrange density is surprisingly simple and it is gauge invariant up to total harmonic derivative. The equations of motion are integrability conditions on the internal space $SU(3)/U(1)\times U(1)$. The jumping over the "$N=3$ barrier" became possible due to the infinite number of auxiliary fields.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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