объединенный
институт
ядерных
исследований
дубна

E2-84-197

R.M.Yamaleev

QUANTUM SPINNING TOP IN FOUR-DIMENSIONAL EUCLIDEAN SPACE AS A MACROSCOPIC ANALOG OF SPIN 1/2

Submitted to "Ядерная физика"

1984

INTRODUCTION

The spin is a pure quantum-mechanical concept and has no classical analog. A formal reason for this is that the spin operators are proportional to the Planck constant and matrices realizing a finite-dimensional representation of the rotation group. However, a macroscopic analog of the spin may be constructed with the aid of operators realizing infinite-dimensional representations of the rotation group[1]. Dynamic quantities corresponding to such operators have a meaning in classical mechanics as well. For instance, an infinite-dimensional analog of the spin-1 operator is the operator of angular momentum, and a macroscopic analog is the usual spinning top. Thus, the spin-1,as compared to other spins, has a distinct status as it bears a tight relation to the three-dimensionality of real space. Formally, this can be interpreted as a consequence of the 3-dimensionality of the spin-operator representation. However, the fact that the structure of the Hamilton operator for a particle with spin-1 is uniquely determined by a covariant coupling of the energy with angular momentum points to a deep content of that coupling[2]. In the general case infinite-dimensional analogs of the spin operators are defined in n-dimensional space, where n is determined by the corresponding spin S: $n = 2S + 1$ for integer and $n = 2(2S + 1)$ for half-integer spins. The most attention is to be paid to the fundamental spin, spin-1/2. A macroscopic analog of spin-1/2 is a model of the spinning top the elements of which move in the 4-dimensional Eucledean space. The model possesses remarkable properties: The direction of angular momentum of the top always coincides with that of angular velocoty, the moment of inertia is a scalar, and the equations of motion are linear. The matrix of finite rotation is also linear which is defined in the 4-dimentional space through the coordinates of 3-dimensional hypersphere, and the formula for angular momentum of the top given on the rotation group coincides with the formula of an infinite-dimensional analog of the spin-1/2 operator. These facts, undoubtedly, give evidence that the theory of rotation including half-integer representations assumes a canonical form in the 4-dimensional Eclidean space. In this space the Hamilton structure for a spin-1/2 particle can be determined on the basis of the covariant coupling of energy with the infinite-dimensional analog of the spin operator [3]. Hence, according to these conclusions the fundamental spin of nature is an occupant of the 4-dimensional Euclidean space* .

1. INFINITE-DIMENSIONAL ANALOGS OF THE SPIN OPERATORS

Consider the behaviour of a scalar function of a vector argument (radius-vector) in the case when the coordinate system rotates around the unit vector $\vec{n}$ by angle θ. It is given by the known formulas:

$$\Phi'(\vec{r}\,') = \exp(i\theta(\vec{n}\vec{M})/h)\,\Phi(\vec{r}) = \Phi(T\vec{r}). \qquad (1.1)$$

Here $\vec{M}$ is the orbital moment

$$\vec{M} = [\vec{r} \times \vec{p}] = -ih[\vec{r} \times \frac{\partial}{\partial \vec{r}}], \qquad (1.2)$$

T is the finite-rotation matrix that is expressed in terms of infinitesimal operators of the group of three-dimensional rotations

$$\hat{T}(\vec{n}) \equiv \exp(\theta(\vec{n}\vec{\tau})). \qquad (1.3)$$

In the Cartesian basis the operator of spin-1 is determined through matrices τ_k $(k = x, y, z)$:

$$S_k = -ih\,\tau_k\,, \qquad (1.4)$$

so that the expression (1.3) can be written in the form

$$\hat{T}(\vec{n}) \equiv \exp(i\theta(\vec{n}\vec{S})/h). \qquad (1.5)$$

Using the properties of τ-matrices the orbital moment (1.2) can be defined as follows:

$$M_k = \sum_{n,m}^{3} r_n\,\tau_k^{nm}\,p_m = \sum_{n,m}^{3} r_n\,S_k^{nm}\frac{\partial}{\partial r_m}, \qquad (1.6)$$

and the commutation relations for M_k in this case are derived as a consequence of commutation relations for r_n and p_m

$$[r_n, p_m] = ih\,\delta_{nm} \qquad (1.7)$$

and τ-matrices

$$[\tau_i, \tau_k] = \sum_{j=1}^{3} \epsilon_{ikj}\,\tau_j\,; \qquad (1.8)$$

---

*In the present work we restrict our consideration to the nonrelativistic theory.

In view of the above relations between M_k and S_k the operator M_k will be called an infinite-dimensional analog of the spin-1 operator. This concept may be equally generalized to the case of an arbitrary spin, once the corresponding representation of the spin operator is found in the Cartesian basis.

To find the infinite-dimensional analog of the 1/2-spin operator, we shall consider generators of a six-parameter group of transformations of the 4-dimensional Euclidean space

$$\vec{M} = [\vec{r} \times \vec{p}], \quad N = r_4\vec{p} - \vec{r}p_4. \qquad (1.9)$$

To the operators (1.9) there correspond finite-dimensional operators τ_M and τ_N represented by antisymmetric 4x4 matrices. We shall now form of the vectors $\vec{M}$ and $\vec{N}$ two 3-dimensional vectors

$$\vec{\mathfrak{M}}_\pm = \vec{M} \pm \vec{N} \qquad (1.10)$$

and introduce the matrices

$$\vec{\tau}_\pm = \vec{\tau}_M \pm \vec{\tau}_N\,. \qquad (1.11)$$

The matrices $-i\vec{\tau}_\pm$ are analogs of the Pauli matrices, and the operator

$$\vec{S}_\pm = -ih\,\tau_\pm/2 \qquad (1.12)$$

is the 1/2-spin operator in the Cartesian basis. In analogy with (1.6) the operators (1.10) can be expressed in terms of the spin matrices (1.11):

$$\mathfrak{M}_{\pm k} = \sum_{\mu,\nu}^{4} r_\mu\,\tau_{k\pm}^{\mu\nu}\,p_\nu\,. \qquad (1.13)$$

Thus, $\vec{\mathfrak{M}}_\pm/2$ is an infinite-dimensional analog of the 1/2-spin operator and obeys the same commutation relations as the orbital momentum. According to the general theory of such operators

$$(\vec{\mathfrak{M}}_\pm/2h)^2\,\Psi = \ell(\ell+1)\,\Psi, \quad (\vec{\mathfrak{M}}_\pm/2h)_z\,\Psi = \pm m\,\Psi. \qquad (1.14)$$

From the theory of hyperspherical functions it follows that

$$\frac{(\vec{N} \pm \vec{M})^2}{h^2}\,\Psi = L(L + (f-2))\,\Psi, \qquad (1.15)$$

where f = 4 is the dimensionality of space, L = 0,1,2,3,...

From (1.15) we obtain

$$\left(\frac{\vec{M} \pm \vec{N}}{2h}\right)^2\Psi = (\vec{\mathfrak{M}}_\pm/2h)^2\,\Psi = \frac{L}{2}\left(\frac{L}{2} + 1\right)\Psi$$

and, as a result, $\ell = L/2$ = 0, 1/2, 1, 3/2 and so on, m = 0, ±1/2, ±1, ±3/2 and so on, i.e., the eigenvalues of the operator $\vec{\mathfrak{M}}_\pm/2$ are multiple to h/2.

2. CONNECTION OF GENERATOR OF THE ROTATION GROUP WITH HAMILTON OPERATORS FOR PARTICLES WITH SPIN-1 AND 1/2

In classical mechanics an important relation is the relation

$$H = p^2/2m. \tag{2.1}$$

That determines the structure of the Hamilton-Jacobi equation. In quantum mechanics the relation (2.1) represents an expression for the Hamilton operator that determines the structure of the Schrödinger equation describing the motion of a spinless particle. It may not be out of place to raise the question: Is it possible to generalize the relation (2.1) so that the resulting relation between dynamic quantities (operators) will uniquely define the structure of wave equations for particles with spin? First of all, we are interested in the spin 1/2 and 1. Let us now consider the case of spin 1. We will express the Hamilton operator (2.1) through the angular momentum. The expression sought can be easily obtained if in (2.1) we pass to the spherical coordinates. However, such a connection is noncovariant as it depends on the choice of the coordinate system. We will be a success if we express the sought relation in the vector form

$$\vec{r}\, 2mH = -[\vec{M} \times \vec{p}] + s\vec{p} \qquad s = (\vec{r}\vec{p}). \tag{2.2}$$

If we take a scalar product of (2.2) and $\vec{r}/\vec{r}^{\,2}$ (the relation (2.2)being multiplied by $\vec{r}/\vec{r}^{\,2}$ from the left) and apply the commutation relations

$$[r_i , p_k] = ih\,\delta_{ik} , \tag{2.3}$$

we get the corresponding expression for the energy operator in spherival coordinates

$$2mH = p_r^2 - 2ihp_r/r + M^2/r^2, \quad p_r = -ih\partial/\partial r, \tag{2.4}$$

where M is the angular-momentum operator dependent on angular variables only.

An important property of the system (2.2) is observed when a particle moves in an external electromagnetic field. To see this, we introduce the field potentials $(\vec{A}, \phi)$ in a canonical gauge-invariant way. In this case

$$\vec{p} \to \vec{p} + \frac{e}{c}\vec{A}, \quad H \to H - e\phi. \tag{2.5}$$

Substituting (2.5) into (2.2) and taking account of the commutation relations

$$[p_i , p_k] = ih\epsilon_{ike}\, \mathcal{H}_e , \tag{2.6}$$

where $\mathcal{H}_e$ are components of the vector of the magnetic-field strength, we obtain

$$\vec{r}H = \vec{r}p^2/2m - (ih\mu)[\vec{r} \times \vec{\mathcal{H}}], \quad \mu = e/2mc. \tag{2.7}$$

Using the matrices $\vec{r}(r_x , r_y , r_z)$ we rewrite the latter expression in the form

$$\begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} [H - (p^2/2m + \mu(-ih\vec{r}\vec{\mathcal{H}}))] = 0. \tag{2.8}$$

Here the square brackets contain the Hamilton operator for a particle with spin+1. To derive the operator H, we multiply (2.7) (or(2.8)) from the left by $\vec{r}/r^2$ in a scalar way, and as a result, we come back to the relation (2.1), the Hamilton operator for a spinless particle. So, the task consists in a correct separation of the operator H from (2.8) without losing an important term, the operator of interaction with the magnetic field.

In the 3-dimensional space we introduce the Cartesian coordinates that compose a real orthonormal basis

$$(\vec{e}_i\, \vec{e}_k) = \delta_{ik} . \tag{2.9}$$

We shall make use of the right-hand sided system of coordinates, so that

$$[\vec{e}_i \times \vec{e}_k] = \epsilon_{ikj}\vec{e}_j , \quad \epsilon_{123} = 1. \tag{2.9a}$$

The homogeneity and isotropic nature of space allow on arbitrary choice of the coordinate basis. Consequently, the components of radius-vector r_k $(k = x, y, z)$ in the expansion

$$\vec{r} = \vec{e}_k r_k \tag{2.10}$$

are arbitrary quantities. Thus, inserting (2.10) into (2.2) and equating expressions of each r_k we get the system of three vector equations

$$\vec{e}_k\, 2mH = [[\vec{e}_k \times \vec{p}] \times \vec{p}] + (\vec{e}_k \vec{p})\vec{p}. \tag{2.11}$$

Multiplying (2.11) from the left by $\vec{e}_i$ and applying (2.9) and (2.9a) we obtain

$$2mH\,\delta_{ik} = p^2\delta'_{ik} + \mu(-ih\,\vec{r}_{ik\ell}\, \vec{\mathcal{H}}_\ell) \tag{2.12}$$

the Hamilton operator for a particle with spin 1. At the same time we have found matrices for spin 1 expressed in terms of basis vectors $\vec{e}_k$ /4/: $(r_k)_{ij} = (\vec{e}_k [\vec{e}_i \times \vec{e}_j])$.

The method presented above for obtaining the energy operator for a spin 1 particle makes the coupling between operators $\vec{M}$ and $\vec{r}$ from (1.6) meaningful. The classical system (2.2) in the known principle of correspondence (between quntum and classical mechanics) plays the same role for a spin 1 particle as the expression (2.1) does for a spinless particle /2/. It seems that such merits the spin 1 should be a fundamental spin, nevertheless this part belongs to the spin 1/2. Thus, our next step is to modify the scheme presented above as to obtain the Hamilton operator for particles with spin 1/2.

Note that the system (2.2) may be written in another form by using the formula

$$(\vec{\sigma}\vec{A})(\vec{\sigma}\vec{B}) = (\vec{A}\vec{B}) + i(\vec{\sigma}[\vec{A}\times\vec{B}]), \qquad (2.12a)$$

where $\sigma(\sigma_x, \sigma_y, \sigma_z)$ is the basis of Pauli matrices. Then, instead of (2.2) we get

$$s + i(\vec{\sigma}\vec{M}) = (\vec{\sigma}\vec{r})(\vec{\sigma}\vec{p}), \qquad (\vec{\sigma}\vec{r})2mH = (s + i(\vec{\sigma}\vec{M}))(\vec{\sigma}\vec{p}). \qquad (2.13)$$

However, the systems (2.2) and (2.13) are equivalent to each other provided there is no external field present, i.e., there holds the equality $(\vec{M}\vec{p}) = 0$. If the electromagnetic field is introduced by (2.5), than

$$(\vec{M}\vec{p}) = (\vec{r}[\vec{p}\times\vec{p}]) = \frac{ihe}{c}(\vec{r}\vec{\mathcal{H}}) \neq 0. \qquad (2.14)$$

Transforming (2.13) with the use of (2.6) and (2.14) we obtain

$$(\vec{\sigma}\vec{r})H = (\vec{\sigma}\vec{r})(p^2/2m + \mu h(\vec{\sigma}\vec{\mathcal{H}})) =$$
$$= (\vec{\sigma}\vec{r})(\vec{p}^2/2m) + i(\vec{\sigma}[\vec{r}\times\vec{\mathcal{H}}]h\mu + h\mu(\vec{r}\vec{\mathcal{H}}). \qquad (2.15)$$

Comparing expressions of the basis units $(I, i\sigma_x, i\sigma_y, i\sigma_z)$ in the left- and right-hand sides of (2.15) we verify that $(\vec{r}\vec{\mathcal{H}}) = 0$. This condition is removed if the fourth dimensional (r_4 and p_4) is introduced and (2.13) is generalized as follows:

$$(r_4 - i(\vec{\sigma}\vec{r}))(p_4 + i(\vec{\sigma}\vec{p})) = s + i(\vec{\sigma}\vec{\mathfrak{M}}),$$
$$(r_4 - i(\vec{\sigma}\vec{r}))2mH = (s + i(\vec{\sigma}\vec{\mathfrak{M}}))(p_4 - i(\vec{\sigma}\vec{p})). \qquad (2.16)$$

Then from (2.16) we obtain the Hamilton operator for a particle with spin 1/2 in the 4-dimensional Euclidean space:

$$2mH = (p_4 + i(\vec{\sigma}\vec{p}))(p_4 - i(\vec{\sigma}\vec{p})).$$

Using the properties of the algebraic basis of σ-matrices and equating the expressions of the basic units in (2.16) we arrive at the following system of equations

$$s = (\vec{r}\vec{p}) + r_4 p_4$$
$$\vec{\mathfrak{M}}_+ = [\vec{r}\times\vec{p}] + r_4\vec{p} - \vec{r}p_4$$
$$r_4 2mH = (\vec{\mathfrak{M}}_+\vec{p}) + sp_4 \qquad (2.17)$$
$$\vec{r}2mH = -[\vec{\mathfrak{M}}_+\times\vec{p}] + s\vec{p} - \vec{\mathfrak{M}}_+ p_4.$$

As is seen, the system (2.17) represents a generalization of (2.2), but with the vector $\vec{\mathfrak{M}}_+ = \vec{M} + \vec{N}$ instead of the vector $\vec{M}$.

We shall next introduce in the 4-dimensional Euclidean space the Cortesian unit vectors $\vec{e}_\mu$, $e_{4\mu}$ ($\mu = 1,2,3,4$) $\vec{e}_\nu\vec{e}_\mu + e_{4\nu}e_{4\mu} = \delta_{\nu\mu}$. Using the expansion $(r_4, r) = (e_{4\mu}, e_\mu)x_\mu$ we transform the system (2.17), equating to zero the coefficients of independent coordinates x_μ. As a result, we obtain the following system:

$$e_{4\mu}2mH = (\vec{\mathfrak{M}}_{+\mu}\vec{p}) + s_\mu p$$
$$\vec{e}_\mu 2mH = -[\vec{\mathfrak{M}}_{+\mu}\times\vec{p}] + s_\mu\vec{p} - \vec{\mathfrak{M}}_{+\mu}p_4$$
$$s_\mu = (\vec{e}_\mu\vec{p}) + e_{4\mu}p_4$$
$$\vec{\mathfrak{M}}_\mu = [\vec{e}_\mu\times\vec{p}] + e_{4\mu}\vec{p} - \vec{e}_\mu p_4.$$

Using the properties of multiplication of unit vector we get from this system the energy operator for spin 1/2 particles

$$2mH(+) = (p_4 + \vec{\tau}\vec{p})(p_4 - \vec{\tau}\vec{p}), \qquad (2.18)$$

where spin matrices τ_k ($k = 1, 2, 3$)

$$\tau_k^{\alpha\beta} = (\{e_4^\alpha e_k^\beta - e_k^\alpha e_4^\beta + \epsilon_{men}(e_e^\alpha e_n^\beta - e_n^\alpha e_e^\beta)\}e_k^m).$$

Equations (2.16) or (2.17) may be rewritten in the algebraic basis of τ-matrices; these obey the formula:

$$-(\vec{A}\vec{\tau})(\vec{B}\vec{\tau}) = (\vec{A}\vec{B})I + (\vec{\tau}[\vec{A}\times\vec{B}]) \qquad (2.19)$$

Applying (2.19) we rewrite equations (2.17) as follows:

$$(s + \vec{\tau}\vec{\mathfrak{M}}_+) = (r_4 - \vec{\tau}\vec{r})(p_4 + \vec{\tau}\vec{p})$$
$$(r_4 - \vec{\tau}\vec{r})2mH = (s + \vec{\tau}\vec{\mathfrak{M}}_+)(p_4 - \vec{\tau}\vec{p}), \qquad (2.20)$$

thus arriving at the energy operator (2.18). An analogous system follows when the use is made of the operator $\vec{\mathfrak{M}}_-$:

$$s + \vec{r}\vec{\mathfrak{M}}_- = (r_4 + \vec{r}\vec{r})(p_4 - \vec{r}\vec{p}) \qquad (2.21)$$
$$(r_4 + \vec{r}\vec{r}) 2mH = (s + \vec{r}\vec{\mathfrak{M}}_-)(p_4 + \vec{r}\vec{p}),$$

that corresponds to the energy operator

$$2mH(-) = (p_4 - \vec{r}\vec{p})(p_4 + \vec{r}\vec{p}). \qquad (2.22)$$

The formulae (2.18)-(2.22) allow the use both of r_+ and r_- matrices, since here their commutation and anticommutation properties of matrices are important rather than their explicit form. When an external electromagnetic field is present, from (2.18) due to noncommutativity of momentum components we obtain

$$2mH(+) = \vec{p}^2 + \vec{p}_4^2 + (([\vec{p}\times\vec{p}] + (\vec{p}p_4 - p_4\vec{p}))\vec{r}), \qquad (2.23)$$

and (2.22) yields

$$2mH(-) = \vec{p}^2 + p_4^2 + (([\vec{p}\times\vec{p}] - (\vec{p}p_4 - p_4\vec{p}))\vec{r}). \qquad (2.24)$$

Here the expression in brackets can be replaced by the magnetic-field strength:

$$[\vec{p}\times\vec{p}] = -ih\frac{e}{c}\vec{\mathcal{H}}. \qquad (2.25)$$

If we assume the fourth dimension to be real and introduce the potential A_4:

$$p_4 \to p_4 + \frac{e}{c}A_4, \qquad (2.26)$$

then

$$\vec{p}p_4 - p_4\vec{p} = -ih\frac{e}{c}\left(\frac{\partial A_4}{\partial \vec{x}} - \frac{\partial \vec{A}}{\partial x_4}\right) = -ih\frac{e}{c}\vec{\mathcal{H}}_4. \qquad (2.27)$$

In this case the operators (2.23) and (2.24) differ from each other and may, in principle, correspond to different electrons.

As is seen from (2.23) and (2.24) the operators H(+) and H(-) transform into each other under spatial reflections. If we pass to a bispinor representation of the wave function using a basis analogous to the basis of Dirac γ-matrices, then me may unite (2.20) and (2.22). We introduce the following real matrices

$$g_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad g_k = \begin{pmatrix} 0 & -r_k \\ r_k & 0 \end{pmatrix}, \quad (k = 1, 2, 3). \qquad (2.28)$$

Then, because of the correspondence of g_μ (μ = 1,2,3,4) and r_k $r_k \Longleftrightarrow g_4 g_k$ we get

$$2mH = (p^\mu g_\mu)(p^\mu g_\mu). \qquad (2.29)$$

In the bispinor representation we may define the operator of inversion: $P = g_4$, and generators of the group O(4): $\sigma_{\mu\nu} = \frac{1}{2}[g_\mu, g_\nu]$ which make it easy to establish the transformation of the equation:

$$ih\frac{\partial}{\partial t}\Psi = H\Psi \qquad (2.30)$$

under dicrete and continuous transformations of the coordinate system/3/.

3. ROTATION IN THE FOUR-DIMENSIONAL EUCLIDEAN SPACE

Consider the behaviour of a scalar function of a 4-dimensional argument $(r_4, \vec{r})$ under the rotation of the coordinate system around the unit vector (0, $\vec{n}$) by angle θ:

$$\Phi'(r') = \exp(i\theta(\frac{\vec{n}\vec{\mathfrak{M}}_\pm}{2})/h)\,\Phi(r_4, \vec{r}) = \Phi(T(r_4, \vec{r})), \qquad (3.1)$$

where generators $\vec{\mathfrak{M}}_\pm$ are determined by (1.10). The finite-rotation matrix $\hat{T}$ is expressed in terms of the infinitesimal operators (1.11):

$$\hat{T}(\vec{n}) = \exp(\theta(\vec{n}\frac{\vec{r}_\pm}{2})). \qquad (3.2)$$

The formulae (3.1) and (3.2) are analogous to (1.1) and (1.3). From (3.1) and (3.2) we obtain the following formulae for a rotation by an infinitesimal angle $\delta\theta$:

$$dr = \frac{1}{2}(\vec{n}\vec{r})\delta\theta, \qquad d\vec{r} = \frac{1}{2}[\vec{n}\times\vec{r}]\delta\theta - \frac{1}{2}\vec{n}r_4\delta\theta. \qquad (3.3)$$

From these formulae it is seen that the generators $\vec{r}_\pm/2$ and $\vec{\mathfrak{M}}_\pm/2$ realise a spinor representation of the group of 3-dimensional rotations in the 4-dimensional Euclidean space, i.e., they are generators of the group SU(2). Relative to the rotation the 4-dimensional vector $(r_4, \vec{r})$ represents a spinor defined in the Cartesian basis. Since in this case we are in the 4-dimensional space, then in the general case we may define the "rotation" around the unit 4-dimensional vector $(n_4, \vec{n})$

with four parameters. Here the generator of "rotation" around the axis $(n_4, 0)$ is

$$s = r_4 p_4 + (\vec{r}\vec{p}), \tag{3.4}$$

to which there is known to correspond the dilatation (scale transformation) parameter ℓ. So, instead of (3.3) we get

$$dr_4 = (n_4 r_4 + \vec{n}\vec{r})\delta\theta/2, \quad d\vec{r} = ([\vec{n}\times\vec{r}] + n_4\vec{r} - \vec{n}r_4)\delta\theta/2. \tag{3.5}$$

With the use of the τ-matrix basis (3.5) can be written as follows:

$$dr_4 + (\vec{\tau}d\vec{r}) = (n_4 - (\vec{\tau}\vec{n}))(r_4 + (\vec{\tau}\vec{r}))\delta\theta/2. \tag{3.6}$$

The angular velocity around each axis of "rotation" is determined in a standard way:

$$\vec{\omega} = \vec{n}\frac{d\theta}{dt}, \quad \omega_4 = n_4\frac{d\ell}{dt}, \tag{3.7}$$

where θ may be taken to be Euler angles, and ℓ represents the dilatation parameter.

4. FOUR-DIMENSIONAL ROTATION OF A SOLID BODY

Now we shall apply the formulae (3.3)-(3.7) to the description of rotation of a solid body in the 4-dimensional Euclidian space. As in the coordinate system attached to the body the components of the vector r_μ are constant, the velocity v_μ is an absolute velocity resulting only from the rotation of the body around a fixed point. Therefore, from (3.3) we obtain the expression:

$$v_4 = \frac{1}{2}(\vec{\omega}\vec{r}), \quad \vec{v} = \frac{1}{2}[\vec{\omega}\times\vec{r}] - \frac{1}{2}\vec{\omega}r_4 \tag{4.1}$$

which may, with account of (3.5)-(3.6), be generalized as follows:

$$v_4 + (\vec{\tau}\vec{v}) = (\omega_4 - (\vec{\tau}\vec{\omega}))(r_4 + (\vec{\tau}\vec{r}))/2. \tag{4.2}$$

The kinetic energy and angular momentum are determined by the formulae:

a) for the rotation around the axis $(0, \vec{n})$

$$T = \sum_n \frac{m_n}{2}(v_{4n}^2 + \vec{v}_n^2) = \frac{1}{2}\,\frac{I}{4}\vec{\omega}^2, \tag{4.3}$$

$$\vec{\mathfrak{M}}_\pm/2 = \sum_n \frac{m_n}{2}\{[\vec{r}_n\times\vec{p}_n] \pm (r_{4n}\vec{p} - \vec{r}p_{4n})\} = \frac{I}{4}\vec{\omega}; \tag{4.4}$$

b) for the rotation around the axis $(n_4, \vec{n})$:

$$T = \sum_n \frac{m_n}{2}(v_4 - (\vec{\tau}\vec{v}))_n (v_4 + (\vec{\tau}\vec{v}))_n = \frac{1}{2}\,\frac{I}{4}(\vec{\omega}^2 + \omega_4^2), \tag{4.5}$$

$$(s + \vec{\tau}\vec{\mathfrak{M}}_\pm)/2 = \frac{I}{4}(\omega_4 + \vec{\tau}\vec{\omega}). \tag{4.6}$$

Here the quantity I represents the moment of inertia of the spinning top and is determined by the formula:

$$I = \sum_n m_n(r_4^2 + \vec{r}^2)_n . \tag{4.7}$$

In three-dimensional space the angular momentum satisfies the equation of motion:

$$\frac{d}{dt}\vec{M} = \vec{N}, \tag{4.8}$$

$\vec{N} = [\vec{r}\times\vec{F}]$ which is a corollary of the Newton equation: $\frac{d}{dt}\vec{p} = \vec{F}$. The time derivative here is taken in a fixed coordinate system. In a moving (rotating) reference frame we obtain for the time derivative

$$\frac{d}{dt}\vec{M} + [\vec{\omega}\times\vec{M}] = \vec{N}. \tag{4.9}$$

If the four-dimensional equations of motion are of the form

$$\frac{d}{dt}(p_4 + (\vec{\tau}\vec{p})) = F_4 + (\vec{\tau}\vec{F}) \tag{4.10}$$

the four-dimensional angular momentum will obey the following equation of motion

$$\frac{d}{dt}(s + (\vec{\tau}\vec{\mathfrak{M}})) = N_4 + 2T + (\vec{\tau}\vec{N}), \tag{4.11}$$

where $N_4 = r_4F_4 + (\vec{r}\vec{F})$, $\vec{N} = r_4\vec{F} - \vec{r}F_4 + [\vec{r}\times\vec{F}]$.

In a moving coordinate system this equation has the form

$$\frac{d}{dt}(s + (\vec{\tau}\vec{\mathfrak{M}})) + (\omega_4 - (\vec{\tau}\vec{\omega}))(s + (\vec{\tau}\vec{\mathfrak{M}}))/2 = N_4 + 2T + (\vec{\tau}\vec{N}). \tag{4.12}$$

Applying the formulae (4.3) and (4.4) we finally obtain the four independent equations

$$\frac{d}{dt}(s + (\vec{\tau}\vec{\mathfrak{M}})) = N_4 + (\vec{\tau}\vec{N}). \tag{4.13}$$

In contrast to (4.9) the equations (4.13) are linear in the angular velocity $(\omega_4, \vec{\omega})$. (In the case of classical equations the τ-matrices are assumed to be time-independent.)

5. THE QUANTUM THEORY OF ANGULAR MOMENTUM IN FOUR-DIMENSIONAL EUCLIDEAN SPACE

If we attach the coordinate system $(\xi\eta\zeta)$ to a solid body, the orientation of the system will be defined by three Euler angles $(\alpha\beta\gamma)$ characterizing the position of the axes $(\xi\eta\zeta)$ with respect to the laboratory system (xyz). The change of a function under a rotation of the coordinate axes by an angle ϕ around the vector $\vec{n}$ is carried out by the operator

$$R^{n}_{\phi} = \exp(i(\vec{n}\frac{\vec{\mathfrak{M}}_{\pm}}{2})\phi). \tag{5.1}$$

Under a rotation of the coordinate axis by three euler angles the rotation operator has the form

$$R(\alpha\beta\gamma) = R^{z}_{\alpha}R^{y}_{\beta}R^{z}_{\alpha}. \tag{5.2}$$

Correspondingly, the Wigner or D-functions are determined as follows [5]:

$$D^{j}_{mm'}(\alpha\beta\gamma) \equiv \langle jm'|R(\alpha\beta\gamma)|jm\rangle. \tag{5.3}$$

The operator of a finite rotation for j = 1/2 in the Cartesian basis is of the form

$$R^{(1/2)} = \exp(-\tau_z\gamma/2)\exp(-\tau_y\beta/2)\exp(-\tau_z\alpha/2), \tag{5.4}$$

and the corresponding rotation matrix is

$$\|D^{1/2}_{mm'}\| = \begin{Vmatrix} \chi_4 & -\chi_1 & -\chi_2 & -\chi_3 \\ \chi_1 & \chi_4 & -\chi_3 & \chi_2 \\ \chi_2 & \chi_3 & \chi_4 & -\chi_1 \\ \chi_3 & -\chi_2 & \chi_1 & \chi_4 \end{Vmatrix}, \tag{5.5}$$

or using the matrices τ_k (k = 1, 2, 3),

$$\|D^{1/2}_{mm'}\| = \chi_4\delta_{mm'} - \chi_k\tau^{k}_{mm'}, \tag{5.6}$$

where χ_μ (μ = 1,2,3,4) are expressed in terms of the Euler angles

$$\chi_1 = \sin\frac{\beta}{2}\sin\frac{\gamma-\alpha}{2}, \qquad \chi_2 = \sin\frac{\beta}{2}\cos\frac{\gamma-\alpha}{2},$$
$$\chi_3 = \cos\frac{\beta}{2}\sin\frac{\gamma+\alpha}{2}, \qquad \chi_4 = \cos\frac{\beta}{2}\cos\frac{\gamma+\alpha}{2}. \tag{5.7}$$

From these relations it is seen that χ_μ can be considered as coordinates of a point on a unit sphere in the 4-dimensional Euclidean space $\chi_1^2+\chi_2^2+\chi_3^2+\chi_4^2=1$. D-function of any order can be expressed by χ_μ-parameters. For instance, the matrix of finite rotaton of a three-dimensional space looks as follows

$$\|D^{1}_{mm'}\| = (2\chi_4^2-1)\delta_{mm'} + 2(\chi_m\chi_{m'} + \chi_4\epsilon_{mm'\ell}\chi_\ell),$$

i.e., it is quadratics on terms of χ_μ [6]. Applying to the form of representation (5.5a) and to the components of τ-matrices $\tau_x\tau_y = -\tau_z$, $\tau_y\tau_z = -\tau_x$, $\tau_z\tau_x = -\tau_y$ it is easy to derive a formula of composition of the rotation group parameters χ_μ: $O(\chi_\mu)O(\chi'_\mu) = O(\chi''_\mu)$, where

$$\chi''_4 = \chi_4\chi'_4 - \chi_k\chi'_k, \qquad \chi''_k = \chi_4\chi'_k + \chi_k\chi'_4 + [\vec{\chi}\times\vec{\chi}']_k. \tag{5.8}$$

In quantum theory of the spinning top it is necessary to find a representation of the projection operator the angular-momentum of a solid body in terms of Euler angles (or in terms of the parameters χ_μ), and of the dilatation operator as a function of scale parameter ℓ. For the spinning top we introduce generalized coordinates as follows $\xi_\mu = \ell\chi_\mu$ and define the wave function $\Psi = \Psi(\ell,\chi_\mu)$. Then we consider a change of the function $\Psi(\ell,\chi_\mu)$ as a result of a small rotaton given by the parameters χ°_μ, where $\chi_\Psi \simeq 1$. Then from (5.8) it follows that $\chi'_4 = \chi_4 - \chi^\circ_k\chi_k$, $\chi'_k = \chi_k + \chi^\circ_k\chi_4 + [\chi^\circ\times\chi]_k$.
Therefore, $\delta\chi_4 = -\chi^\circ_k\chi_k$, $\delta\chi_k = \chi^\circ_k\chi_4 + [\vec{\chi}^\circ\times\vec{\chi}]_k$.
The change of the function $\Psi(\ell,\chi_\mu)$ can be written in the following form

$$\Psi(\ell,\chi_\mu+\delta\chi_\mu) = (1+\delta\chi_\mu\frac{\partial}{\partial\chi_\mu})\Psi(\ell,\chi_\mu) = (1+2\chi^\circ_k J_k)\Psi(\ell,\chi_\mu),$$

where the generators of group J_k have the form

$$J_k = \frac{1}{2}(\chi_4\frac{\partial}{\partial\chi_k} - \chi_k\frac{\partial}{\partial\chi_4} + [\vec{\chi}\times\frac{\partial}{\partial\vec{\chi}}]_k). \tag{5.9}$$

Substituting the functions $\chi_\mu = \chi_\mu(\alpha,\beta,\gamma)$ from (5.7) into the expression (5.9) for J_k we get

$$-ihJ_1 = -ih[\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} - \cos\gamma\frac{\partial}{\partial\beta} - \sin\gamma\,\mathrm{ctg}\beta\frac{\partial}{\partial\gamma}]$$
$$-ihJ_2 = -ih[-\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} - \sin\gamma\frac{\partial}{\partial\beta} + \cos\gamma\,\mathrm{ctg}\beta\frac{\partial}{\partial\gamma}] \tag{5.10}$$
$$-ihJ_3 = -ih\frac{\partial}{\partial\gamma},$$

i.e., the known relations for the angular momentum of a solid body in terms of the Euler angles. In this case the formulas for the operator

$$-ihJ_k = \mathfrak{M}_{+k}/2 \tag{5.11}$$

coincide with the formulas for the projection of angular momentum in terms of the Euler angles into the moving system. $D^J_{mm'}(\alpha, \beta, \gamma)$ are eigenfunctions of three operators defining a three-dimensional rotation:

$$J_z D^J_{mm'} = -mD^J_{mm'}, \quad J_{z'} D^J_{mm'} = -m'D^J_{mm'}, \quad J^2 D^J_{mm'} = J(J+1)D^J_{mm'}. \tag{5.12}$$

According to the rotation laws (3.3) and (3.5), the boundary conditions look as follows

$$D(\alpha \pm 4\pi, \beta, \gamma) = D(\alpha, \beta, \gamma), \; D(\alpha, \beta \pm 4\pi, \gamma) = D(\alpha, \beta, \gamma), \; D(\alpha, \beta, \gamma \pm 4\pi) = D(\alpha, \beta, \gamma). \tag{5.13}$$

As a result we get a half-integer-range of values J:

$J = 0, 1/2, 1, ...; \; m, m' = -J, -J+1, ... J-1, J.$

$D(\chi_\mu)$-functions constitute a complete orthogonal set of functions on a unit three-dimensional sphere and the basis of the irreducible representation (J, J) of the group O(4)[7,8].

The fourth rotation component is generated by the dilatation operator

$$s = -ih\,\xi_\mu \frac{\partial}{\partial \xi_\mu} = -ih\ell \frac{\partial}{\partial \ell}. \tag{5.14}$$

Here ℓ is a generalized rotational coordinate analogous to the Euler angles. After Dirac[9] we shall define the wave function in four-dimensional space as a homogeneous function of ξ_μ of the degree n. Then, in accordance with the Euler theorem on homogeneous functions

$$-ih\xi_\mu \frac{\partial}{\partial \xi_\mu} \Psi = -ihn\Psi, \quad n = 0, \pm 1, \pm 2, ... \tag{5.15}$$

From the analysis of the classical formulae (4.5) and (4.6) we determine the energy operator in the quantum case as follows:

$$H = (s_1 + \vec{r}\vec{\mathfrak{M}})(s_2 - \vec{r}\vec{\mathfrak{M}})/2I, \tag{5.16}$$

where $s_1 = \xi^\mu \pi_\mu$, $s_2 = \pi^\mu \xi_\mu$. Removing brackets in (5.16) and taking account of the commutation relations

$$[s, \vec{\mathfrak{M}}] = 0, \qquad [\vec{\mathfrak{M}} \times \vec{\mathfrak{M}}] = 2hi\vec{\mathfrak{M}}, \tag{5.17}$$

we get

$$2IH = \ell^2 \pi_e^2 - 2ih\ell\pi_e + \mathfrak{M}^2 - 2hi(\vec{r}\vec{\mathfrak{M}}). \tag{5.18}$$

Provided the Hamilton operator is known, we can obtain the equations of motion for the operators s, $\mathfrak{M}$ and $\vec{r}$:

$$\frac{d}{dt}(s + \vec{r}\vec{\mathfrak{M}}) = [H, s + \vec{r}\vec{\mathfrak{M}}]\frac{i}{h}. \tag{5.19}$$

Determining the commutator in (5.19) we arrive at the equations

$$\frac{d}{dt}(\vec{\mathfrak{M}}/2) = -[\frac{\vec{\mathfrak{M}}}{2} \times \vec{S}]/(I/4) \tag{5.20}$$

$$\frac{d}{dt}\vec{S} = -[\vec{S} \times \frac{\vec{\mathfrak{M}}}{2}]/(I/4), \quad \vec{S} = -ih\vec{r}/2. \tag{5.21}$$

Hence it follows that

$$\frac{d}{dt}(\frac{\vec{\mathfrak{M}}}{2} + \vec{S}) = 0, \quad \frac{d}{dt}(\vec{\mathfrak{M}}\vec{S}) = 0. \tag{5.22}$$

Combining (5.22) with (5.21) we also find that

$$\frac{d}{dt}\frac{\vec{\mathfrak{M}}}{2} + [\frac{\vec{\mathfrak{M}}}{2} \times \frac{\vec{\mathfrak{M}}}{2}]/(I/4) = 0. \tag{5.23}$$

Further, from (5.19) $ds/dt = 0$ follows. The fourth component of rotation connected with the dilatation parameter gives, perhaps, a slight contribution. The three-dimensional part of rotation with the angular momentum $\vec{\mathfrak{M}}_\pm/2$ and Euler angles should, on the contrary, provide an essential contribution on physical phenomena involving particles with spin 1/2.

As to the spin 1/2 itself, the considered variant of the spinning top may serve as a macroscopic analog of the spin 1/2. As is known, if the electon is modelled by a rotating ball with radius $R_e = e^2/mc$, the velocity of its surface is about 1000 times as high as the light velocity. In the given model the moment of inertia of the top is determined by the formula

$$I = mR^2, \tag{5.24}$$

where R is a fundamental constant. Therefore, from the equality $-ih\vec{r}/2 = I\vec{\omega}/2$ we find that

$$\vec{\omega} = -ih\vec{r}/I. \tag{5.25}$$

The magnitude of the angular velocity is defined as an eigenvalue of (5.25) and is sufficiently small as R is large enough[10].

REFERENCES

1. Maiorana E. Nuovo Cimento, 1982, vol.9, p.335.
2. Yamaleev R.M. JINR, E2-81-322, Dubna, 1981.
3. Yamaleev R.M. JINR, P2-81-302, Dubna, 1981; JINR, P2-81-910, Dubna, 1982.

4. Fedorov F.I. Lorentz Group.Moscow, "Nauka", 1979 (in Russian).
5. Varshalovich D.A., Moskalev A.N., Chersonski V.K. Quantum Theory of Angular Momentum. Moscow, "Nauka",1975(in Russian).
6. Lukach I., Smorodinski J.A. JINR, P2-7465, Dubna, 1973.
7. Pestov A.B. JINR, P2-8418, Dibna, 1974; Pestov A.B., Shavochina N.S. JINR, P2-11022, Dubna, 1977.
8. Bogolubov N.N., Logunov A.A., Todorov I.G. Faundation of Axiomatic Way in Quantum Field Theory. Moscow, "Nauka", 1969, p. 117 (in Russian).
9. Dirac P.A.M. Annals of Mathematics, 1935, vol.36, No.3, p.656.
10. Yamaleev R.M. JINR, P2-80-732, Dubna, 1980.

Received by Publishing Department
on March 29, 1984.

Ямалеев Р.М. E2-84-197

Квантовый волчок в четырехмерном эвклидовом пространстве как макроскопический аналог спина 1/2

Развивается подход, согласно которому обычный волчок является макроскопическим аналогом спина 1. Макроскопический аналог спина 1/2 конструируется в 4-мерном эвклидовом пространстве. Найдены законы вращательного движения для твердого тела и уравнения движения для волчка в 4-мерном пространстве. Предложенная модель обладает замечательным свойством: направление углового момента совпадает с направлением угловой скорости независимо от формы волчка.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1984

Yamaleev R.M. E2-84-197

Quantum Spinning Top in Four-Dimensional Euclidean Space as a Macroscopic Analog of Spin 1/2

An approach is developed according to which a usual spinning top is a macroscopic analog of spin 1. The macroscopic analog of spin 1/2 is constructed in the 4-dimensional Euclidean space. Equations are found for the spinning-top motion and for the rotational motion of a solid body in the 4-dimensional space. The model proposed for the spinning top has an interesting property: the direction of angular momentum coincides with that of angular velocity irrespective of the form of the top.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1984