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ANALYTICAL METHODS FOR MULTILOOP CALCULATIONS

(Two lectures on the method of uniqueness)

1984

CONTENTS

Introduction

In the present lectures we give a description of the method for analytical calculation of multiloop Feynman diagrams. It is called the "method of uniqueness" and is aimed at exact calculation of massless Feynman integrals depending on a single dimensional variable (momentum or coordinate). Integrals of the type, on the one hand, can be evaluated exactly and on the other hand many problems like determination of anomalous dimensions of the operators and of renormalization group functions, calculation of total cross section of e^+e^--annihilation and of structure functions of deep-inelastic scattering can be reduced to the calculation of the very integrals.

Remarkable property of the described method is that it involves neither integration of elementary or special functions, nor expansion in and summation of infinite series of any kind. Rather it is based on a number of simple and obvious formulas and is reduced to the manipulation with the diagrams. Below we give the examples of its efficiency.

Lecture I. Basic Notions of the Method of Uniqueness

1.1. Notation

All the calculations will be performed in the coordinate space of dimension D. The dependence of the integral on external argument is determined on pure dimensional grounds and is power-like. The aim of the calculation is the coefficient function $F(D)$. For $D=4-2\varepsilon$ it is the Laurent series in ε, and of interest, as a rule, are the coefficients of negative powers of ε.

The integration is held over all internal vertices of a diagram. The lines of the graphs are associated with simple powers like $1/(x-y)^{2\alpha}$, α being called the index of a line and depicted above the line:

$$x \xrightarrow{\ \alpha\ } y \quad \Rightarrow \quad \frac{1}{[(x-y)^2]^{\alpha}} .$$

The index of ordinary line is $\frac{D-2}{2}$ due to the well-known Fourier-transform

$$\int \frac{d^D p \, e^{ipx}}{p^2} = \frac{\pi^{D/2}\, 2^{D-2}}{[x^2]^{\frac{D-2}{2}}} \cdot \frac{\Gamma(\frac{D}{2}-1)}{\Gamma(1)}, \qquad (1.1)$$

where Γ is Euler Γ-function. For arbitrary index ν eq. (1.1) reads:

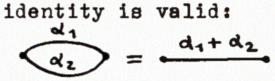
$$\int \frac{d^D p \, e^{ipx}}{[p^2]^{\nu}} = \frac{\pi^{D/2}\, 2^{D-2\nu}}{[x^2]^{D/2-\nu}}\, a(\nu); \quad a(\nu) \equiv \frac{\Gamma(D/2-\nu)}{\Gamma(\nu)}. \qquad (1.2)$$

Here $\nu \neq D/2, D/2+1, \ldots$ In x-space zero index means the absence of the line.

We will also need the concept of the index of a vertex, of a triangle and of a diagram - the sum of indices of constituent lines. The line, vertex, and triangle will be called "unique" if their indices are equal to 0, D, and $D/2$, respectively.

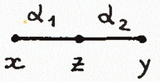
1.2. Integration of chains and simple loops

Simple loops in x-space contrary to the p-space do not contain any integration being an ordinary product of propagators. The following graphical identity is valid:



(1.3)

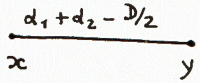
As for the chains, they should be integrated. To do this we note that they are ordinary products in p-space. Performing transformation to p-space for every line and multiplying them there, we go back to x-space. Namely:



$$\equiv \int \frac{d^D z}{[(x-z)^2]^{\alpha_1}[(z-y)^2]^{\alpha_2}} = \int \frac{d^D z\, d^D p\, d^D q \; a(\alpha_1)\, a(\alpha_2)}{[p^2]^{D/2-\alpha_1}\,[q^2]^{D/2-\alpha_2}} \cdot$$

$$\cdot \frac{\pi^{-D}}{2^{2\alpha_1+2\alpha_2}}\, e^{ip(x-z)+iq(z-y)} = \int \frac{d^D p\; e^{ip(x-y)}}{[p^2]^{D-\alpha_1-\alpha_2}} \frac{a(\alpha_1)\, a(\alpha_2)}{2^{2\alpha_1+2\alpha_2-D}} =$$

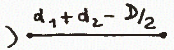
$$= \frac{\pi^{D/2} a(\alpha_1)\, a(\alpha_2)\, a(D-\alpha_1-\alpha_2)}{[(x-y)^2]^{\alpha_1+\alpha_2-D/2}} \equiv$$



$$\pi^{D/2} a(\alpha_1)\, a(\alpha_2) \cdot a(D-\alpha_1-\alpha_2),$$

where we have used eq. (1.2). Hence, we have the following graphical equality:

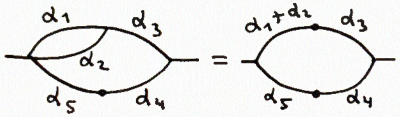
$= v(\alpha_1,\alpha_2,\alpha_3)$



, (1.4)

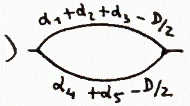
$$v(\alpha_1,\alpha_2,\alpha_3) = \pi^{D/2} \prod_{i=1}^{3} a(\alpha_i), \quad \alpha_3 = D-\alpha_1-\alpha_2 .$$

Any diagram consisting of a sequence of chains and simple loops can be calculated due to eqs. (1.3), (1.4). For example,

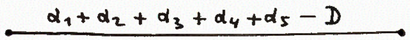


$$= v(\alpha_1+\alpha_2, \alpha_3, D-\alpha_1-\alpha_2-\alpha_3)\cdot v(\alpha_4,\alpha_5,$$

$D-\alpha_4-\alpha_5)$

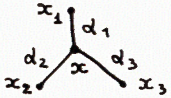


$$= v(\alpha_1+\alpha_2,\alpha_3, D-\alpha_1-\alpha_2-\alpha_3)\, v(\alpha_4,\alpha_5, D-\alpha_4-\alpha_5)\,\cdot$$



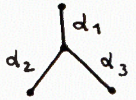
1.3. Uniqueness relation

Consider a triple vertex with arbitrary indices. It corresponds to the integral

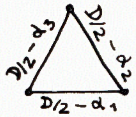


$$\equiv \int \frac{d^D x}{[(x-x_1)^2]^{\alpha_1}[(x-x_2)^2]^{\alpha_2}[(x-x_3)^2]^{\alpha_3}} . \qquad (1.5)$$

Calculation is straightforward if some index $\alpha_i = 0$, i.e., when one of the lines is unique. However it is not the only case. The integral is exactly calculated as well if the vertex is unique. In this case the following identity holds:



$$\overset{\sum \alpha_i = D}{=\!=\!=} \; v(\alpha_1,\alpha_2,\alpha_3)\cdot$$



, (1.6)

which is called the uniqueness relation. In the r.h.s. of (1.6) there is a simple product of propagators, connecting the points x_1, x_2 and x_3 with the indices being the Fourier-transform of initial ones. The triangle obtained is also unique. So, the uniqueness relation connects a unique vertex with a unique triangle. It is also called the "star-triangle" relation.

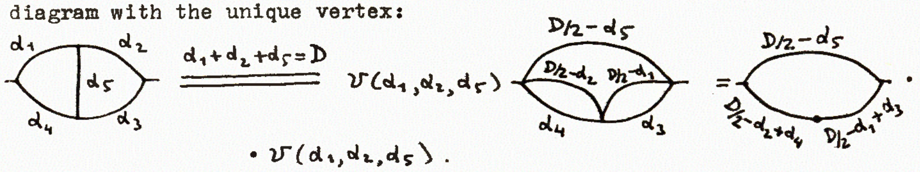
Uniqueness relation can be obtained in the following way: Shifting the integration variable in (1.5) $x_\mu \to x'_\mu + x'_{1\mu}$, we perform the inversion $x'_\mu \to x''_\mu/(x'')^2$. Now, the integral becomes

$$\int \frac{d^D x \; [x^2]^{\alpha_1+\alpha_2+\alpha_3}}{[x^2]^D \left[x + \frac{x_1-x_2}{(x_1-x_2)^2}\right]^{\alpha_2} \left[x + \frac{x_1-x_3}{(x_1-x_3)^2}\right]^{\alpha_3} [(x_1-x_2)^2]^{\alpha_2}\, [(x_1-x_3)^2]^{\alpha_3}} .$$

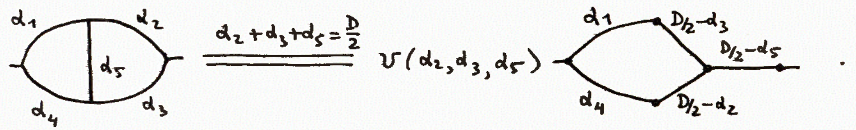
If $\sum \alpha_i = D$ the integral is simplified and according to eq. (1.4) is equal to

$$v(\alpha_1,\alpha_2,\alpha_3)\cdot\frac{1}{[(x_1-x_2)^2]^{D/2-\alpha_3}\,[(x_1-x_3)^2]^{D/2-\alpha_2}\,[(x_2-x_3)^2]^{D/2-\alpha_1}}\,.$$

In the case when the diagram contains a unique vertex or unique triangle it can be immediately reduced. Consider, for example, the diagram with the unique vertex:



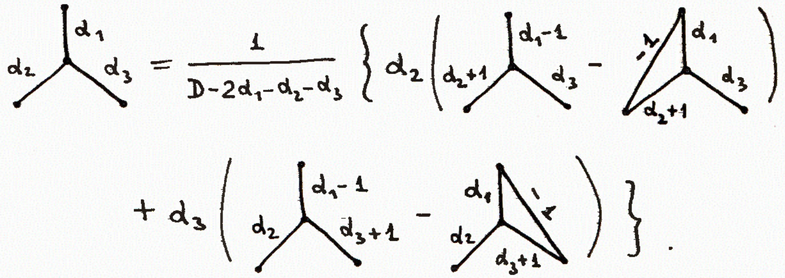
For a unique triangle we have



The remained diagrams are easily integrated due to eqs. (1.3), (1.4).

1.4. Integration by parts

For a vertex which is not unique of great use is the following equality obtained by integration by parts

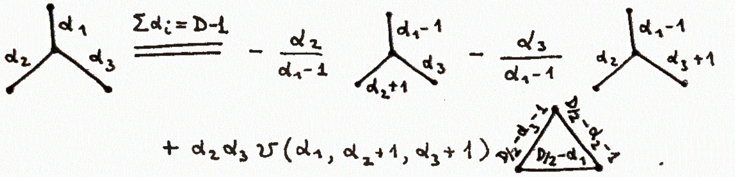


(1.7)

This equality is valid for arbitrary indices and enables us to change them by unity. Eq. (1.7) by itself does not provide us with the calculation of the integral. However, the resulting diagrams may contain unique elements, that is what we need. This happens when the diagram contains the elements with the indices deviating by unity from uniqueness. We call it one-step deviation from uniqueness.

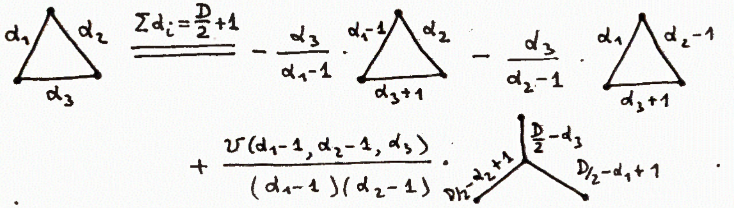
1.5. One-step from uniqueness

The line, vertex, and triangle are called one-step deviating from uniqueness if their indices are equal to 1, $D-1$, and $D/2+1$, respectively. For a vertex one-step deviating from uniqueness eq. (1.7) can be rewritten as



(1.8)

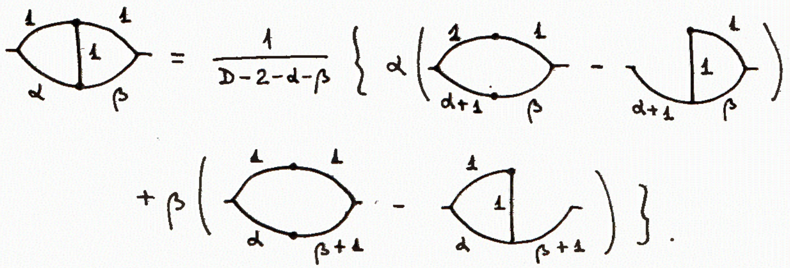
For the triangle one-step deviating from uniqueness an analogous equality holds



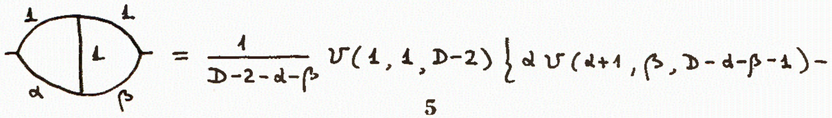
(1.9)

Eqs. (1.7-1.9) enable us to calculate the diagrams if their elements one-step deviate from uniqueness. As an example, consider the diagram with three lines one-step deviating from uniqueness

Application of eq. (1.7) to the lower vertex gives

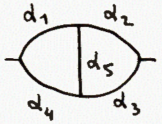


Resulting diagrams are reduced to a sequence of chains and simple loops and are easily calculated. The result is



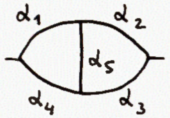
$$-\alpha\, v(\alpha+1, \beta+2-D/2, \tfrac{3D}{2}-\alpha-\beta-3) + \beta\, v(\alpha, \beta+1, D-\alpha-\beta-1) - \beta\, v(\alpha+2-D/2, \beta+1, \tfrac{3D}{2}-\alpha-\beta-3)\} . \quad (1.10)$$

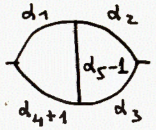
Consider now a diagram with two triangles and one vertex one-step deviating from uniqueness:

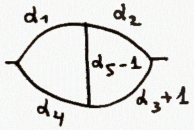


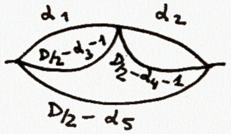
$$\alpha_3+\alpha_4+\alpha_5 \equiv S_2 = D-1 ;$$
$$\alpha_1+\alpha_4+\alpha_5 \equiv t_1 = D/2+1 ;$$
$$\alpha_2+\alpha_3+\alpha_5 \equiv t_2 = D/2+1 .$$

Applying eq. (1.8) to the lower vertex we have



$= -\frac{\alpha_4}{\alpha_5-1}$


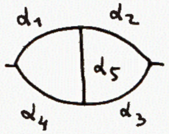
$-\frac{\alpha_3}{\alpha_5-1}$


$+\alpha_3\alpha_4\, v(\alpha_5, \alpha_3+1, \alpha_4+1)$


.

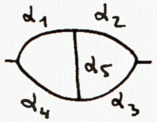
The first diagram is now easily evaluated due to uniqueness of right triangle, the second one due to uniqueness of left triangle, and the third one is reduced to the chains and simple loops.

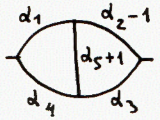
The example of application of eq. (1.9) comes from the diagram with two vertices and one triangle one-step deviating from uniqueness:

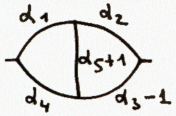


$$\alpha_1+\alpha_2+\alpha_5 \equiv S_1 = D-1 ;$$
$$\alpha_3+\alpha_4+\alpha_5 \equiv S_2 = D-1 ;$$
$$\alpha_2+\alpha_3+\alpha_5 \equiv t_2 = \tfrac{D}{2}+1 .$$

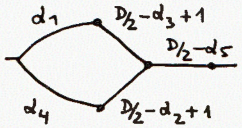
Using eq. (1.9) for the right triangle, we get



$= -\frac{\alpha_5}{\alpha_2-1}$


$-\frac{\alpha_5}{\alpha_3-1}$


$+$

$+\frac{v(\alpha_2-1, \alpha_3-1, \alpha_5)}{(\alpha_2-1)(\alpha_3-1)}$


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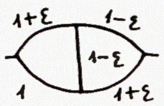
The resulting diagrams are again easily evaluated due to arisen uniquenesses.

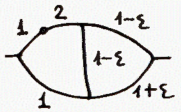
Thus, in the presence of one uniqueness the diagram is immediately reduced. In the presence of simultaneously three one-step deviations from uniqueness the diagram is reduced after integration by parts.

1.6. Transformation of indices

In a real calculation, when using dimensional regularization, space-time dimension is $D=4-2\varepsilon$. The indices of ordinary line, triple vertex and triangle are then $1-\varepsilon$ and $3-3\varepsilon$, that differs by an order of ε from the values corresponding to one-step deviation from uniqueness, i.e., 1, $3-2\varepsilon$, $3-\varepsilon$, respectively. As we have already seen, one can change the index by unity using integration by parts. The change of the index by an order of ε can be achieved with the help of transformations forming a group including the following elements:

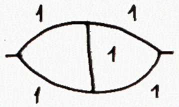
a) Insertion of a point into a line. Inserting a point into some line of a diagram and using the uniqueness relation, we come to a diagram with changed values of indices. For example,



$= \frac{1}{v(1, 2, 1-2\varepsilon)}$


$=$

We fit the index of a new line (equal to 2) in such a way as to create the unique vertex. Now, we have

$= \frac{v(1-\varepsilon, 1-\varepsilon, 2)}{v(1, 2, 1-2\varepsilon)}$


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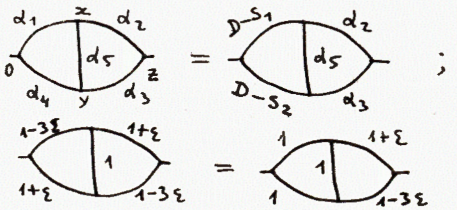
The resulting diagram now has three lines one-step deviating from uniqueness and is easily evaluated.

b) Conformal transformation of inversion. Fixing external arguments we perform the inversion $x_\mu \to x_\mu/x^2$ of all integration variables and of external coordinate. The propagators remain unchanged

$$[(x-y)^2]^\alpha \to \left[\left(\frac{x}{x^2}-\frac{y}{y^2}\right)^2\right]^\alpha = \left[\frac{x^2}{x^4}-\frac{2xy}{x^2y^2}+\frac{y^2}{y^4}\right]^\alpha =$$
$$= \frac{[(x-y)^2]^\alpha}{[x^2]^\alpha\,[y^2]^\alpha}$$

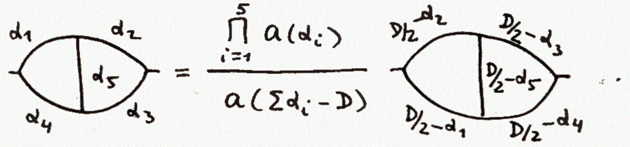
and the measure is transformed as $d^Dx \to \frac{d^Dx}{[x^2]^D}$. Finally, we come to a diagram with changed indices. For example,



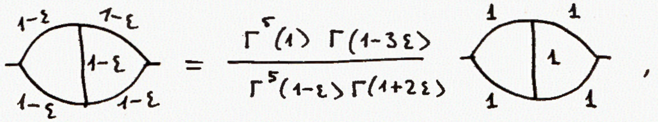
; $D = 4-2\varepsilon$.

The equality is always assumed in a sense of coefficient functions.

c) Transition to a dual diagram. Performing the Fourier-transform of coordinate diagram, one can treat the obtained momentum diagram as a coordinate one with changed indices. This trick always works for planar diagrams because in this case the propagators in p -space are the inverse squares of the difference of two momenta. The obtained diagram is called a dual one (in a sense of Fourier-transform). Arising multiplier is equal to the product of multipliers associated with the Fourier-transform of the lines divided by inverse transformation of the diagram itself. For example,



This transformation is extremely useful for the diagram with ordinary lines. Indeed

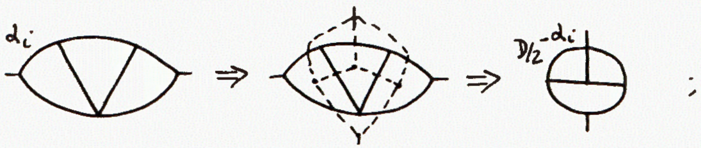


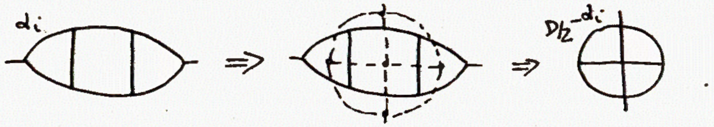
i.e., we create the needed one-step deviations from uniqueness at once. This expression is valid in all orders in ε .

For arbitrary planar diagram the dual one is constructed according to the following rule:

Put a point inside every loop of the diagram, and two points outside it. Connect all the points by the lines so that every line of the initial diagram be crossed only once. New lines produce the dual diagram with the indices dual to that of the crossed old lines.

We demonstrate this rule by examples:





The transformation of indices listed above enables us to change their values creating the needed uniquenesses or one-step deviations from uniqueness. After that the diagrams are evaluated due to eqs. (1.3), (1.4), (1.6-1.9).

1.7. Examples of two- and three-loop calculations

Thus, the procedure of calculation by the method of uniqueness is the following:

i) To be sure whether there are any uniquenesses. If so, the diagram is immediately simplified.

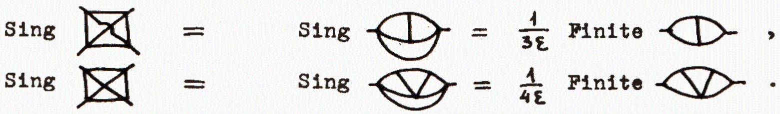
ii) If not, to be sure whether there are any one-step deviations from uniqueness. If they are, the diagram is simplified after integration by parts.

iii) If there is neither uniqueness nor one-step deviations from uniqueness, one should try to create them artificially using the transformation of indices.

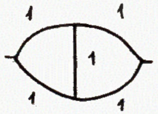
iv) If one has not succeeded to do this, then extremely useful is the implementation of the tables. We will describe this in the next lecture.

To give some examples of concrete calculations, we consider the φ^4 theory. To find the β -function in three- and four-loop approximation, one has to calculate, in particular, the following vertex diagrams:

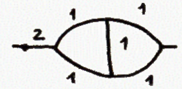
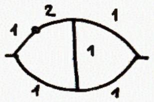
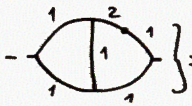
We are interested in singular (in ε) contributions. Symbolically

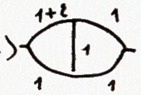
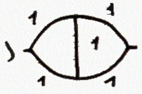


Hence, we are interested in finite parts (in ε) of two- and three-loop integrals. This means that we can take the indices of the lines to be $1+\alpha_i\varepsilon$, choosing α_i in order to create the needed one-step deviations from uniqueness. Clearly, for our purposes in a given case it is usefull to have all the indices equal to 1. Then, we have

 $= \left(\text{eq. (1.10)},\ \alpha=\beta=1\right) = 6\zeta(3)+O(\varepsilon)$;

 = (eq. (1.7) for the left upper vertex) =

$= -\frac{1}{2\varepsilon}\Big\{$  $-$  $+$  $-$  $\Big\} =$

$= -\frac{1}{2\varepsilon}\Big\{ v(1,2,1-2\varepsilon)$  $- v(1+2\varepsilon,2,1-4\varepsilon)$  $\Big\} =$

$$= (\text{eq. (1.10)}) = 20\zeta(5)+O(\varepsilon).$$

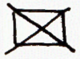
We have used here the well-known expansion of Γ-function:

$$\Gamma(1+x) = \exp\Big\{-\gamma x + \sum_{n=2}^{\infty}\frac{(-)^n}{n}\zeta(n)x^n\Big\}, \tag{1.11}$$

where γ is the Euler constant and $\zeta(n)=\sum\limits_{k=1}^{\infty}\frac{1}{k^n}$ is the Riemann ζ-function.

So,

KR'  $= \frac{1}{3\varepsilon}\,6\zeta(3)$,

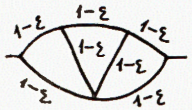
KR'  $= \frac{1}{4\varepsilon}\,20\zeta(5)$.

L e c t u r e II

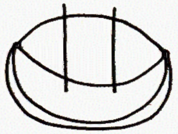
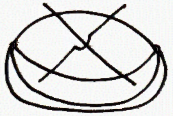
CALCULATION OF COMPLICATED DIAGRAMS

2.1. Three-, four-, and five-loop integrals

Consider now more complicated integrals than in the first lecture. For example, the same V-like diagram with ordinary lines

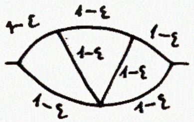
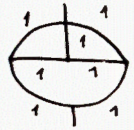
 .

This diagram arises from the above considered four-loop diagram when we are interested not only in the singular part of it but in the finite part as well. The following five-loop diagrams can be also reduced to it:

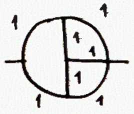
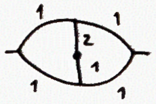
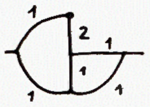
 ,  .

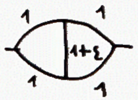
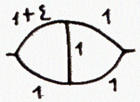
In both cases one has to know the V-like diagram with $O(\varepsilon)$ precision. Hence, one cannot choose the indices of the lines but has to perform the transformation.

In a given case it is useful first to go to the dual diagram

 $= \dfrac{\Gamma^3(1)\Gamma(1-4\varepsilon)}{\Gamma^3(1-\varepsilon)\Gamma(1+3\varepsilon)}$  .

Now, applying eq. (1.7) to the central vertex, we get

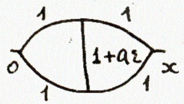
 $= -\frac{1}{2\varepsilon}\,2\Big\{$  $-$  $\Big\} =$

$= -\frac{1}{\varepsilon}\,v(1,2,1-2\varepsilon)\Big\{$  $-$  $\Big\}$.

The second diagram is easily calculated due to eq. (1.10). As for the first one, it is a problem. This is because it has no uniquenesses or one-step deviations from uniqueness and it is impossible to create them by index transformations. Thus, the described methods do not give us exact calculation of V-like diagram. However, we are not interested in exact calculation, rather in the expression with $O(\varepsilon)$ accuracy, i.e., one has to know the two-loop diagram up to ε^3. (The reduction of every loop is equivalent to two orders in ε). To calculate the obtained diagram up to $O(\varepsilon^3)$ we use the method of the tables.

2.2. Tables for multiloop calculations

To illustrate the idea of table implementation to multiloop calculations, consider the diagram



$$\equiv \frac{F_\varepsilon(1+a\varepsilon)}{[x^2]^{1+2\varepsilon+a\varepsilon}} \qquad (2.1)$$

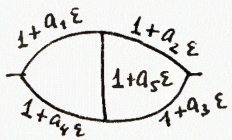
This diagram is easily calculated exactly (i.e., in all orders in ε) for $a = 0, -1, -2, -3$. These correspond to the set of three one-step deviations from uniqueness. Consider the Teylor expansion of $F_\varepsilon(1+a\varepsilon)$. It is not an expansion over a because ε is contained in the measure of integration as well. However, the coefficients will be polynomials in a of the power less or equal to that of ε . We have

$$F_\varepsilon(1+a\varepsilon) = c_0 + (c_1 a + c_2)\varepsilon + (c_3 a^2 + c_4 a + c_5)\varepsilon^2 + \\ + (c_6 a^3 + c_7 a^2 + c_8 a + c_9)\varepsilon^3 + O(\varepsilon^4). \qquad (2.2)$$

With the value of $F_\varepsilon(1+a\varepsilon)$ for four values of a , we find out four terms of ε -expansion, i.e., the coefficients $c_0 \div c_9$. The result is

$$F_\varepsilon(1+a\varepsilon) = \frac{1}{1-2\varepsilon}\Big\{ 6\zeta(3) + 9\zeta(4)\varepsilon + [21(a+1)(a+2) - 6a(a+3)]\zeta(5)\varepsilon^2 \\ + [45(a+1)(a+2) - \frac{15}{2}a(a+3)]\zeta(6)\varepsilon^3 - [23(a+1)(a+2) - 8a(a+3)]\cdot \\ \cdot \zeta^2(3)\varepsilon^3 + O(\varepsilon^4)\Big\}. \qquad (2.3)$$

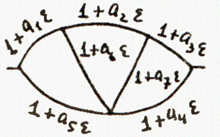
Thus, without knowing how to calculate the function $F_\varepsilon(1+a\varepsilon)$ for $a=1$, we can find its value up to ε^3 using eq. (2.3) as a table. In an analogous way one can obtain more general table for the diagram



(2.4)

up to ε^3 for arbitrary $a_1 \div a_5$. We will write it down below.

With the help of this table one can also construct the table for more complicated three-loop diagrams with $O(\varepsilon)$ accuracy. They are



$$= \frac{\exp[-3(\gamma\varepsilon + \frac{\zeta(2)}{2}\varepsilon^2)]}{1-2\varepsilon}\Big\{ 20\zeta(5) + \\ + \varepsilon[50\zeta(6) + (20 + 6(a_4+a_5+a_6+a_7))\zeta^2(3)] + O(\varepsilon^2)\Big\} \qquad (2.5)$$

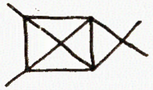
and

$$= \frac{\exp[-3(\gamma\varepsilon + \frac{\zeta(2)}{2}\varepsilon^2)]}{1-2\varepsilon}\Big\{ 20\zeta(5) + \\ + \varepsilon[50\zeta(6) - (4 + 6(a_3+a_4+a_5+a_7))\zeta^2(3)] + O(\varepsilon^2)\Big\}. \qquad (2.6)$$

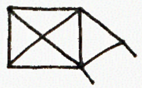
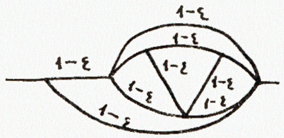
This accuracy is quite enough for calculations up to 4-5 loops in scalar theories. Note, that the exponents in front of the braces in (2.5), (2.6) do not give any contribution in what follows and can be omitted.

2.3. Examples of table application

As an example of application of the obtained tables we consider the following diagram



We are interested in the contribution to the renormalization constant, i.e., in pole terms proportional to $1/\varepsilon^2$ and $1/\varepsilon$. Due to the independence of KR' for the diagram in MS scheme of external lines, it is equivalent to the calculation of polar terms of the following diagram

 = 

Hence, to find polar terms one has to know the V -like diagram with $O(\varepsilon)$ accuracy. This is exactly what the table (2.5) does. After that we have a sequence of chains and simple loops, that gives

$$(C_0 + C_1\varepsilon)\frac{\Gamma(4\varepsilon)\,\Gamma(1)\,\Gamma(1-5\varepsilon)\,\Gamma(5\varepsilon)}{\Gamma(2-5\varepsilon)\,\Gamma(1-\varepsilon)\,\Gamma(1+4\varepsilon)\,\Gamma(2-6\varepsilon)},$$

where C_0 and C_1 are given by eq. (2.5) and are equal to

$$C_0 + C_1\varepsilon = \frac{1}{1-2\varepsilon}\left[20\zeta(5) + \varepsilon(50\zeta(6) - 4\zeta^2(3))\right].$$

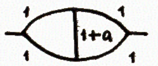
Substracting the divergent subgraph, we finally find for the KR'

$$KR'$$  $$= -\frac{4\zeta(5)}{\varepsilon^2} + \frac{3\zeta(5) + \frac{5}{2}\zeta(6) - \frac{1}{5}\zeta^2(3)}{\varepsilon}. \qquad (2.7)$$

2.4. Functional equations

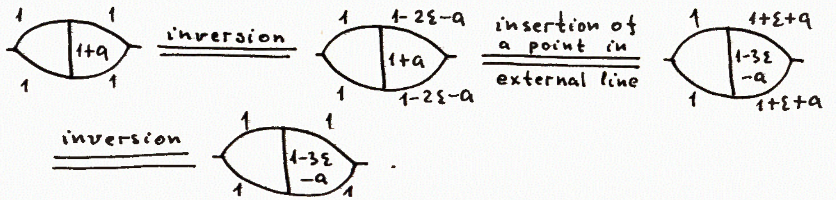
Up to now all the calculations have been performed due to the fact that after some transformations including integration by parts, uniqueness relation, transformation of indices, the diagrams in the r.h.s. have been reduced to the sequence of chains and simple loops. However, sometimes it cannot be done, especially when the number of loops is large enough ($\geqslant 5$). Otherwise, the obtained equality can be treated as a functional equation for the coefficient function. Solving this equation, we can find the function of interest.

We demonstrate this possibility by an example of already discussed two-loop diagram with an arbitrary central line



$\equiv F_\varepsilon(1+a)$.

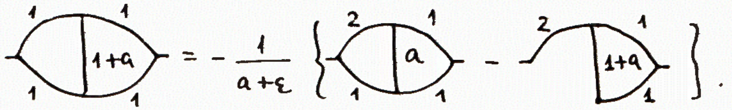
Perform with the diagram the following transformation



Thus, we obtain the first equation for $F_\varepsilon(1+a)$:

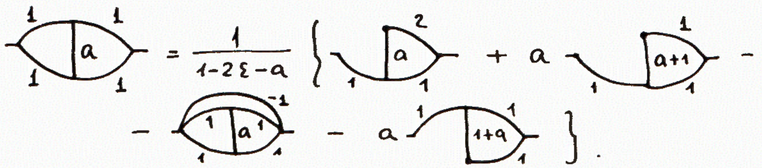
$$F_\varepsilon(1+a) = F_\varepsilon(1-3\varepsilon-a). \tag{2.8}$$

To get another equation we apply integration by parts (1.7) to the upper vertex. This gives



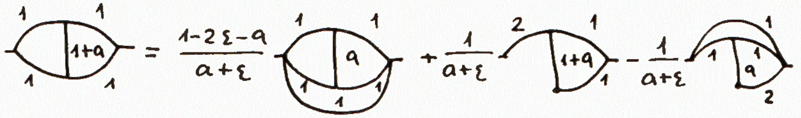
(2.9)

Using the same eq. (1.7) but with another isolated line, we obtain



(2.10)

Combining eqs. (2.9) and (2.10), we come to the equation



(2.11)

or analytically

$$F_\varepsilon(1+a) = \frac{1-2\varepsilon-a}{a+\varepsilon} F_\varepsilon(a) + \frac{2(2a-1+3\varepsilon)\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)\Gamma^2(1-\varepsilon)}{(a+\varepsilon)\Gamma(a+1)\Gamma(2-3\varepsilon-a)\Gamma^2(1)}, \tag{2.12}$$

where we have used eqs. (1.3) and (1.4). Eqs. (2.8) and (2.12) are the desired functional equations for $F_\varepsilon(1+a)$.

2.5. Solution of functional equations

To simplify the inhomogeneous part of eq. (2.12), we make the substitution

$$F_\varepsilon(1+a) = 2\,\frac{\Gamma^2(1-\varepsilon)\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)\; G_\varepsilon(1+a)}{\Gamma^2(1)\,\Gamma(1+a)\,\Gamma(1-a-3\varepsilon)}, \tag{2.13}$$

where the function $G_\varepsilon(1+a)$ obeys the equations

$$G_\varepsilon(1+a) = G_\varepsilon(1-a-3\varepsilon), \tag{2.14}$$

$$G_\varepsilon(1+a) = -\frac{a}{a-1+3\varepsilon} G_\varepsilon(a) + \frac{1}{a-1+3\varepsilon}\left(\frac{1}{a+\varepsilon} + \frac{1}{a-1+2\varepsilon}\right).$$

To find the solution, consider the analytical properties of G_ε. It is known, e.g., on the basis of α-representation, that the function $F_\varepsilon(1+a)$ is a meromorphic function, regular at $a=0$ with simple poles at $a=\pm n-2\varepsilon$ and $a=\pm n-\varepsilon$, where $n=1,2,\ldots$ The same conclusion follows from the inhomogeneous term in eq. (2.12). The function G_ε obtains additional poles due to the Γ-functions in denominator of eq. (2.13). That is why we look for the solution of eqs. (2.14) in the form of infinite series of poles

$$G_\varepsilon(1+a) = \sum_{n=1}^{\infty} f_n\left(\frac{1}{n+a+\varepsilon} + \frac{1}{n-a-2\varepsilon}\right) + \sum_{n=1}^{\infty} \phi_n\left(\frac{1}{n+a} + \frac{1}{n-a-3\varepsilon}\right), \tag{2.15}$$

where we have automatically satisfied eq. (2.8). Substituting now eq. (2.15) into eq. (2.14) and equalizing residues at the poles, we get the equations for f_n and ϕ_n:

$$f_n = -f_{n+1}\frac{n+\varepsilon}{n+1-2\varepsilon}, \quad \phi_n = -\phi_{n+1}\frac{n}{n+1-3\varepsilon}.$$

Their solution is

$$f_n = (-)^n\frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)} C_1(\varepsilon), \quad \phi_n = (-)^n\frac{\Gamma(n+1-3\varepsilon)}{\Gamma(n)} C_2(\varepsilon). \tag{2.16}$$

The inhomogeneous part of eq. (2.14) fixes C_1:

$$C_1(\varepsilon) = \frac{\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)} .$$

To find C_2 we compare the obtained solution with the known one for particular value of a. As has already been mentioned, the function $F_\varepsilon(1+a)$ is known exactly for $a=0$. Comparison of eqs. (2.13), (2.15) and (2.16) with $F_\varepsilon(1)$ gives

$$C_2(\varepsilon) = -\frac{\Gamma(\varepsilon)\,\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)} .$$

As a result we have

$$F_\varepsilon(1+a) = 2\,\frac{\Gamma^2(1-\varepsilon)\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)\Gamma(\varepsilon)}{\Gamma^2(1)\Gamma(1+a)\Gamma(1-a-3\varepsilon)\Gamma(2-2\varepsilon)} \Big\{ \sum_{n=1}^{\infty} (-)^n \frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)} \cdot$$
$$\left(\frac{1}{n+a+\varepsilon} + \frac{1}{n-a-2\varepsilon}\right) - \frac{\Gamma(1-\varepsilon)\,\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)} \sum_{n=1}^{\infty} (-)^n \frac{\Gamma(n+1-3\varepsilon)}{\Gamma(n)} \cdot \tag{2.17}$$
$$\cdot \left(\frac{1}{n+a} + \frac{1}{n-a-3\varepsilon}\right) \Big\} .$$

For the ultimate conclusion about the validity of solution (2.17), one has to be convinced that it is impossible to add an arbitrary solution of homogeneous equation. Indeed, such a solution vanishes at integer points, is an analytic function and exponentially bounded at imaginary axis. Hence, due to the Karlson theorem it is identically zero.

The last sum in (2.17) is equal to $-\Gamma(1+a)\Gamma(1-a-3\varepsilon)$. Thus, $F_\varepsilon(1+a)$ can be also rewritten as

$$F_\varepsilon(1+a) = 2\,\frac{\Gamma^2(1-\varepsilon)\,\Gamma(\varepsilon)}{\Gamma^2(1)\Gamma(2-2\varepsilon)} \Big\{ \frac{\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)}{\Gamma(1+a)\Gamma(1-a-3\varepsilon)} \sum_{n=1}^{\infty} (-)^n \frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)} \cdot \tag{2.18}$$
$$\cdot \left(\frac{1}{n+a+\varepsilon} + \frac{1}{n-a-2\varepsilon}\right) + \frac{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma(-a-2\varepsilon)\Gamma(a+2\varepsilon)}{\Gamma(1-2\varepsilon)\,\Gamma(1+2\varepsilon)} \Big\} .$$

For $\varepsilon=0$ eq. (2.17) gives

$$F_0(1+a) = \frac{2}{a}\sum_{n=1}^{\infty} (-)^n \left[\frac{1}{(n+a)^2} - \frac{1}{(n-a)^2}\right] = \tag{2.19}$$
$$= -8\sum_{n=1}^{\infty} (-)^n \frac{n}{(n^2-a^2)^2} = \frac{2}{a}\left[\beta'(1+a) - \beta'(1-a)\right] ,$$

where

$$\beta(1+x) = \frac{1}{2}\left[\psi\left(1+\frac{x}{2}\right) - \psi\left(\frac{1}{2}+\frac{x}{2}\right)\right] .$$

So, solving the functional equations we find the coefficient function not yielded by other calculation methods. For $\varepsilon=0$, we succeded in setting the closed expression. Analogous equations can be also obtained for more complicated diagrams. The solution is presented in the form of one-fold series like (2.17). These solutions happen to be very usefull, enlarging the class of exactly calculable diagrams. We demonstrate this by an example of calculation of the most complicated diagram in the five-loop approximation of the φ^4-theory.

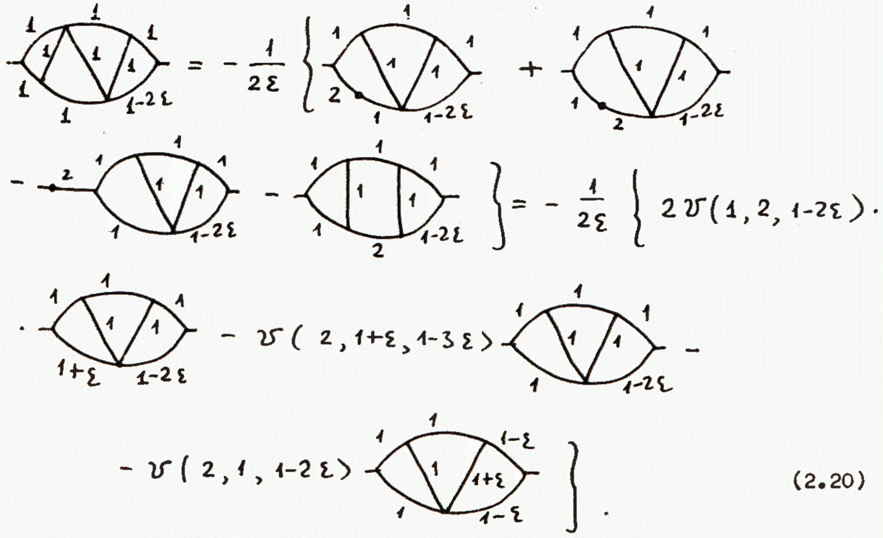
2.6. Five-loop calculations in the φ^4 theory

The most complicated for calculation is the following vertex diagram:

.

To evaluate it one has to find an N-like diagram

up to $O(1)$. Hence, we can choose the indices of the lines in a suitable way. Choosing them in the following we apply eq. (1.7) to the lower triple vertex:



(2.20)

We have used here eqs. (1.3), (1.4), and (1.6). Now for the calculation of $\mathcal{N}$-like diagram up to $O(1)$ we have to calculate three V-like diagrams up to $O(\varepsilon^2)$ or several two-loop diagrams up to $O(\varepsilon^4)$. Unfortunately, not all of them can be explicitly calculated by the method of uniqueness. As for the tables (2.5), (2.6) they contain expansions up to $O(\varepsilon)$ and $O(\varepsilon^3)$, respectively. So, we need to expand the tables by one order in ε. For this purpose, we use the obtained solution of functional equations (2.17).

Return to the two-loop diagram (2.1). The expansion of $F_\varepsilon(1+a\varepsilon)$ over ε up to ε^4 contains three structures (we take into account the symmetry properties (2.8)): (cf. eq. (2.3))

$$F_\varepsilon(1+a\varepsilon)=\frac{1}{1-2\varepsilon}\Big\{6\zeta(3)+\dots+\varepsilon^4\big[C_{10}(a+1)(a+2)+ \\ +C_{11}a(a+3)+C_{12}a(a+1)(a+2)(a+3)\big]+O(\varepsilon^5)\Big\}. \tag{2.21}$$

The known expressions for $F_\varepsilon(1+a\varepsilon)$ for $a=0,-1$ give us the coefficients C_{10} and C_{11}. As for the C_{12}, it is not determined from the particular values of a. However, as is easy to see, it is equal to

$$C_{12}=\frac{1}{4!}\left.\frac{d^4F_0(1+a)}{da^4}\right|_{a=0}.$$

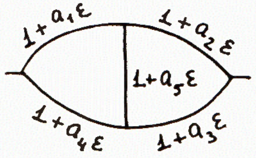
To find it, we expand the function $F_0(1+a)$ (2.19) into a series in a^2. We have

$$F_0(1+a)=8\sum_{n=0}^{\infty}a^{2n}(n+1)\left(1-\frac{1}{2^{2n+2}}\right)\zeta(2n+3), \tag{2.22}$$

that leads to

$$C_{12}=\frac{189}{8}\zeta(7).$$

The number obtained enables us to complete expansion (2.21) and also to construct the expansion up to $O(\varepsilon^4)$ for an arbitrary two-loop diagram (2.4) and up to $O(\varepsilon^2)$ for an arbitrary V-like diagram, i.e., to continue the tables obtained earlier by one order of ε. They are



$$=\frac{\exp\left[-2\left(\gamma\varepsilon+\frac{\zeta(2)}{2}\varepsilon^2\right)\right]}{1-2\varepsilon}\Big\{A_0\zeta(3)+ \\ +A_1\zeta(4)\varepsilon+A_2\zeta(5)\varepsilon^2+A_3\zeta(6)\varepsilon^3-A_4\zeta^2(3)\varepsilon^3+ \\ +A_5\zeta(7)\varepsilon^4-A_6\zeta(3)\zeta(4)\varepsilon^4+O(\varepsilon^5)\Big\}, \tag{2.23}$$

$$A_0=6,$$

$$A_1=9,$$

$$A_2=42+30(a_1+a_2+a_3+a_4)+45a_5+10(a_1^2+a_2^2+a_3^2+a_4^2) \\ +15a_5^2+15a_5(a_1+a_2+a_3+a_4)+10(a_1a_2+a_3a_4+a_1a_4+ \\ +a_2a_3)+5(a_1a_3+a_2a_4),$$

$$A_3=\frac{5}{2}(A_2-6),$$

$$A_4=46+42(a_1+a_2+a_3+a_4)+45a_5+14(a_1^2+a_2^2+a_3^2+a_4^2) \\ +15a_5^2+33a_5(a_1+a_2+a_3+a_4)+50(a_1a_2+a_3a_4)+31(a_1a_3 \\ +a_2a_4)+14(a_1a_4+a_2a_3)+6a_5(a_1^2+a_2^2+a_3^2+a_4^2)+6a_5^2\cdot \\ (a_1+a_2+a_3+a_4)+24a_5(a_1a_2+a_3a_4)+12a_5(a_1a_3+a_2a_4) \\ +12(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)+12(a_1^2a_2+a_2^2a_1 \\ +a_3^2a_4+a_4^2a_3)+6(a_1^2a_3+a_3^2a_1+a_2^2a_4+a_4^2a_2),$$

$$A_5=294+402(a_1+a_2+a_3+a_4)+\frac{2223}{4}a_5+260(a_1^2+a_2^2+a_3^2+a_4^2) \\ +\frac{3183}{8}a_5^2+516a_5(a_1+a_2+a_3+a_4)+386(a_1a_2+a_3a_4+a_1a_4+a_2a_3) \\ +\frac{575}{2}(a_1a_3+a_2a_4)+84(a_1^3+a_2^3+a_3^3+a_4^3)+\frac{567}{4}a_5^3+168\cdot \\ \cdot(a_1^2a_2+a_2^2a_1+a_3^2a_4+a_4^2a_3+a_1^2a_4+a_4^2a_1+a_2^2a_3+a_3^2a_2)+\frac{441}{4}\cdot \\ \cdot(a_1^2a_3+a_3^2a_1+a_2^2a_4+a_4^2a_2)+\frac{945}{4}a_5(a_1^2+a_2^2+a_3^2+a_4^2)+252a_5^2\cdot \\ \cdot(a_1+a_2+a_3+a_4)+\frac{693}{2}a_5(a_1a_2+a_3a_4+a_1a_4+a_2a_3)+\frac{945}{4}a_5\cdot \\ \cdot(a_1a_3+a_2a_4)+210(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)+ \\ +14(a_1^4+a_2^4+a_3^4+a_4^4)+\frac{189}{8}a_5^4+42a_5(a_1^3+a_2^3+a_3^3+a_4^3) \\ +\frac{189}{4}a_5^3(a_1+a_2+a_3+a_4)+\frac{525}{8}a_5^2(a_1^2+a_2^2+a_3^2+a_4^2)+\frac{357}{4}a_5^2\cdot \\ \cdot(a_1a_2+a_3a_4+a_1a_4+a_2a_3)+\frac{105}{2}a_5^2(a_1a_3+a_2a_4)+84a_5\cdot$$

$$
\cdot\left(a_1^2a_2+a_2^2a_1+a_3^2a_4+a_4^2a_3+a_1^2a_4+a_4^2a_1+a_2^2a_3+a_3^2a_2\right)+
$$

$$
+\frac{189}{4}a_5\left(a_1^2a_3+a_3^2a_1+a_2^2a_4+a_4^2a_2\right)+\frac{357}{4}a_5\left(a_1a_2a_3+a_1a_2a_4\right.
$$

$$
\left.+a_1a_3a_4+a_2a_3a_4\right)+28\left(a_1^3a_2+a_2^3a_1+a_3^3a_4+a_4^3a_3+\right.
$$

$$
\left.+a_1^3a_4+a_4^3a_1+a_2^3a_3+a_3^3a_2\right)+14\left(a_1^3a_3+a_3^3a_1+a_2^3a_4\right.
$$

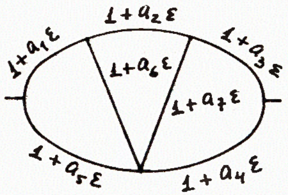
$$
\left.+a_4^3a_2\right)+42\left(a_1^2a_2^2+a_3^2a_4^2+a_1^2a_4^2+a_2^2a_3^2\right)+\frac{189}{8}\cdot
$$

$$
\cdot\left(a_1^2a_3^2+a_2^2a_4^2\right)+42\left(a_1^2a_2a_3+a_1^2a_2a_4+a_1^2a_3a_4+\right.
$$

$$
+a_2^2a_1a_4+a_2^2a_1a_3+a_2^2a_3a_4+a_3^2a_1a_4+a_3^2a_2a_4+a_3^2a_1a_2
$$

$$
\left.+a_4^2a_2a_3+a_4^2a_1a_3+a_4^2a_1a_2\right)+\frac{315}{4}a_1a_2a_3a_4\,,
$$

$$
A_6=3\left(A_4-1\right)\;;
$$



$$
=\frac{\exp\left[-3\left(\gamma\varepsilon+\frac{\zeta(2)}{2}\varepsilon^2\right)\right]}{1-2\varepsilon}\,.
$$

$$
\Big\{20\zeta(5)+\varepsilon\left[50\zeta(6)+\Big(20+6(a_4+a_5+a_6+a_7)\Big)\zeta^2(3)\right]
$$

$$
+\varepsilon^2\Big[\zeta(7)\cdot7\Big(\frac{380}{7}+24(a_1+a_3)+32a_2+17(a_4+a_5)
$$

$$
+33(a_6+a_7)+6(a_1^2+a_3^2)+8a_2^2+4(a_4^2+a_5^2)
$$

$$
+8(a_6^2+a_7^2)+8(a_1+a_3)a_2+2(a_1a_4+a_3a_5)
$$

$$
+6(a_1a_5+a_3a_4)+10(a_1a_6+a_3a_7)+6(a_1a_7+
$$

$$
+a_3a_6)+4a_1a_3+4(a_4+a_5)a_2+12(a_6+a_7)a_2+
$$

$$
+2a_4a_5+4(a_4a_6+a_5a_7)+6(a_4a_7+a_5a_6)
$$

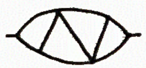
$$
+10a_6a_7+\frac{1}{4}(a_4+a_5+a_6+a_7)+
$$

$$
+\frac{1}{8}(a_4+a_5+a_6+a_7)^2\Big)+
$$

$$
+\zeta(3)\zeta(4)\cdot3\Big(20+6(a_4+a_5+a_6+a_7)\Big)\Big]+
$$

$$
+O(\varepsilon^3)\Big\}. \tag{2.24}
$$

Eq. (2.24) enables us to complete the calculation of the $\mathcal{N}$-like diagram (2.20). The result is



$$
=\frac{441}{8}\zeta(7). \tag{2.25}
$$

Expansions (2.23), (2.24) obtained once can be used further like tables for the calculation of multiloop integrals. The task is only to reduce the integral of interest to the table one. This can be achieved with the help of a number of tricks of the method of uniqueness as we have demonstrated above.

The method of uniqueness happened to be an extremely usefull and powerful technique and enables us to complete the five-loop calculations of the β-function in the φ^4 theory. It became possible to calculate the diagrams that are not yielded by other methods. For the β-function the following expression has been obtained:

$$
\left(\mathcal{L}_{int}=-\frac{16\pi^2}{4!}h\varphi^4\right)
$$

$$
\beta_{MS}(h)=\frac{3}{2}h^2-\frac{17}{6}h^3+\left(\frac{145}{16}+6\zeta(3)\right)h^4
$$

$$
-\left(\frac{3499}{96}+39\zeta(3)-9\zeta(4)+60\zeta(5)\right)h^5
$$

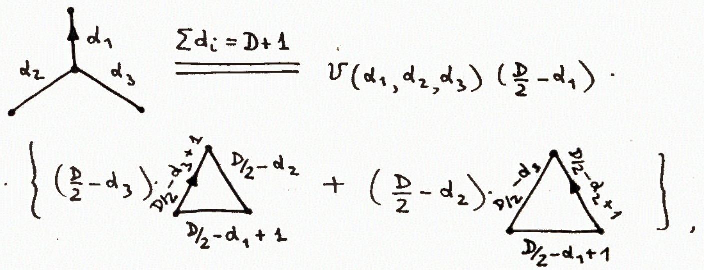
$$
+\left(\frac{767261}{4608}+\frac{7965}{32}\zeta(3)-\frac{1177}{16}\zeta(4)+\frac{2049}{4}\zeta(5)\right. \tag{2.26}
$$

$$
\left.-\frac{771}{4}\zeta(6)+\frac{45}{2}\zeta^2(3)+\frac{1323}{2}\zeta(7)\right)h^6+O(h^7).
$$

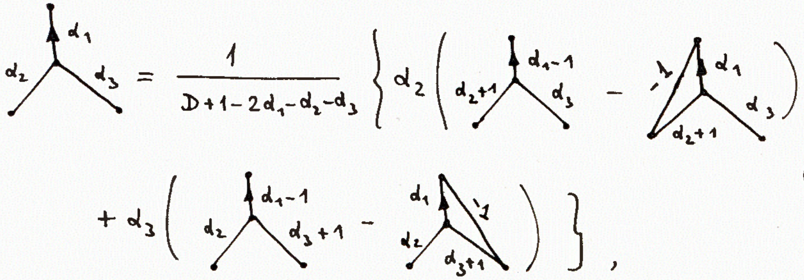
2.7. Discussion

In the present lectures we described the basic notions of the method of uniqueness, demonstrated its relative simplicity and efficiency. The method enables us to get the result for multiloop Feynman integrals without complicated integration and/or summation. This is true not only for scalar theories but also for vector and spinor ones with the nominators in the propagators. Here, we usually reduce the integral to the scalar one using the corresponding projectors. The obtained scalar integral is calculated by the method of uniqueness.

There are, however, direct generalizations of the formulas to the case of nominators. Thus, for instance, uniqueness relation (1.6) and integration by part, eq. (1.7), have the following generalizations:



(2.27)



(2.28)

where the arrow on the line means the corresponding four-vector. Maybe these formulas will be useful for the calculations in gauge theories.

In spite of wide possibilities the method of uniqueness has natural limitations. When the number of loops is large it is not always possible to create the needed uniquenesses in all diagrams. Very useful here are the tables, but the problem arises to reduce the integrals to the table ones. All that contains an element of skill and needs a separate consideration of every diagram. One should

note, however, that multiloop calculatinns always represent some puzzle. The method of uniqueness helps us to solve many of them.

References to Lecture I

[1] The Basic notions of the method of uniqueness were introduced in A.N.Vassiliev, Yu.M.Pis'mak, Yu.R.Honkonen. " $1/n$ expansion: calculation of indices η and ν to $1/n^2$ order in arbitrary dimension"., TMF, 1981, 47, p. 291.

[2] The uniqueness relation known also as the startriangle or Yang-Baxter relation is used in the theories with conformal symmetry. To calculate an integral in 3-dimensional space it was used in M.D'Eramo, L.Pelitti, G.Parisi. "Theoretical predictions for critical exponents at λ-point of Bose Liquids", Let. Nuovo Cim., 1971, 2, p. 878.

[3] The method of integration by parts in multiloop calculations was proposed in K.G.Chetyrkin, F.V.Tkachov. "Integration by parts: the algorithm to calculate 4-loop β-function". Nucl.Phys., 1981, B192, p.159, where eqs. like (1.7), (1.10) were used but for the triangle in p-space.

[4] Calculations of the diagrams with uniqueness or one-step deviations from uniqueness with the help of eqs. (1.6), (1.8), (1.9) were firstly carried out in [1] and in:
N.I.Ussyukina. "On the calculation of multiloop diagrams in perturbation theory", TMF, 1983, 54, p. 124.

[5] The operations of index transformation were proposed in [1]. Here also a table of such transformations applied to the two-loop diagram was given.

[6] To the calculation of multiloop diagrams in scalar theories the method of uniqueness was applied in:
D.I.Kazakov
"Calculation of Feynman integrals by the method of uniqueness", Preprint JINR, E2-83-323, Dubna, 1983 and TMF, 1984, 58, p. 343.

References to Lecture II

[7] The idea of the tables was proposed in [6]. Here also the tables (2.4), (2.5), and (2.6) were constructed up to ε^3 and ε, respectively.

[8] To simplify a diagram and to reduce it to a table one the infrared R^*-operation can be used. It was introduced in:
K.G.Chetyrkin, F.V.Tkachov.
"Infrared R-operation and ultraviolet counterterms", Phys.Lett., 1982, 114B, p. 340.

[9] The functional equations for the coefficient functions of the diagrams were proposed in:
D.I.Kazakov
"Multiloop Calculations: Method of Uniqueness and Functional Equations", Preprint JINR, E2-83-839, Dubna, 1983.

Here also the solutions given above were obtained.

[10] For the calculation of ladder diagrams the functional relations were used in:
V.V.Belokurov, N.I.Ussyukina
"Calculation of ladder diagrams in arbitrary order", J.Phys. A: Math.Gen., 1983, 16, p. 2811.

[11] Five-loop calculations in the φ^4 theory were performed in
K.G.Chetyrkin, S.G.Gorishny, S.A.Larin, F.V.Tkachov.
"Five-loop renormalization group calculations in the $g\varphi^4_{(4)}$ theory", Phys.Lett., 1983, 132B, p. 351 on the basis of integration by parts [3] and R^*-operation [8]. Part of the diagrams was calculated numerically.
Analytical calculations have been completed in [6] except for the N-like diagram for which the answer was predicted.

[12] Analytical expression for the β-function in 5-loop approximation was first published in:
D.I.Kazakov
"The method of uniqueness, a new powerful technique for multiloop calculations", Phys.Lett., 1983, 133B, p. 406.

[13] Exact calculation of N-like diagram was performed in [9]. Here also the expanded tables (2.23), (2.24) were constructed.

[14] The formulas of the method of uniqueness with numerators were first given in [12].

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Казаков Д.И. E2-84-410
Аналитические методы вычисления диаграмм высоких порядков /две лекции по методу уникальностей/

В лекциях содержится последовательное изложение одного из методов аналитического вычисления многопетлевых интегралов, встречающихся в квантовой теории поля. Этот метод носит название "метода уникальностей" и направлен на точное аналитическое вычисление безмассовых интегралов, зависящих от одного размерного параметра. Применение метода иллюстрируется большим количеством примеров.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1984

Kazakov D.I. E2-84-410
Analytical Methods for Multiloop Calculations
(Two lectures on the method of uniqueness)

In the present lectures we give a step-by-step description of the method for analytical calculation of multiloop integrals one is met with in QFT. It is called the method of "uniqueness" and is aimed at the exact analytical calculation of massless integrals depending on a single dimensional parameter. Application of the method is illustrated by numerous examples.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1984