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N.P.Ilieva

VACUUM PROBLEM IN THE SCHWINGER MODEL

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1. Introduction

The problem of vacuum structure in QCD is a matter of current interest. In the non-Abelian gauge theories (as QCD) approximate calculation methods and analysis used are usually motivated by different assumptions (sometimes only because of their symplicity) and often lead to inconsistent results. An experimental test of most of these assumptions is still not possible. Important conclusions may be drawn by comparing the results of exact and approximate calculations in exactly solvable field-theoretical models.

From such a point of view the vacuum structure of the Schwinger model/1/ is very attractive. This model, as a theory of fermions in one spatial dimension, may be bosonized/2,3,4/ thus loosing the chiral symmetry of the initial classical Lagrangian. For its restitution the so-called θ -vacuum/3/ is introduced, so the theory acquires two parameters - e and θ . Later on it has been pointed out/5/ that this procedure is equivalent to the introduction of a constant classical electric field defined by this new parameter θ . The same θ -vacuum is used when considering the massive Schwinger model. However, there arises the question/5/: why is it necessary to introduce such an additional parameter θ in this case too, if the chiral invariance is broken from the beginning by the mass term for $\Psi(x)$?

In paper/6/ a new interpretation of θ -vacuum in the Schwinger model was proposed which is not connected with an artificial restitution of the classical Lagrangian chiral invariance. It is shown that the existence of the θ -vacuum is caused by nontrivial topological properties of the gauge field. In such an approach Coleman's constant electric field appears as a quantum physical observable because of discontinuity of the phase of the gauge field state function. The eigenvalues of this field are labelled by the Brillouin zone number k and an angle θ :

$$F = p\frac{e}{2\pi} = (2\pi k + \theta)\frac{e}{2\pi} \quad , \quad k = 0, \pm 1, \pm 2, \ldots$$

Here p is a canonical momentum conjugated to the topological variable $\mathcal{N}$

$$\mathcal{N}[A] = \frac{e}{2\pi}\int dx\, A_1(x).$$

This variable describes the gauge field infrared (longitudinal) dynamics and is a continuous analog of the Pontryagin index

$$\nu = \frac{e}{2\pi}\int dx\, dt\, \epsilon_{\mu\nu}F^{\mu\nu} = \int dt\, \dot{\mathcal{N}} = \mathcal{N}\Big|_{t=\infty} - \mathcal{N}\Big|_{t=-\infty} \neq 0.$$

So, dynamical treatment of this quantity appears as a main feature of the approach proposed in /6/.

In the present paper it is shown that in the Schwinger model a nontrivial topology leads to a physical vacuum with a structure of a coherent state of observable fields. Part 2 is devoted to the model-Hamiltonian construction when quantum motions of bosonic and fermionic vacua are taken into account. In part 3 the ground-state structure proposed is motivated. This ground state is used to obtain the quark condensates and Green's functions for the currents that is a subject of part 4. Conservation of the cluster property and validity of Wilson's expansion in the Schwinger model are discussed therein.

2. The Role of Fermionic and Bosonic Vacua Motions in the Schwinger Model

The Schwinger model, massless quantum electrodynamics in two-dimensional space-time, is given by

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(x)\left(i\gamma^{\mu}\partial_{\mu} - e\gamma^{\mu}A_{\mu}\right)\Psi(x) \tag{1}$$
$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \quad , \mu,\nu = 0,1.$$

Consideration of a simple example - free gauge field in the gauge $A_0 = 0$, is enough to convince us that the nontrivial gauge field topology has an essential influence on model properties. In this gauge we have

$$\mathcal{L}_{em} = \frac{1}{2}\left(\partial_0 A_1\right)^2. \tag{2}$$

This Lagrangian is invariant under stationary gauge transformations

$$A_1'(x,t) = e^{i\lambda(x)}\left(A_1(x,t) + \frac{i}{e}\partial_1\right)e^{-i\lambda(x)}. \tag{3}$$

In the classical theory this leads to

$$A_1'(x,t) = A_1(x,t) + \frac{1}{e}\partial_1\lambda(x). \tag{4}$$

Gauge transformations are given by the function $\lambda(x)$, determined on the line $R(1)$, with the boundary condition

$$\lim_{|x|\to\infty}\lambda(x) = 0.$$

However, in the quantum theory the right-hand sides of eqs. (3) and (4) are not always equal to each other, because as a basic element of gauge transformations there appears the Weyl phase $\exp\{i\lambda(x)\}$ determined on the circle. The requirement of the state-function gauge invariance leads to the following boundary condition for this phase

$$\lim_{|x|\to\infty} e^{i\lambda(x)} = 1$$

or

$$\lambda^{(n)}(\infty) - \lambda^{(n)}(-\infty) = 2\pi n \;, \quad n = 0, \pm 1, \pm 2, \dots \tag{5}$$

Thus, we have a map of the line $R(1)$ onto the circle $U(1)$ characterized by an integer n (which indicates the number of rotations of $R(1)$ around $U(1)$). This nontrivial topology disappears in the classical limit but in the quantum theory the gauge-field configuration space $\{A_1^{(n)}(x)\}$ is not simply connected and the "points"

$$A_1^{(n)}(x,t) = e^{i\lambda^{(n)}(x)}\left(A_1(x,t) + \frac{i}{e}\partial_1\right)e^{-i\lambda^{(n)}(x)} \;, \quad n = 0, \pm 1, \pm 2, \dots \tag{6}$$

are physically identical that leads to the state-function phase discontinuity

$$\Psi[A^{(n+1)}] = e^{i\theta}\Psi[A^{(n)}] \;, \quad |\theta| < \pi . \tag{7}$$

Condition (7) determines the constant electric field spectrum mentioned above

$$F\Psi = p\frac{e}{2\pi}\Psi = \frac{e}{2\pi}(2\pi k + \theta)\Psi \;, \quad k = 0, \pm 1, \pm 2, \dots \tag{8}$$

and points to the existence of a field analogy of Josephson's effect (k is the Brillouin zone number). This undamped collective motion of the gauge vacuum is described by the following Hamiltonian [6]

$$H_{coll} = \frac{1}{2}\left(\frac{ep}{2\pi}\right)^2 V = \frac{p^2}{2\mu} \;, \quad V = \int dx \tag{9}$$

which coincides with the one for a free particle with a mass

$$\mu = \frac{1}{V}\left(\frac{2\pi}{e}\right)^2 .$$

The collective-motion Hamiltonian may be directly obtained projecting the action and Pontryagin index as a dynamical variable onto the general solution of the constraint equation

$$\frac{\delta S}{\delta A_0} = 0 \longrightarrow A_0 = \partial_1^{-1}\partial_0 A_1 + Cx .$$

This method is used in paper [6] to obtain the Schwinger-model Hamiltonian, which differs from Coleman's one [5] by the last term.

(9). This is due to the choice of the gauge $A_1 = 0$. In this case the gauge field is not a dynamical variable, that is why it is more difficult to establish its longitudinal dynamics. For this reason one must take into account that the connection between two gauges $A_0 = 0$ and $A_1 = 0$ is determined up to an integration constant which may be fixed by the gauge field topology. This may be done by the following substitution in the Hamiltonian of [5]

$$e\partial_1^{-1}j_0 \longrightarrow e\partial_1^{-1}j_0 - E \;, \quad E = -\frac{pe}{2\pi} .$$

Thus, Coleman's Hamiltonian coincides with the one from [6]:

$$H = \int dx\left[i\bar\Psi(x)\gamma_1\partial_1\Psi(x) - \frac{e^2}{2}j_0(x)\left(\partial_1^{-2}\right)j_0(x) - e^2\frac{p}{2\pi}x\,j_0(x) + \frac{e^2}{2}\left(\frac{p}{2\pi}\right)^2\right] . \tag{10}$$

It is not difficult to obtain the equivalent bose-form of the theory. As is known, in the axial current commutator there appears an anomalous term

$$\left[j_{50}(x), j_{51}(y)\right] = \frac{1}{\pi i}\partial_y\delta(x-y) \tag{11}$$

due to the negative-energy-state filling , required for the positive definiteness of the free-fermion Hamiltonian [1,5].

The substitution

$$j_{5\mu}(x) = \frac{1}{\sqrt{\pi}}\partial_\mu\phi(x) \tag{12}$$

transforms relation (11) into the scalar field $\phi(x)$ commutator. The equivalent bosonic Hamiltonian has the form

$$H_B = \frac{1}{2}\int dx\left[\pi^2 + (\partial_1\phi)^2 + M^2\left(\phi - \frac{p}{2\sqrt{\pi}}\right)^2\right] , \tag{13}$$

where $\pi = \partial_0\phi$ is a canonical momentum for the field $\phi(x)$, $M^2 = e^2/\pi$.

The chiral charge in the model

$$Q_5 = \int dx\, j_{50}(x) = \frac{1}{\sqrt{\pi}}\int dx\,\partial_0\phi(x) \tag{14}$$

is not conserved. Thus, the quantum-theory chiral invariance is broken by the "polarization" of the Dirac vacuum caused by the gauge field rather than by the topological degeneration of the latter.

Hamiltonian (13) is not invariant under chiral transformations. However, there appears an invariance under simultaneous transformations of the observable fields

$$\sigma^n H_B \sigma^{-n} = H_B \;, \quad \sigma^n = \exp\{i\pi n(Q_5 - 2N)\} . \tag{15}$$

3. The Vacuum Structure

As is mentioned above, the gauge field topological degeneration is not connected with the chiral-symmetry breaking of the initial classical Lagrangian. Nevertheless, this degeneration plays an essential role in the Schwinger model. It is just this role we are now going to study.

It follows from (13), that the theory describes a massive scalar field

$$\tilde{\phi}(x) = \phi(x) - \frac{p}{2\sqrt{\pi}}.$$

It is this compound field that vanishes when $x \to \pm\infty$. Hence, our task is to construct a vacuum state that ensures the validity of the relation

$$\langle vac|\, \tilde{\phi}(x)\, |vac\rangle = 0. \tag{16}$$

Let us write the field operator $\phi(x)$ as a sum of a positive- and negative-frequency parts

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

and denote by $a^{\pm}(k)$ the creation and annihilation operators of a ϕ - boson with momentum k. The vacuum state in the Fock space of these operators is defined as

$$\phi^-|0\rangle = 0, \quad \langle 0|\phi^+ = 0, \quad \langle 0|0\rangle = 1,$$

i.e.,

$$\langle 0|\phi|0\rangle = 0. \tag{17}$$

So, vectors

$$|n\rangle = \frac{a^+(k)^n}{\sqrt{n!}}|0\rangle$$

contain the whole information about the ϕ -bosons in a fixed state.

However, the vacuum state of the whole theory must also satisfy one more condition (see (8)):

$$p|vac\rangle = (2\pi k + \theta)|vac\rangle. \tag{18}$$

From definition (4) and substitution (12) the relation

$$[Q_5, \phi] = -i/\sqrt{\pi} \tag{19}$$

follows.

Owing to (19) and to the commutativity of topological momentum p and chiral charge Q_5 we arrive at the following vacuum structure in the Schwinger model

$$|vac\rangle = exp\{-\tfrac{1}{2} i p Q_5\}|0\rangle. \tag{20}$$

Thus, the vacuum represents a coherent state of the observable fields. It is not difficult to check that this structure ensures the validity of (16) what means that there takes place

$$\langle vac|\, \phi - \frac{p}{2\sqrt{\pi}}\, |vac\rangle = 0.$$

4. Matrix Elements and Two-Point Green's Functions for the Currents

Now we are going to calculate some quantities, for example, quark condensates $\langle J(x)\rangle$ and $\langle J_5(x)\rangle$, using the vacuum structure obtained above. Here

$$\begin{aligned} J(x) &= \bar{\Psi}(x)\Psi(x), \\ J_5(x) &= \bar{\Psi}(x)\gamma_5\Psi(x). \end{aligned} \tag{21}$$

We need explicit expressions for these currents in terms of the bosonic field $\phi(x)$.

Let the two-component spinors $\Psi(x), \bar{\Psi}(x)$ have the form

$$\Psi(x) = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix}, \quad \bar{\Psi}(x) = (\Psi_L^+, \Psi_R^+)$$

and γ-matrices are

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then currents are written as

$$\begin{aligned} J(x) &= \Psi_L^+(x)\Psi_R(x) + \Psi_R^+(x)\Psi_L(x) \\ J_5(x) &= \Psi_L^+(x)\Psi_R(x) - \Psi_R^+(x)\Psi_L(x) \\ j_{50}(x) &= \Psi_L^+(x)\Psi_L(x) - \Psi_R^+(x)\Psi_R(x) \end{aligned} \tag{22}$$

The spinors $\Psi(x)$ and $\bar{\Psi}(x)$ satisfy canonical equal-time commutation relations. So, we find

$$\begin{aligned} [j_{50}(x), \Psi_L^+(y)\Psi_R(y)] &= 2\delta(x-y)\Psi_L^+(y)\Psi_R(y), \\ [j_{50}(x), \Psi_R^+(y)\Psi_L(y)] &= -2\delta(x-y)\Psi_R^+(y)\Psi_L(y). \end{aligned} \tag{23}$$

From Lagrangian (1) and relation (12) it follows that $j_{50}(x)$ is proportional to the momentum, conjugated to the field $\phi(x)$

$$j_{50}(x) = \frac{1}{\sqrt{\pi}}\partial_0\phi(x) = \frac{1}{\sqrt{\pi}}\Pi(x).$$

Introducing the notation

$$\begin{aligned} \Psi_L^+(x)\Psi_R(x) &= F_+[\phi(x)], \\ \Psi_R^+(x)\Psi_L(x) &= F_-[\phi(x)] \end{aligned} \tag{24}$$

(these are just the terms we want to express by $\phi(x)$) we may rewrite (22) as

$$[\Pi(x), F_\pm[\phi(y)]] = \pm 2\sqrt{\pi}\,\delta(x-y)F_\pm[\phi(y)].$$

This is equivalent to the following functional equations

$$\frac{\delta}{\delta\phi(x)} F_\pm[\phi(y)] = \pm 2i\sqrt{\pi}\,\delta(x-y)F_\pm[\phi(y)].$$

Their solutions are

$$F_\pm[\phi(x)] = \mathcal{K}\exp\{\pm 2i\sqrt{\pi}\,\phi(x)\}. \qquad (25)$$

The value of integration constant $\mathcal{K}$ may be easily obtained (see /8/).

Relations (22),(24), and (25) give us the following correspondence formulas

$$J(x) = 2\mathcal{K}\cos 2\sqrt{\pi}\,\phi(x), \qquad (26)$$
$$J_5(x) = 2i\mathcal{K}\sin 2\sqrt{\pi}\,\phi(x).$$

Note that the first of these relations allows us to bosonize the massive Schwinger model too, because the mass term has the same structure as $J(x)$. All other considerations (concerning the topological degeneration and Dirac vacuum "polarization") are valid in this case, too.

So, we obtain the equivalent bosonic Hamiltonian for the massive model

$$H_B = \frac{1}{2}\int dx\left[\Pi^2 + (\partial_1\phi)^2 + M^2\left(\phi - \frac{\rho}{2\sqrt{\pi}}\right)^2 + 2\mathcal{K}m\cos 2\sqrt{\pi}\phi\right].$$

It has the same invariance (15) as the bosonized massless one. Thus, Coleman's question about the third-parameter introduction in the massive model takes no longer place.

Using definition (20) and relations (17-19) we find

$$\langle vac|J(x)|vac\rangle = 2\mathcal{K}\cos\theta, \qquad (27)$$
$$\langle vac|J_5(x)|vac\rangle = 2i\mathcal{K}\sin\theta.$$

Then it follows that

$$\langle vac|J^\pm(x)|vac\rangle = \mathcal{K}e^{\pm i\theta}, \qquad (28)$$

where

$$J^\pm(x) = \frac{1}{2}\left(J(x) \pm J_5(x)\right).$$

It is not difficult to obtain vacuum expectation values of different products of currents $J^\pm(0)$ (with the assumption that a product of two operators at one and the same point is equal to their normal product):

$$\langle vac|J^\pm(0)J^\pm(0)|vac\rangle = \langle vac|:e^{\pm 2i\sqrt{\pi}\,2\phi(0)}:|vac\rangle = \mathcal{K}^2e^{\pm 2i\theta} \qquad (29)$$
$$\langle vac|J^\pm(0)J^\mp(0)|vac\rangle = \mathcal{K}^2.$$

These results allow us to made some conclusions:

1. Both condensates $\langle J\rangle$ and $\langle J_5\rangle$ differ from 0 because of the nontrivial gauge field topology. The variable θ has a dynamical content, it is a quantum physical observable. So, it is not correct to put it equal to 0. This is the reason for the difference between (27) and the corresponding result in paper /9/.

2. There is a difference between vacuum expectation values of current products when $x=0$ (see (29)) just for the same reason. However, at the same time a conclusion made in /9/ is confirmed about the contradiction of these results with those obtained under the assumption in /10/.

3. Cluster property is conserved, i.e.,

$$\lim_{x\to\infty}\langle vac|T(J^+(x)J^-(0))|vac\rangle = \langle vac|J^+|vac\rangle\langle vac|J^-|vac\rangle.$$

Indeed, we have for the right-hand side

$$\lim_{x\to\infty}\langle vac|\mathcal{K}^2:e^{2i\sqrt{\pi}(\phi(x)-\phi(0))}:|vac\rangle e^{-4\pi i\Delta(x)} = \lim_{x\to\infty}\mathcal{K}^2e^{-4\pi i\Delta(x)} = \mathcal{K}^2$$

that agrees with the left-hand side value according to (28). Here $\Delta(x)$ is Green's function of the massive scalar field, which tends to 0 when $x\to\infty$.

4. From (16), (18) it follows that the vacuum expectation value of the field $\phi(x)$ is

$$\langle vac|\phi(x)|vac\rangle = \frac{1}{2\sqrt{\pi}}(2\pi k + \theta).$$

So, for any function of $\phi(x)$ the vacuum expectation value of each term of Taylor's series and of each its finite sum will depend on the Brillouin zone number k, when the vacuum expectation value of the whole series will not depend on it. Hence, the Wilson expansion is not always applicable in this model (except the case of zero zone, when $k=0$).

Conclusion

Quantization of a two-dimensional Abelian gauge field in accordance with its topological properties gives rise to an additional infrared physical observable, which depends on an integer k (the Brillouin zone number) and on an angle θ /5/.

The influence of this observable (hence, of the gauge field topology) on the vacuum structure in the Schwinger model is shown. The ground state of the system is constructed to satisfy the requirement of an assymptotical-vanishing value of the massive bosonic field. This state represents a coherent state of the observable fields characterized by quantum numbers mentioned above. Quark-condensates values and Green's functions for currents are obtained. Some differences from the results when the variable θ is not considered as an observable are discussed. The vacuum structure proposed preserves cluster property but at the same time shows that in the Schwinger model the method suggested in /10/ is not applicable and Wilson's expansion is valid only in the Brillouin zone with $k=0$.

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Проблема вакуума в модели Швингера

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Препри... ...Дубна 1984

Ilieva N.P. E2-84-228

Vacuum Problem in the Schwinger Model

The Schwinger model with quantum motions of bosonic and fermionic vacua is considered. The physical vacuum is defined as a coherent state of the observable fields. Quark-condensate values are found, which depend on the gauge field topological degeneration parameter.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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