сообщения
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дубна

E2-84-244

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ON A COMPLETE SET OF IRREDUCIBLE HIGHEST-WEIGHT REPRESENTATIONS FOR sl(3,C)

1984

1. Introduction

The highest-weight representations of semisimple Lie algebras are interesting from both the mathematical and physical point of view (see, e.g., the references given in the papers[1,2]). There is a classification theorem according to which the irreducible representation with a given highest weight Λ of a complex semisimple Lie algebra L is unique up to equivalence. However, not every member of such an equivalence class is suitable for practical calculations. It motivates interest to various constructions of irreducible highest-weight representations.

In the recent series of papers[2-4], we have studied this problem for the Lie algebras $A_n \sim sl(n+1,\mathbb{C})$, with a particular attention paid to the case $n=2$. The method used for the construction employed a family of canonical (or boson) realizations of $gl(n+1,\mathbb{C})$ obtained earlier[5,6]. For $sl(3,\mathbb{C})$, one can get in this way two sets of suitable candidates for the role of highest-weight representations : the so-called maximal and mixed representations. Another highest-weight representations may be obtained by combining the latter with suitable automorphisms of $sl(3,\mathbb{C})$ and of the corresponding Weyl algebra. As a result, we have a family of infinite-dimensional representations, at least one to each Λ including those cases where the irreducible highest-weight representations are finite-dimensional (and well known - cf., e.g., Ref.7, Section 10.1).

Most of these infinite-dimensional representations are irreducible : the proof for the maximal representations was given in Ref.2, while the remaining ones were treated in Refs.3,4 . However, the claim made in Ref.4, namely that one obtains in this way a complete set of (infinite-dimensional) irreducible highest-weight representations for $sl(3,\mathbb{C})$, appears to be not fully justified. It was overlooked in Ref.3 that there are two subsets in the set of all weights Λ (the sets $\Omega(1,12)$ and $\Omega(2,12)$ - cf.below) for which the irreducibility proof presented there failed to work. Even worse, for some of these cases the constructed representations are in fact reducible, as we shall see later.

It is the main aim of the present paper to rectify the situation. Fortunately, it can be done in the same framework. We shall be able to select explicitly a subspace in the representation space V of a given mixed representation ρ_Λ that corresponds to its irreducible component. It will be achieved by decomposing V with respect to the irreducible representations of a subalgebra $gl(2,\mathbb{C})$ contained in ρ_Λ. Moreover, the decomposition procedure works even for those weights for which the irreducible highest-weight representations are finite-dimensional (the set Ω_{fin} - see below). Hence the method of canonical (boson) realizations is actually able to yield a complete set of irreducible highest-weight representations of $sl(3,\mathbb{C})$ (of the form suitable for practical calculations). We are convinced (having various positive indications) that the extension of the last assertion to $sl(n+1,\mathbb{C})$, $n>2$, and eventually to other semisimple Lie algebras, is mainly a technical matter.

2. Preliminaries

Here we shall resume briefly some notions needed in the following ; for a more detailed information see Refs.2,6 . The Lie algebra $gl(n+1,\mathbb{C})$ has the standard basis formed by $(n+1)^2$ elements e_{ij}, $i,j=1,\dots,n+1$, that fulfil the relations

$$[e_{ij},e_{kl}] = \delta_{kj}e_{il} - \delta_{il}e_{kj} \quad . \tag{1}$$

The simple subalgebra $sl(n+1,\mathbb{C})$ is generated by the elements e_{ij}, $i\neq j$, and $h_i = e_{i+1,i+1} - e_{ii}$, $i=1,2,\dots,n$.

Let $\Lambda = (\Lambda_1,\dots,\Lambda_n)$ with Λ_i's being complex numbers. A representation ρ of $sl(n+1,\mathbb{C})$ on a vector space W is called representation with the highest weight Λ if there is a non-zero vector $x_0 \in W$ such that

$$\rho(e_{ij})x_0 = 0 \quad , \quad j<i \quad , \tag{2a}$$

$$\rho(h_i)x_0 = \Lambda_i x_0 \quad , \quad i=1,2,\dots,n \quad , \tag{2b}$$

and x_0 is cyclic for ρ, i.e., $\rho(UL)x_0 = W$, where UL means the universal enveloping algebra of $L = sl(n+1,\mathbb{C})$. In view of the relations (1), it is enough to check the condition (2a) for $i=j+1$ only.

The representations treated below are expressed in terms of the standard creation and annihilation (boson) operators. Let, e.g., V_N be the vector space spanned by the vectors $|n_1,n_2,\dots,n_N\rangle$ with $n_i \in \mathbb{N}_0 \equiv \{0,1,2,\dots\}$. Then the action of these operators on V_N is given by the relations

$$\bar{a}_i|n_1,\dots,n_i,\dots,n_N\rangle = (n_i+1)^{1/2}\,|n_1,\dots,n_i+1,\dots,n_N\rangle \quad , \tag{3a}$$

$$a_i|n_1,\dots,n_i,\dots,n_N\rangle = n_i^{1/2}\,|n_1,\dots,n_i-1,\dots,n_N\rangle \quad . \tag{3b}$$

Since they fulfil the canonical commutation relations, one can construct to each $\Lambda = (\Lambda_1,\Lambda_2)$ the representation ρ_Λ of $sl(3,\mathbb{C})$ by the formulae[4-6]

$$\rho_\Lambda(h_1) = -\bar{a}_1a_1 + \bar{a}_2a_2 + \tau(h_1) \quad , \tag{4a}$$

$$\rho_\Lambda(h_2) = -\bar{a}_1a_1 - 2\bar{a}_2a_2 - \tfrac{1}{2}\tau(h_1) + \Lambda_2 + \tfrac{1}{2}\Lambda_1 \quad , \tag{4b}$$

$$\rho_\Lambda(e_{21}) = \bar{a}_2a_1 + \tau(e_{21}) \quad , \tag{4c}$$

$$\rho_\Lambda(e_{31}) = -a_1 \quad , \tag{4d}$$

$$\rho_\Lambda(e_{32}) = -a_2 \quad , \tag{4e}$$

$$\rho_\Lambda(e_{12}) = \bar{a}_1a_2 + \tau(e_{12}) \quad , \tag{4f}$$

$$\rho_\Lambda(e_{13}) = \bar{a}_1(\bar{a}_1a_1 + \bar{a}_2a_2 - \tfrac{1}{2}\tau(h_1) - \Lambda_2 - \tfrac{1}{2}\Lambda_1) + \bar{a}_2\tau(e_{12}) \quad , \tag{4g}$$

$$\rho_\Lambda(e_{23}) = \bar{a}_2(\bar{a}_1a_1 + \bar{a}_2a_2 + \tfrac{1}{2}\tau(h_1) - \Lambda_2 - \tfrac{1}{2}\Lambda_1) + \bar{a}_1\tau(e_{21}) \quad , \tag{4h}$$

where τ is a representation of $sl(2,\mathbb{C})$ on a vector space W. Strictly speaking, one should write here $a_1 \otimes I_W$ instead of a_1, etc., but there is presumably no danger of misunderstanding. In this sense therefore, the relations

$$[a_i,\tau(g)] = [\bar{a}_i,\tau(g)] = 0 \quad , \quad i=1,2 \quad , \tag{5}$$

are valid for all $g \in sl(2,\mathbb{C})$.

It is easy to see that if $\tau(e_{12})y_0 = 0$ and $\tau(h_1)y_0 = \Lambda_1 y_0$ for some $y_0 \in W$, then the representation ρ_Λ fulfils the conditions (2) with $x_0 = |0,0\rangle \otimes y_0$. Two cases are of a particular interest :

(a) if $W = V_1$ and τ is given by

$$\tau(h_1) = -2\bar{a}_3 a_3 + \Lambda_1 \quad , \tag{6a}$$

$$\tau(e_{21}) = -a_3 \quad , \tag{6b}$$

$$\tau(e_{12}) = \bar{a}_3(\bar{a}_3 a_3 - \Lambda_1) \tag{6c}$$

for some $\Lambda_1 \in \mathbb{C}$, then the representation (4) is called <u>maximal representation</u>,

(b) if W is finite-dimensional with the basis $|s\rangle_W$, $s = 0,1,\dots,\Lambda_1$ for some $\Lambda_1 \in \mathbb{N}_0$ and

$$\tau(h_1)|s\rangle_W = (\Lambda_1 - 2s)|s\rangle_W \quad , \tag{7a}$$

$$\tau(e_{21})|s\rangle_W = -s\,|s-1\rangle_W \quad , \tag{7b}$$

$$\tau(e_{12})|s\rangle_W = (s-\Lambda_1)|s+1\rangle_W \quad , \tag{7c}$$

then the formulae (4) yield the so-called <u>mixed representations</u> of $sl(3,\mathbb{C})$.

It is useful to decompose the set $\Omega = \mathbb{C}^2$ of all weights Λ into disjoint subsets depending on whether the components Λ_i and their combination $1 + \Lambda_1 + \Lambda_2$ belong to $\mathbb{N}_0 \equiv \{0,1,2,\dots\}$ or not (for real Λ , it is sketched on Fig.1 together with the corresponding root diagram of $sl(3,\mathbb{C})$). In particular, we denote

$$\Omega_{fin} = \Omega(1,2,12) = \{\Lambda : \Lambda_1 \in \mathbb{N}_0 \ , \ \Lambda_2 \in \mathbb{N}_0 \ , \ 1 + \Lambda_1 + \Lambda_2 \in \mathbb{N}_0\} \ ,$$

$$\Omega_{max} = \Omega(\emptyset) = \{\Lambda : \Lambda_1 \notin \mathbb{N}_0 \ , \ \Lambda_2 \notin \mathbb{N}_0 \ , \ 1 + \Lambda_1 + \Lambda_2 \notin \mathbb{N}_0\} \ ,$$

and furthermore

$$\Omega(1,12) = \{\Lambda : \Lambda_1 \in \mathbb{N}_0 \ , \ \Lambda_2 \notin \mathbb{N}_0 \ , \ 1 + \Lambda_1 + \Lambda_2 \in \mathbb{N}_0\}.$$

etc. (we mark the sets by the symbols of the conditions that are fulfilled). Hence we have the decomposition

$$\Omega = \Omega_{max} \cup \Omega(1) \cup \Omega(2) \cup \Omega(12) \cup \Omega(1,12) \cup \Omega(2,12) \cup \Omega_{fin} \quad , \tag{8}$$

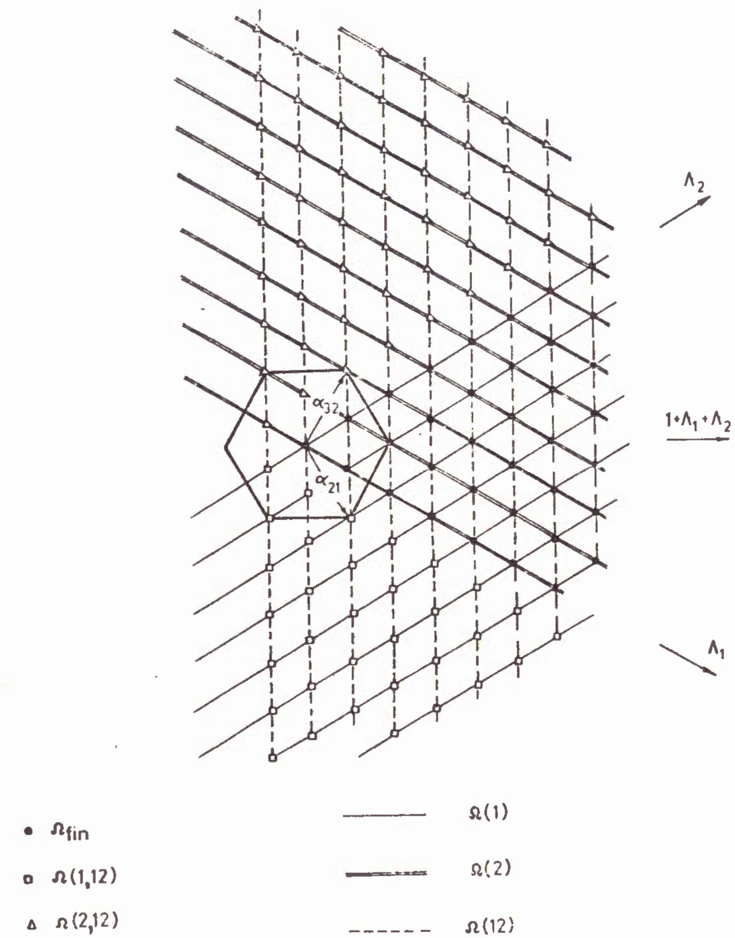

Fig.1. The decomposition (8) for real weights.

because $\Omega(1,2)=\emptyset$; the notation used here differs in part from that employed in Ref.4 .

It is easy to see[2,3] that in order to prove irreducibility of the representations ρ_Λ , one has just to check cyclicity, i.e., the remaining requirement from the definition of the highest-weight representations. The following assertions are valid :

Proposition 2.1 : (a) If $\Lambda\in\Omega_{max}$, then the maximal representation ρ_Λ given by (4) and (6) is irreducible and has the highest weight Λ . (b) If $\Lambda\in\Omega(1)$, then the mixed representation ρ_Λ given by (4) and (7) is irreducible and has the highest weight Λ .

The part follows from Theorem 4.3 of Ref.2, while (b) has been proven in Ref.3 . As mentioned in the introduction, the last proof fails to work for $\Lambda\in\Omega(1,12)$, because it uses linear combinations with the coefficients which are eventually zero in this case. Using further an automorphism of $sl(3,\mathbb{C})$ that permutes h_1 and h_2 (see Ref.3), one can construct from (4) and (7) irreducible highest-weight representations referring to $\Lambda\in\Omega(2)$, while the case $\Lambda\in\Omega(2,12)$ is again left open. Finally, irreducible highest-weight representations for $\Lambda\in\Omega(12)$ have been constructed in Ref.4 .

3. Decomposition of the representation space

Hence we are left with the task to construct irreducible highest-weight representations for $\Lambda\in\Omega(1,12)$. To this end, we shall use the mixed representations ρ_Λ given by (4) and (7), with the representation space $V=V_2\otimes W$. To each $x\in V$, it is easy to find an element $g\in UL$, $L=sl(3,\mathbb{C})$, such that $\rho_\Lambda(g)x=x_0\equiv|0,0\rangle\otimes|0\rangle_W$. Since the relations (2) are valid with this x_0 , it is enough to select the irreducible component of ρ_Λ acting on $\rho_\Lambda(UL)x_0$. Notice that the mixed representations themselves may be similarly "reduced out" from the maximal ones (up to the normalization of the basis vectors) : the relation (7c) shows that we cannot get the vectors $|m,n\rangle\otimes|s\rangle$ with $s>\Lambda_1$ starting from the "vacuum" if $\Lambda_1\in N_0$.

We are going to perform the reduction by decomposing V to a direct sum of subspaces referring to irreducible representations of the subalgebra $gl(2,\mathbb{C})$ generated by e_{12},e_{21},h_1 and h_2 . With such a decomposition at hand, we shall pick up those subspaces which are contained in $\rho_\Lambda(UL)x_0$.

The space V is spanned by the vectors $|n_1,n_2,s\rangle=|n_1,n_2\rangle\otimes|s\rangle_W$ with $n_i\in N_0$ and $s=0,1,\dots,\Lambda_1\in N_0$. First we shall pass to another

basis. For arbitrary $t\in N_0$ and $r=0,1,\dots,\min(t,\Lambda_1)$, we denote

$$x_{t,r}=\sum_{n=0}^{r}\binom{r}{n}[n!(t-n)!]^{1/2}\,|n,t-n,r-n\rangle\ . \tag{9a}$$

Proposition 3.1 : The vectors

$$x_{t,r,q}=\rho_\Lambda(e_{12})^q x_{t,r} \tag{9b}$$

with $t\in N_0$, $r=0,1,\dots,\min(t,\Lambda_1)$ and $q=0,1,\dots,\Lambda_1+t-2r$ span the vector space V .

Proof : First we check that all the vectors (9b) are non-zero. In view of their definition, it is sufficient to verify that $\rho_\Lambda(e_{12})^{\Lambda_1+t-2r}x_{t,r}\neq 0$. By induction, one finds easily

$$\rho_\Lambda(e_{12})^t|n_1,n_2,s\rangle=\sum_{k=0}^{t}(-1)^{t-k}\binom{t}{k}\frac{(\Lambda_1-s)!}{(\Lambda_1-s-t+k)!}\times\left[\frac{n_2!(n_1+k)!}{(n_2-k)!n_1!}\right]^{1/2}|n_1+k,n_2-k,s+t-k\rangle\ , \tag{10}$$

where the terms containing "negative factorials" in the denominator are supposed to be zero. Thus we have

$$\rho_\Lambda(e_{12})^q x_{t,r}=\sum_{k=0}^{q}\sum_{n=0}^{r}(-1)^{q-k}\binom{q}{k}\binom{r}{n}\frac{(\Lambda_1-r+n)!(t-n)!}{(\Lambda_1-r+n-q+k)!}\times\left[\frac{(n+k)!}{(t-n-k)!}\right]^{1/2}|n+k,t-n-k,r-n+q-k\rangle\ . \tag{11}$$

Changing further the summation indices k,n to $l=n+k$, n , one obtains

$$\rho_\Lambda(e_{12})^q x_{t,r}=\sum_{l=\max(0,r+q-\Lambda_1)}^{\min(t,q+r)}(-1)^{q-l}\,c_l^{t,r,q}\,\frac{q!}{(\Lambda_1-r-q+l)!}\times\left[\frac{l!}{(t-l)!}\right]^{1/2}|l,t-l,r+q-l\rangle \tag{12a}$$

with

$$c_l^{t,r,q}=\sum_{n=\max(0,l-q)}^{\min(r,l)}(-1)^n\binom{r}{n}\frac{(\Lambda_1-r+n)!}{(q-l+n)!}\,\frac{(t-n)!}{(r-n)!}\ .$$

Writing the two quotients as the derivatives of suitable power functions, we arrive after a short calculation at the expression

$$c_1^{t,r,q} = \frac{d^{1-r-q+\Lambda_1}}{d\xi^{1-r-q+\Lambda_1}} \frac{d^{t-1}}{d\eta^{t-1}} \left[\xi^{\Lambda_1 - r} \eta^{t-n} (\eta - \xi)^r \right]_{(\xi,\eta)=(1,1)} \qquad (12b)$$

These relations show, in particular, that for any given t,r we have $\rho_\Lambda(e_{12})^q x_{t,r} = 0$ if $q > \Lambda_1 + t - 2r$, because in that case all terms of the above derivative contain certain positive powers of $\eta - \xi$, and consequently, $c_1^{t,r,q} = 0$ for each 1. Furthermore,

$$c_1^{t,r,\Lambda_1+t-2r} = (-1)^{r+1-t} \, r!$$

is non-zero so $x_{t,r,q} \neq 0$ holds for $q = \Lambda_1 + t - 2r$. In view of (9b), the same is true for $q = 0,1,\dots,\Lambda_1+t-2r-1$.

Next one has to check linear independence of (any finite set of) the vectors (9b). The relations (3), (4) and (7) imply

$$\rho_\Lambda(h_1) x_{t,r,q} = (\Lambda_1 + t - 2r - 2q) x_{t,r,q} , \qquad (13a)$$

$$\rho_\Lambda(h_2) x_{t,r,q} = (\Lambda_2 + r + q - 2t) x_{t,r,q} , \qquad (13b)$$

$$\rho_\Lambda(h_1^2 + 2h_1 + 4e_{12}e_{21}) x_{t,r,q} = (\Lambda_1 + t - 2r)(\Lambda_1 + t - 2r + 2) x_{t,r,q} \; ; \qquad (13c)$$

in the last case one can use commutativity of the operator involved with $\rho_\Lambda(e_{12})$ and calculate the eigenvalue for $q = 0$ only (see (15a) below). It is not difficult to see that the values of t,r,q are determined by the triplet of eigenvalues uniquely, and therefore the linear independence follows.

Finally, we shall verify that the vectors (9b) span V. Let V_t denote the subspace of V generated by all $|n, t-n, s\rangle$ with $0 \le n \le t$ and $s = 0,1,\dots,\Lambda_1$, and similarly, let U_t be spanned by the vectors $x_{t,r,q}$ with $0 \le r \le \min(t,\Lambda_1)$ and $q = 0,1,\dots,\Lambda_1+t-2r$. In view of (12a), we have $U_t \subset V_t$ for each t; we are going to show that their dimensions are the same so $U_t = V_t$. Clearly $\dim V_t = (\Lambda_1+1)(t+1)$; on the other hand, the linear independence of $x_{t,r,q}$ yields

$$\dim U_t = \sum_{r=0}^{\min(t,\Lambda_1)} (\Lambda_1 + t - 2r + 1) = \dim V_t .$$

Consequently,

$$V = \sum_{t=0}^{\infty}{}^{\oplus} \, V_t = \sum_{t=0}^{\infty}{}^{\oplus} \, U_t . \qquad (14a)$$

■

The decomposition can be carried out further, in particular, we can write

$$V = \sum_{t=0}^{\infty}{}^{\oplus} \sum_{r=0}^{\min(t,\Lambda_1)}{}^{\oplus} \, V_{t,r} , \qquad (14b)$$

where $V_{t,r}$ is spanned by the vectors $x_{t,r,q}$ with $q = 0,1,\dots,\Lambda_1+t-2r$.

<u>Proposition 3.2</u> : The restriction of the representation ρ_Λ to the subalgebra $gl(2,\mathbb{C})$ generated by e_{12}, e_{21}, h_1, h_2 acts irreducibly on each $V_{t,r}$.

<u>Proof</u> : Due to the definition, we have $\rho_\Lambda(e_{12}) x_{t,r,q} = x_{t,r,q+1}$ for $q = 0,1,\dots,\Lambda_1+t-2r-1$ and $\rho_\Lambda(e_{12}) x_{t,r,\Lambda_1+t-2r} = 0$. It is easy to see that

$$\rho_\Lambda(e_{21}) x_{t,r} = 0 \qquad (15a)$$

and

$$[e_{21}, e_{12}^q] = q \, e_{12}^{q-1} (h_1 - q + 1) \qquad (15b)$$

so $\rho_\Lambda(e_{21}) x_{t,r,q} = q \, \rho_\Lambda(e_{12})^{q-1} (\rho_\Lambda(h_1) - q + 1) x_{t,r} = q(\Lambda_1 + t - 2r - q + 1) x_{t,r,q-1}$. In combination with the relations (13a,b), this yields the desired result. ■

Concluding this section, we are going to visualize the action of the representation ρ_Λ given by (4) and (7) in the newly introduced basis. It is convenient to change its normalization in the following way

$$\tilde{x}_{t,r,q} = (\Lambda_1 + t - 2r - q)! \; x_{t,r,q} , \qquad (16)$$

for each $t \in \mathbb{N}_0$, $r = 0,1,\dots,\min(t,\Lambda_1)$ and $q = 0,1,\dots,\Lambda_1+t-2r$. Then we have

<u>Proposition 3.3</u> : The representation ρ_Λ acts on the vectors (16) as

$$\rho_\Lambda(h_1) \tilde{x}_{t,r,q} = (\Lambda_1 + t - 2r - 2q) \tilde{x}_{t,r,q} , \qquad (17a)$$

$$\rho_\Lambda(h_2) \tilde{x}_{t,r,q} = (\Lambda_2 + r + q - 2t) \tilde{x}_{t,r,q} , \qquad (17b)$$

$$\rho_\Lambda(e_{12}) \tilde{x}_{t,r,q} = (\Lambda_1 + t - 2r - q) \tilde{x}_{t,r,q+1} , \qquad (17c)$$

$$\rho_\Lambda(e_{21}) \tilde{x}_{t,r,q} = q \tilde{x}_{t,r,q-1} , \qquad (17d)$$

$$\rho_\Lambda(e_{13})\tilde{x}_{t,r,q} = \frac{t-r-\Lambda_2}{t-2r+\Lambda_1+1}\,\tilde{x}_{t+1,r,q+1} - \tag{17e}$$
$$- \frac{t-2r+\Lambda_1-q}{t-2r+\Lambda_1+1}\,(1+\Lambda_1+\Lambda_2-r)(\Lambda_1-r)\,\tilde{x}_{t+1,r+1,q} \;,$$

$$\rho_\Lambda(e_{23})\tilde{x}_{t,r,q} = \frac{t-r-\Lambda_2}{t-2r+\Lambda_1+1}\,\tilde{x}_{t+1,r,q} + \tag{17f}$$
$$+ \frac{q}{t-2r+\Lambda_1+1}\,(1+\Lambda_1+\Lambda_2-r)(\Lambda_1-r)\,\tilde{x}_{t+1,r+1,q-1} \;,$$

$$\rho_\Lambda(e_{31})\tilde{x}_{t,r,q} = -\frac{r}{t-2r+\Lambda_1+1}\,\tilde{x}_{t-1,r-1,q} - \tag{17g}$$
$$- \frac{q}{t-2r+\Lambda_1+1}\,(t-r)(t-r+\Lambda_1+1)\,\tilde{x}_{t-1,r,q-1} \;,$$

$$\rho_\Lambda(e_{32})\tilde{x}_{t,r,q} = \frac{r}{t-2r+\Lambda_1+1}\,\tilde{x}_{t-1,r-1,q+1} - \tag{17h}$$
$$- \frac{t-2r+\Lambda_1-q}{t-2r+\Lambda_1+1}\,(t-r)(t-r+\Lambda_1+1)\,\tilde{x}_{t-1,r,q} \;.$$

Proof : Notice first that $t-2r+\Lambda_1+1$ is always non-zero since $r \leq \min(t,\Lambda_1)$. One must find first $\rho_\Lambda(e_{ij})x_{t,r}$ from (4),(7) and (9a) ; it is a tedious but straightforward calculation. Using then

$$\tilde{x}_{t,r,q} = (\Lambda_1 + t - 2r - q)!\,\rho_\Lambda(e_{12})^q x_{t,r}$$

together with (13a,b),(15b), we get the first four of the relations (17). The remaining ones are obtained with the help of the identities

$$[e_{13}, e_{12}^q] = [e_{32}, e_{12}^q] = 0 \;, \tag{18a}$$

$$[e_{23}, e_{12}^q] = -q\, e_{12}^{q-1} e_{13} \;, \tag{18b}$$

$$[e_{31}, e_{12}^q] = q\, e_{12}^{q-1} e_{32} \;, \tag{18c}$$

which follow from (1) by induction. ■

4. The irreducible representations

Now we are going to show how one can select the irreducible component of the representation (17). To this end, we introduce for the basis (16) a different notation that calls to mind the Gel'fand-Zetlin patterns (see Ref.7, Section 10.1) : the vectors $\tilde{x}_{t,r,q}$ will be written as

$$m = \begin{vmatrix} m_{13} & m_{23} & m_{33} \\ & m_{12} \quad m_{22} & \\ & m_{11} & \end{vmatrix} = \begin{vmatrix} \Lambda_1+\Lambda_2 & \Lambda_1 & 0 \\ & \Lambda_1+t-r \quad r & \\ & r+q & \end{vmatrix} \;, \tag{19}$$

where, of course, $m_{12} = \Lambda_1, \Lambda_1+1, \ldots$, $m_{22} = 0,1,\ldots,\Lambda_1$ and $m_{11} = m_{22}, m_{22}+1,\ldots,m_{12}$. Let us stress that (16) and (19) differ just by the numeration of the basis vectors, because there is a bijective correspondence between the triplets (m_{12},m_{22},m_{11}) and (t,r,q) . The formulae (17) now acquire the following form :

$$\rho_\Lambda(h_1)m = (r_2-2r_1)m \;, \tag{20a}$$

$$\rho_\Lambda(h_2)m = (r_3-2r_2+r_1)m \;, \tag{20b}$$

$$\rho_\Lambda(e_{k,k-1})m = \sum_{j=1}^{k-1} a_{k-1}^j(m)\, m_{k-1}^j \;, \quad k = 2,3 \;, \tag{20c}$$

$$\rho_\Lambda(e_{k-1,k})m = \sum_{j=1}^{k-1} b_{k-1}^j(m)\, \hat{m}_{k-1}^j \;, \quad k = 2,3 \;, \tag{20d}$$

where m_l^j , $\hat{m}_l^j$ abbreviate the subtraction (addition) of 1 in the appropriate index, symbolically

$$m_l^j = m(m_{jl} \to m_{jl}-1) \;, \quad \hat{m}_l^j = m(m_{jl} \to m_{jl}+1)$$

and the numerical factors in (20) are the following

$$r_k = \sum_{j=1}^{k} m_{jk} \;, \tag{21a}$$

$$a_1^1(m) = m_{11} - m_{22} \;, \quad b_1^1(m) = m_{12} - m_{11} \;, \tag{21b}$$

$$a_2^1(m) = -\frac{(m_{12}-m_{23})(m_{12}+1)}{m_{12}-m_{22}+1}\, b_1^1(m) \;, \tag{21c}$$

$$a_2^2(m) = \frac{m_{22}}{m_{12}-m_{22}+1} \;, \tag{21d}$$

$$b_2^1(m) = \frac{m_{12}-m_{13}}{m_{12}-m_{22}+1} \quad , \tag{21e}$$

$$b_2^2(m) = \frac{(m_{13}-m_{22}+1)(m_{23}-m_{22})}{m_{12}-m_{22}+1}\, a_1^1(m) \quad . \tag{21f}$$

The relations (20),(21) describe the representation fully, the action of $\rho_\Lambda(e_{13})$, $\rho_\Lambda(e_{31})$ being obtained from the commutation relations.

As we have explained above, in order to obtain an irreducible highest-weight representation corresponding to a given Λ (with $\Lambda_1 \in N_0$) one has to take the restriction $\tilde{\rho}_\Lambda$ of ρ_Λ to the subspace $\rho_\Lambda(UL)x_0$, where x_0 is the would-be highest-weight vector,

$$x_0 \equiv \tilde{x}_{0,0,0} \equiv \left| \begin{matrix} \Lambda_1+\Lambda_2 & & \Lambda_1 & & 0 \\ & \Lambda_1 & & 0 & \\ & & 0 & & \end{matrix} \right| \quad . \tag{22}$$

The following cases are to be distinguished. If $\Lambda_2 \in N_0$, then the relations (20d),(21e) imply $\Lambda_1 \leq m_{12} \leq m_{13} = \Lambda_1+\Lambda_2$. In another words, one cannot reach in this case the vectors with $m_{12} > m_{13}$ applying successively the operators (20) to the "vacuum" (22). On the other hand, there are no additional restrictions to m_{12} if $\Lambda_2 \notin N_0$. As to m_{22} , the relations (20d) and (21f) show that $m_{22} = 0,1,\ldots,\Lambda_1$ if $1+\Lambda_1+\Lambda_2 \notin N_0$, or if $1+\Lambda_1+\Lambda_2 \in N_0$ and $\Lambda_2 \geq -1$. On the other hand, if $1+\Lambda_1+\Lambda_2 \in N_0$ and $\Lambda_2 \leq -2$, the values $m_{22} = 0,1,\ldots,1+\Lambda_1+\Lambda_2$ are admitted only. The remaining relations impose no restrictions to the indices m_{12}, m_{22}, m_{11} of the vectors contained in $\rho_\Lambda(UL)x_0$.

The described procedure of selecting the irreducible component of ρ_Λ is sketched on Fig.2 . The range of m_{11} is finite (cf.(19)) and may be eventually plotted at the third axis. Let us summarize the above discussion :

<u>Theorem 4.1</u> : (a) Let $\Lambda_2 \in N_0$, then $V_\Lambda \equiv \rho_\Lambda(UL)x_0$ is spanned by the vectors (19) with $\Lambda_1 \leq m_{12} \leq \Lambda_1+\Lambda_2$, $0 \leq m_{22} \leq \Lambda_1$ and $m_{22} \leq m_{11} \leq \leq m_{12}$. Consequently, ρ_Λ is reducible and the irreducible highest-weight representation $\tilde{\rho}_\Lambda = \rho_\Lambda \restriction V_\Lambda$ is finite-dimensional.
(b) If $1+\Lambda_1+\Lambda_2 \notin N_0$ or $\Lambda_2 = -1$, then $V_\Lambda = V$ so ρ_Λ is irreducible.
(c) If $1+\Lambda_1+\Lambda_2 \in N_0$ and $\Lambda_2 \leq -2$, then V_Λ is spanned by the vectors (19) with $m_{12} \geq \Lambda_1$, $0 \leq m_{22} \leq 1+\Lambda_1+\Lambda_2 \leq \Lambda_1 - 1$ and $m_{22} \leq m_{11} \leq m_{12}$. Consequently, ρ_Λ is reducible and the irreducible highest-weight representation $\tilde{\rho}_\Lambda$ is infinite-dimensional.

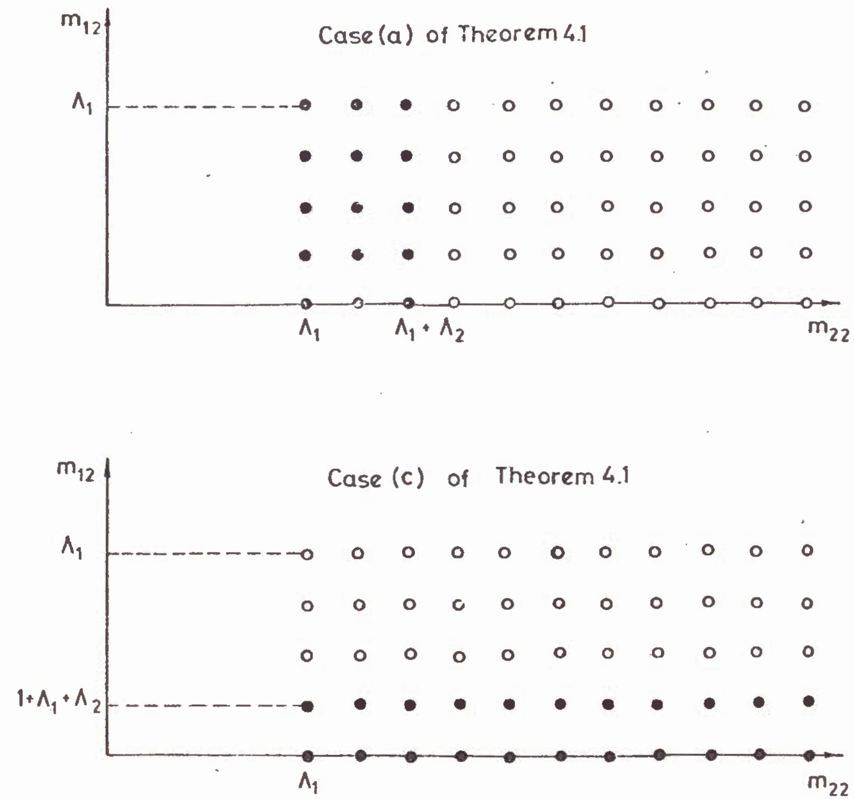

Fig.2. Illustration to the reduction process.

Let us comment on these results. First of all, the assertion (a) shows how one can get the standard irreducible highest-weight representations from ρ_Λ for $\Lambda \in \Omega_{fin}$. Actually, the formulae (20),(21) are almost the same as those obtained by Gel'fand-Zetlin method (see Ref.7, Section 10.1), the differences being due to different normalization.

Furthermore, the assertions (b),(c) make it possible to answer the main question we addressed ourselves in this paper. Consider a given $\Lambda \in \Omega(1,12)$. If $\Lambda_2 = -1$, then ρ_Λ itself is the sought irreducible highest-weight representation. On the other hand, if $\Lambda_2 \leq -2$, one has to restrict ρ_Λ to the infinite-dimensional subspace V_Λ specified in the assertion (c). The case $\Lambda \in \Omega(2,12)$ may be dealt with using the automorphism τ of sl(3,C) (cf.the relations (16) of Ref.3) that permutes h_1 and h_2. The irreducible highest-weight representations referring to $\Lambda_1 = -1$ and $\Lambda_1 \leq -2$ are then $\rho_\Lambda \circ \tau$ and $\tilde{\rho}_\Lambda \circ \tau$, respectively.

Acknowledgement

The authors are grateful to Drs.Č.Burdík and M.Havlíček for useful discussions.

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Received by Publishing Department
on April 13, 1984.

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Экснер П., Навратил О. E2-84-244

О полном наборе неприводимых представлений со старшим весом для sl(3,**C**)

Обсуждается проблема построения неприводимых представлений со старшим весом для алгебры Ли sl(3,**C**). Метод, основанный на канонических реализациях, который рассматривался в недавней серии работ, пополнен здесь процедурой приведения по отношению к подалгебре gl(2,**C**). Этим путем получены неприводимые представления со старшим весом /выраженные явным образом при помощи матричных элементов генераторов/ для всех весов, включая конечномерные случаи.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1984

Exner P., Navrátil O. E2-84-244

On a Complete Set of Irreducible Highest-Weight Representations for sl(3,**C**)

The problem of constructing irreducible highest-weight representations of the Lie algebra sl(3,**C**) considered in a recent series of papers is revisited. The construction based on canonical (boson) realizations that has been presented there is amended by a reduction procedure with respect to a subalgebra gl(2,**C**). It yields irreducible highest-weight representations (expressed explicitly through matrix elements of the generators) for all weights, including the finite-dimensional cases.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1984