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ON AN APPROACH TO THE CALCULATION OF MULTILOOP MASSLESS FEYNMAN INTEGRALS

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1. Recently, in calculations of renormalization group functions in quantum field theory [1], a remarkable progress has been made[2-5]. As is well known, within the dimensional regularization and the minimal subtraction scheme such calculations can be reduced to the evaluation of massless integrals with only one external momentum (the p -integrals). Recent powerful calculational methods considerably expand the class of exactly calculable integrals. We mean first of all the Gegenbauer polynomial x -space technique (GPXT) [2], the integration-by-parts technique (IP)[3], and the "uniqueness" method (UM)[5,8,9]. However, in spite of the apparent progress, the problem of multiloop calculations is far from being resolved. Indeed, neither of the above mentioned methods is universal. For example, it is difficult to use GPXT for calculating the integrals with complicated structures of angular integrations [2]; some two-loop integrals (with arbitrary indices of lines) are not calculable by means of IP; the same is true for UM *). Moreover, even in favourable cases, one cannot use the intrinsic symmetry of the Feynman integrals, which is the reason for the cancellations in the final results [8,5]. Futhermore, the use of these methods (especially UM) usually requires skill and inventiveness, so that we often cannot decide beforehand, whether a given integral is calculable by this method or not.

The main idea of our approach is inspired by the paper [10] where a complicated integral has been calculated by deriving and solving a system of functional equations.

In this paper we suggest a generalization of such functional equations to the case of arbitrary two-loop (and, in principle, multiloop) integrals. From technical point of view our approach is based on the generalization of the well-known identity of the dimensional regularization $\int d^Dk\,(k^2)^{-\alpha} = 0$, $\alpha \neq D/2$ (D is a space-time dimentionality). Using this generalization we reduce the problem to consideration of only vacuum integrals (v -integrals) and reveal a hidden symmetry of p -integrals, which turns out to be useful in deriving functional equations.

*) In this work we always have in view the problem of calculating the dimensionally and analytically regularized p -integrals.

The paper is organized as follows. In section 2 the generalization of the identity $\int d^Dk\, (k^2)^{-\alpha} = 0,\ \alpha \neq D/2$ is given. Then, we use it to go to the vacuum diagrams and, as a result, all p -integrals turn out to be grouped into classes of equal ones. This trick allows one to calculate a number of new integrals. In the third section the developed technique is applied to deriving functional equations for an arbitrary two-loop p -integral. In doing this we use IP and UM. In section 4 we discuss possible extensions of our method.

2. Let us derive the relation, on which our method is based.

As is well known, in the framework of the dimensional regularization the following identity holds /6,11/:

$$f(\alpha) \equiv \int \frac{d^Dp}{(p^2)^{\alpha}} = 0, \quad \alpha \neq D/2. \tag{2.1}$$

If $\alpha = D/2$, then $f(\alpha)$ is ill-defined. Nevertheless, one can try to define it as a distribution over α, choosing appropriately the space of test functions.

To this end consider the following linear functional:

$$\int_{-\infty}^{\infty} dz\, \delta(z) \varphi(z) = \varphi(0). \tag{2.2}$$

It is easy to relate δ and f :

$$\int_{-\infty}^{\infty} dz\, f(iz + D/2)\, \varphi(z) = \int_{-\infty}^{\infty} dz\, \varphi(z) \left\{ \int \frac{d^Dp}{(p^2)^{iz+\frac{D}{2}}} \right\} =$$

$$= \int_{-\infty}^{\infty} dz\, \varphi(z) \left\{ \Omega_D \int_0^{\infty} d|p|\, |p|^{D-1-2iz-D} \right\} = \left\{ |p|^2 = e^x \right\} =$$

$$= \int_{-\infty}^{\infty} dz\, \varphi(z) \left\{ \frac{1}{2} \Omega_D \int_{-\infty}^{\infty} dx\, e^{-izx} \right\} = \pi \Omega_D \varphi(0),$$

so that

$$\int \frac{d^Dp}{(p^2)^{i\alpha + \frac{D}{2}}} = \pi \Omega_D \cdot \delta(\alpha). \tag{2.3a}$$

Here $\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ is the surface of the unit hypersphere in the D -dimensional Euclidean space. To simplify formulae we will use (2.3a) in the form

$$\int \frac{d^Dp}{(p^2)^{D/2+\alpha}} = \pi \Omega_D i \cdot \delta(\alpha). \tag{2.3}$$

In a sense, (2.3) generalizes (2.1). Unlike the latter, (2.3) provides a nontrivial information on massless vacuum integrals (v -integrals). We will show below that (2.3) allows one to reduce the evaluation of p -integrals to the vacuum ones (and find them analytically in some cases), and also establish equalities between Feynman integrals of different topological structures.

Consider an arbitrary p -integral. In the position space such an integral has the following form:

0  $x \;=\; \dfrac{X(\vec{\alpha})}{(x^2)^{\beta}}.$ (2.4)

Here $0, x$ are external points; $\vec{\alpha} = \alpha_1, \ldots, \alpha_n$ are internal line "indices" of the diagram. An internal line with the index α_i , linking the points y and z , corresponds to the $(y-z)^{-2\alpha_i}$; each of the m internal vertex argument is integrated in; $\beta = \sum_{i=1}^{n} \alpha_i - \frac{mD}{2}$ is the dimensionality of the integral in the units of x^2. Note, that indices α_i can be arbitrary complex numbers. Such an assumption turns out to be useful in various applications (see, /5,8/).

Now let us see how (2.3) allows one to reduce the integral (2.4) to v -integrals, corresponding to some vacuum diagrams. Indeed, we multiply (2.4) by $(x^2)^{-\sigma}$ and integrate it over x . From (2.3) we get the vacuum diagram:

0  $x = X(\vec{\alpha}) \displaystyle\int \frac{d^Dx}{(x^2)^{\beta+\sigma}} = X(\vec{\alpha})\, \pi \Omega_D i\, \delta(\beta + \sigma - D/2).$ (2.5)

Due to the presence of the δ -function, we have information on the coefficient function $X(\vec{\alpha})$ only at $\beta = \frac{D}{2} - \sigma$. However, owing to the freedom in choosing σ , this bound does not impose any limitations on β: if we succeed in calculating the v -integral (2.5) at a given σ , we immediately get the result for the integral (2.4) in which $\beta = \frac{D}{2} - \sigma$. Of course, the reverse statement is also correct.

The first result from this consideration is the following: all the integrals of the type (2.4), which can be reduced by the described procedure to the same v -integral, are equal. In what follows we will say that such p -integrals form a G -invariant set.

Generally speaking, the problem of enumerating the elements of the G -set resulting from a given vacuum diagram arises.

First of all it is clear, that such elements can be obtained by "cutting" lines of the original vacuum diagram in all possible ways. Let this vacuum diagram have $m+2$ vertices (one of these points is not integrated in and owing to translation invariance can be set to zero) and n lines connecting them. Choose any two lines with indices α_i and α_j. We have the following chain of identities (see (2.5)):

$$\equiv X(\vec{\alpha})\pi i\delta\left(\sum_{i=1}^{n}\alpha_i-\frac{(m+1)D}{2}\right)=\int\frac{d^Dx}{(x^2)^{\alpha_i}}$$

$$=\int\frac{d^Dx}{(x^2)^{\alpha_j}}$$

$$, \qquad (2.6)$$

so that $X_1(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n)=X_2(\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_n)$,

where

$$=\frac{X_1(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n)}{(x^2)^{D/2-\alpha_i}},$$

$$=\frac{X_2(\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_n)}{(x^2)^{D/2-\alpha_j}}.$$

Furthermore, any two points of a vacuum diagram in the position space can be considered as connected by a line with zero index. Cutting this line,we get,the propagator-type diagram (p-diagram) with the same number of internal lines as in the initial vacuum graph (in the previous case this number has been reduced by one). Such propagator diagrams also contribute to the G-set; their dimensionality is fixed and equals $D/2$.

Another transformation, which allows one to find elements of a G-set, is easier to formulate in momentum space. Performing the Fourier transformation according to the relation

$$\int\frac{d^Dp\,e^{ipx}}{(p^2)^{\alpha}}=\frac{\pi^{D/2}\,2^{D-2\alpha}}{(x^2)^{D/2-\alpha}}\cdot\frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}, \qquad (2.7)$$

we come to the diagram of the same topology but with the changed indices $\alpha_i\to\frac{D}{2}-\alpha_i$ and with integrations over loops instead of vertices. The Fourier-transformed coefficient function is connected with the function $X(\vec{\alpha})$ (see (2.6)) via the relation:

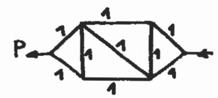
$$X_F\left(\overrightarrow{\frac{D}{2}-\alpha}\right)=\frac{a\left(\sum\limits_{i=1}^{n-1}\alpha_i-\frac{mD}{2}\right)}{\prod\limits_{i=1}^{n-1}a(\alpha_i)\,\pi^{(m+1-\frac{n}{2})D}}\cdot X(\vec{\alpha}), \qquad (2.8)$$

where $a(\delta)=\frac{\Gamma(D/2-\delta)}{\Gamma(\delta)}$, and n, $m+1$ are the numbers of lines and vertices, respectively. Furthermore, in momentum space any vertex can be decomposed into two vertices with a less number of legs by introducing the line with zero index (as in the previous case the introduction of such a spurious line corresponds to multiplying the whole diagram by 1). Cutting this line, we get a p-diagram with the dimensionality $D/2$. This p-diagram enters the G-set of the initial vacuum diagram in momentum space.

Note, that the last two transformations (cutting the line with zero indices in the position and momentum spaces) are ill-defined in the Fourier-conjugated spaces. This is due to the fact that the Fourier transform of $(x^2)^{-D/2}$ is ill-defined (see eq. (2.7)). The problem of the correct definition of these procedures is considered in the Appendix.

In fig. 1 an example is given of how the described three ways of obtaining the G-set work for the vacuum diagram (a). The resulting graphs are those with the topology (b)-(h). The diagrams (b) and (e) are in momentum space, the other is in the position space. A similar method (the technique of "gluing") has been used in ref. [2] to obtain the equality of the diagrams (b)-(e), where all indices equal one in momentum space at $D=4$. Our analysis shows that at the expense of small (disappearing at $D=4$) changes in some indices the exact equality of all these diagrams at any D can be achieved, the result of [2] being reproduced as the equality of first terms in the expansion of the corresponding coefficient functions over $\epsilon=\frac{4-D}{2}$.

To give one more example of the application of the notion of the G-set, we list the elements of the G-set which contains the recently evaluated (see [10]) diagram (at $D=4$):

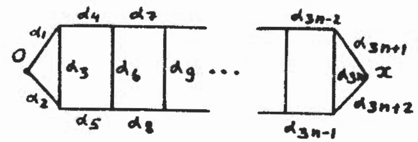


$$=\frac{441}{8}\cdot\frac{\zeta(7)}{p^2}(4\pi)^{-8}.$$

First, there are in this G-set the propagator diagrams with the topology given in fig. 2a): any two vertices of each graph can be chosen to be "external". Second, we can obtain some more diagrams (see fig. 2b)-2d) by decomposing the four-leg vertices in momentum space. We emphasize once again, that the equality of these diagrams for the fixed choice of indices remains valid at any D. These diagrams will be needed,for example, in evaluating the six-loop renormalization group β-function in the $g\varphi^4_{(4)}$ model.

To conclude this section, we will demonstrate that reducing p-integrals to the vacuum ones with the help of (2.3) allows one to calculate easily some classes of Feinman integrals.

Consider, for example, the integral



$$=\frac{Z(\vec{\alpha})}{(x^2)^{\sum_i\alpha_i-nD}}. \qquad (2.9)$$

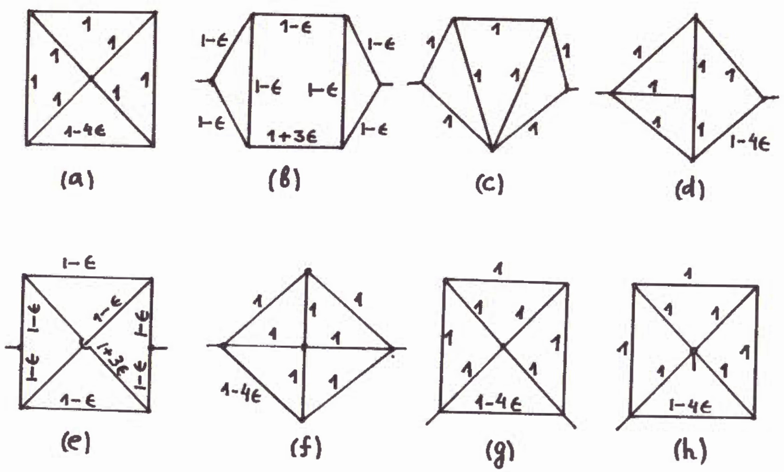


Fig.1

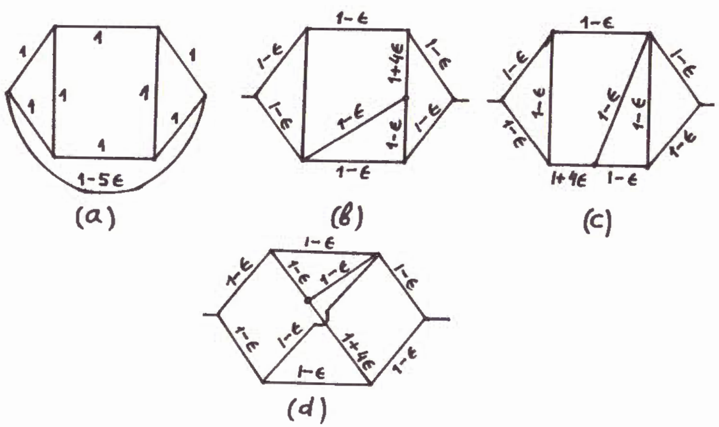
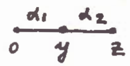


Fig.2

Let us integrate both parts of this equation over x . After a repeated application of the formula



$$\equiv \int \frac{d^D y}{(y^2)^{\alpha_1} (y-z)^{2\alpha_2}} = v(\alpha_1, \alpha_2, D-\alpha_1-\alpha_2)(z^2)^{\frac{D}{2}-\alpha_1-\alpha_2} \qquad (2.10)$$

(where $v(\alpha_1, \alpha_2, \alpha_3) = \pi^{D/2}\, a(\alpha_1)\, a(\alpha_2)\, a(\alpha_3)$) the l.h.s. of (2.9) can be reduced to the expression:

$$v(\alpha_{3n+2}, \alpha_{3n+1}, D-\alpha_{3n+2}-\alpha_{3n+1})\, v(\alpha_{3n-1}, \alpha_{3n+2}+\alpha_{3n+1}+\alpha_{3n}-\tfrac{D}{2}, \tfrac{3D}{2}-\alpha_{3n+2}-\alpha_{3n+1}-\alpha_{3n}-\alpha_{3n-1}) \dots \qquad (2.11)$$

$$\dots v(\alpha_2, \textstyle\sum_i \alpha_i - \alpha_1 - \alpha_2 - (n-\tfrac{1}{2})D, (n+\tfrac{1}{2})D - \sum_i \alpha_i + \alpha_1) \cdot$$

$$\cdot\, \pi i \Omega_D \delta\left(\textstyle\sum_i \alpha_i - (n+\tfrac{1}{2})D\right) \equiv \pi i \Omega_D I(\vec{\alpha})\, \delta\left(\textstyle\sum_i \alpha_i - (n+\tfrac{1}{2})D\right)$$

whereas the r.h.s. is:

$$Z(\vec{\alpha})\, \pi i \Omega_D \delta\left(\textstyle\sum_i \alpha_i - (n+\tfrac{1}{2})D\right)$$

and we immediately conclude that

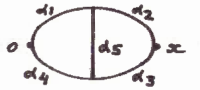
$$Z(\vec{\alpha})\Big|_{\sum_i \alpha_i = (n+\frac{1}{2})D} = I(\vec{\alpha}). \qquad (2.12)$$

In full analogy with the previous example some other integrals can be evaluated, for example, the integral

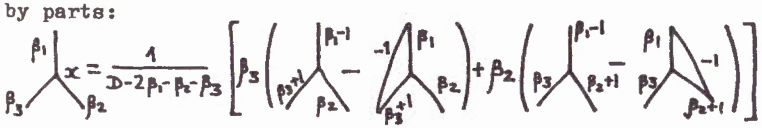
provided that $\sum_{i=1}^{4n+1} \alpha_i = (n+\frac{1}{2})D$. Unfortunately, integrals which are needed, for example, in the renormalization group calculations, are reduced to the v -integrals not amenable to simple computation. Apparently, it is insufficient to use formulae of the type (2.10) to evaluate them (note, that the success of IP and UM is ultimately connected with the possibility of transforming the integral to those calculable by the straightforward application of the one-loop integration formula (2.10)). Thus, the search for more adequate methods is in order.

3. Thus, we have seen that the relation (2.3) allows one to establish various identities between Feymnam integrals with one external momentum. In this section we will show that v -integrals may be of help in deriving the set of functional equations for the coefficient function (2.4).

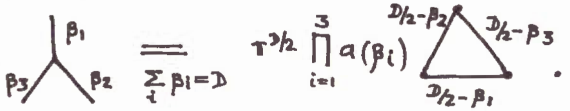
Consider the two-loop diagram

 $= \dfrac{F(\alpha_1,\dots,\alpha_5)}{(x^2)^{\sum_i \alpha_i - D}}$. (3.1)

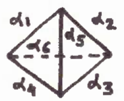
The integrals of this type are rather often encountered in applications but a convenient and sufficiently general representetion for the function $F(\vec{\alpha})$ is still lacking. The explicit form of $F(\vec{\alpha})$ is known, for example, when $\alpha_1, \alpha_2, \alpha_5$ are integer. In this case $F(\vec{\alpha})$ can be expressed via a finite sum of products of the Euler functions. Such a representation can be obtained with the help of integration by parts:

 (3.2)

(x is integrated in). It is not difficult to see that this relation is of little help in the case of arbitrary indices. Further, if, for instance, $\beta_1+\beta_2+\beta_3 = D$, then the well-known uniqueness relation can be applied *):

 (3.3)

The possibilities of obtaining the explicit form of $F(\vec{\alpha})$ using relations (3.2) and (3.3) are rather limited. But at arbitrary α_i these relations impose a number of functional constraints on $F(\vec{\alpha})$**). To obtain the complete set of functional equations for $F(\vec{\alpha})$ following from eqs. (3.2)-(3.3) we will apply the method of the previous section, namely, we will pass to the consideration of vacuum diagram. According to eq. (2.3) we have:

 $= \pi i \Omega_D X(\alpha_1,\dots,\alpha_6)\cdot\delta\left(\sum_{i=1}^{6}\alpha_i - \frac{3D}{2}\right)$. (3.4)

Then $F(\vec{\alpha}) = X(\alpha_1,\dots,\alpha_6)\Big|_{\alpha_6 = \frac{3D}{2} - \sum_{i=1}^{5}\alpha_i}$.

All transformations of the diagram (3.4) resulting from the relations (3.2) and (3.3) and their compositions form a set which we

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*) It is interesting to note that eq. (3.3) has the structure of the Yang-Baxter equations /12/.

**) At first sight, it is rather difficult to apply eq. (3.3) to an arbitrary diagram. But we can represent, for example, the line in the form $\xrightarrow{\alpha_2} = \xrightarrow{D-\alpha_1-\alpha_5}\xrightarrow{\alpha_1+\alpha_2+\alpha_5-D/2}\cdot\nu(\alpha_1+\alpha_5-D/2,\, D-\alpha_1-\alpha_2-\alpha_5,\, \alpha_2)$ and after that eq. (3.3) can be applied to the triangle $\alpha_1, \alpha_5, D-\alpha_1-\alpha_5$.

denote as I. Each element of this set gives rise to a functional equation on $F(\vec{\alpha})$. Obviously, independent equations are those comprising the system of generating elements for the set I. Let us try to extract this system.

Notice the nontrivial symmetry of $F(\vec{\alpha})$ which follows from the representation (3.4). Namely, $X(\alpha_1,\dots,\alpha_6)$ is invariant under the permutations of α_i corresponding to the elements of the group of rotations of the tetrahedron on the l.h.s. of (3.4) together with reflections. This group consists of 24 elements and has three generating elements; for example, rotations around two axes connecting some vertex and the centre of the opposite side, and one reflection. Each transformation reveals some symmetry of the function $X(\vec{\alpha})$ and, therefore, results in some functional equation for the function $F(\vec{\alpha})$. Choosing, for example, as generating elements rotations around axes passing through the vertices $(\alpha_1\,\alpha_5\,\alpha_2)$ and $(\alpha_1\,\alpha_6\,\alpha_4)$ and also the reflection, which can be reduced to the permutation of two vertices $(\alpha_1\,\alpha_2\,\alpha_5)$ and $(\alpha_3\,\alpha_4\,\alpha_5)$, we get:

$$
\begin{aligned}
X(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) &= X(\alpha_2,\alpha_5,\alpha_4,\alpha_6,\alpha_1,\alpha_3)\\
&= X(\alpha_4,\alpha_5,\alpha_2,\alpha_6,\alpha_3,\alpha_1)\\
&= X(\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_5,\alpha_6),
\end{aligned} \tag{3.5}
$$

or:

$$
\begin{aligned}
F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= F\left(\alpha_2,\alpha_5,\alpha_4,\tfrac{3D}{2}-\sum_{i=1}^{5}\alpha_i,\alpha_1\right)\\
&= F\left(\alpha_4,\alpha_5,\alpha_2,\tfrac{3D}{2}-\sum_{i=1}^{5}\alpha_i,\alpha_3\right)\\
&= F(\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_5).
\end{aligned} \tag{3.5a}
$$

Eqs. (3.5) can also be considered as a consequence of the G-invariance, because they can be obtained by cutting lines in (3.4) and using the explicit permutation symmetry of the expression (3.1).

It is clear that the group with generating elements (3.5) is contained in the set I. Add to (3.5) one more transformation corresponding to the relation (3.3) as applied to the vertex $(\alpha_1\,\alpha_2\,\alpha_5)$ (see the last foot note):

$$
\begin{aligned}
F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= a(D-\alpha_1-\alpha_2-\alpha_5)\,a(\alpha_2)\,a(\alpha_5)\,a(\alpha_1)\,\cdot\\
&\cdot F\left(\tfrac{D}{2}-\alpha_2,\tfrac{D}{2}-\alpha_1,\alpha_3,\alpha_4,\alpha_1+\alpha_2+\alpha_5-\tfrac{D}{2}\right).
\end{aligned} \tag{3.6}
$$

Transformations (3.5a) and (3.6) form the set of generating elements of the finite order group entering I. Clearly, this group contains all the equations imposed on $F(\vec{\alpha})$ by the uniqueness relation. Indeed, each line of the diagram (3.4) can be rotated into the position of any other line (taking into account their orientations). It only remains to note that there exists equivalence between oriented lines and relations resulting from the single application of the uniqueness relation. Thus, compositions of all transformations of the diagram (3.4) with the help of the relation (3.3) can be obtained by rotating (or reflecting) diagram (3.4) into the given position and than applying eq. (3.6). It is interesting to note, that all the eleven basic transformations of the integral (3.1) listed in ref.[8], can be derived from only <u>four</u> relations (3.5a) and (3.6). Particularly, these four relations allow one to derive all constraints imposed by such transformations as, for example, Fourier transformation and then transition to the dual representation, and also imposed by conformal transformations (or inversions).

At last, analogous considerations lead to the conclusion that the system of generating elements for the whole set I consists of the transformations (3.5a), (3.6) and relation which can be obtained by applying eq. (3.2) to some set of indices, for example, α_3, α_5 and α_4 :

$$F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=(D-2\alpha_5-\alpha_3-\alpha_4)^{-1}\Big[\alpha_3\big(F(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4,\alpha_5-1)- \qquad (3.7)$$
$$-F(\alpha_1,\alpha_2-1,\alpha_3+1,\alpha_4,\alpha_5)\big)+\alpha_4\big(F(\alpha_1,\alpha_2,\alpha_3,\alpha_4+1,\alpha_5-1)-F(\alpha_1-1,\alpha_2,\alpha_3,\alpha_4+1,\alpha_5)\big)\Big]$$

Below we give some consequences from the relations (3.5a) and (3.6):

$$F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=\prod_{i=1}^{6}a(\alpha_i)\,F\left(\tfrac{D}{2}-\alpha_2,\tfrac{D}{2}-\alpha_3,\tfrac{D}{2}-\alpha_4,\tfrac{D}{2}-\alpha_1,\tfrac{D}{2}-\alpha_5\right); \qquad (3.8)$$

$$F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=F(D-\alpha_1-\alpha_2-\alpha_5,\alpha_2,\alpha_3,D-\alpha_3-\alpha_4-\alpha_5,\alpha_5); \qquad (3.9)$$

$$F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=a(D-\alpha_1-\alpha_2-\alpha_5)\,a(\alpha_1+\alpha_4+\alpha_5-\tfrac{D}{2})\,a(\alpha_2+\alpha_3+\alpha_5-\tfrac{D}{2})\cdot \qquad (3.10)$$
$$\cdot a(D-\alpha_3-\alpha_4-\alpha_5)\,F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6); \qquad \alpha_6=\frac{3D}{2}-\sum_{i=1}^{5}\alpha_i.$$

Eq. (3.8) corresponds to the Fourier transformation and transition to the dual representation, eq. (3.9) to inversion. Eq. (3.10) is interesting because it is a functional equation on unknown function $\tilde{F}(\alpha_5)$ depending on variable α_5 (the other variables are parameters here).

Many enough solutions of the system of eqs. (3.5a)-(3.7) are known for the particular choice of α_i (see [3,5,8]). One can conclude from studying these equations, that $F(\vec{\alpha})$ is a meromorphic function

with simple poles at the points (they are defined by the singularities of the Γ -functions):

$$\begin{aligned}
&1)\ \alpha_i=D/2+n;\quad i=1,2,\dots,5\\
&2)\ \alpha_1+\alpha_2+\alpha_5=D/2-n;\\
&\quad\ \alpha_3+\alpha_4+\alpha_5=D/2-n; \qquad\qquad (3.11)\\
&3)\ \alpha_1+\alpha_4+\alpha_5=D+n;\\
&\quad\ \alpha_2+\alpha_3+\alpha_5=D+n;\\
&4)\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=D-n;\qquad n=0,1,2,\dots
\end{aligned}$$

This distribution of the poles agrees with the general theorems on singularities of the dimensionally (and analytically) regularized Feynman integrals (cases 1) and 3) correspond to ultraviolet divergences of integrals in the x -space; 2) and 4), to infrared devergences[13]). It is clear that these poles can overlap or cancel each other for particular values of α_i. Residues at all these poles can be found with the help of the formulae such as (A.1), (A.4), and also the relation

$$\left.\frac{1}{(x-y)^{2\alpha_1}(y-z)^{2\alpha_2}(z-x)^{2\alpha_3}}\right|_{\sum_i\alpha_i\to D}\longrightarrow\frac{\pi^{\frac{D}{2}}\,v(\alpha_1,\alpha_2,D-\alpha_1-\alpha_2)}{\Gamma(D/2)\,(D-\alpha_1-\alpha_2-\alpha_3)}\delta^D(x-y)\,\delta^D(z-x) \qquad (3.12)$$

and formulae, derived from the above relations by means of differential operators ∂_x^2, ∂_y^2, etc.

It is natural (see (3.11)) to seek function $F(\vec{\alpha})$ in the form

$$F(\vec{\alpha})=f(\vec{\alpha},\epsilon)\prod_{i=1}^{6}\Gamma\left(\tfrac{D}{2}-\alpha_i\right)\Gamma(D-\alpha_1-\alpha_5-\alpha_4)\,\Gamma(D-\alpha_2-\alpha_3-\alpha_5)\cdot \qquad (3.13)$$
$$\cdot\Gamma\left(\alpha_3+\alpha_4+\alpha_5-\tfrac{D}{2}\right)\Gamma\left(\alpha_1+\alpha_2+\alpha_5-\tfrac{D}{2}\right);\quad \alpha_6=\frac{3D}{2}-\sum_{i=1}^{5}\alpha_i,\ D=4-2\epsilon.$$

Here $f(\vec{\alpha},\epsilon)$ is an integral function of all its arguments, and the system of eqs. (3.5a)-(3.7) takes the form

$$\begin{aligned}
f(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)&=f(\alpha_2,\alpha_5,\alpha_4,\alpha_6,\alpha_1)=\\
&=f(\alpha_4,\alpha_5,\alpha_2,\alpha_6,\alpha_3)=\\
&=f(\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_5);\qquad \alpha_6=\frac{3D}{2}-\sum_{i=1}^{5}\alpha_i;
\end{aligned}$$

$$f(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=f\left(\tfrac{D}{2}-\alpha_2,\tfrac{D}{2}-\alpha_1,\alpha_3,\alpha_4,\alpha_1+\alpha_2+\alpha_5-\tfrac{D}{2}\right);$$

$$f(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = (D-2\alpha_5-\alpha_3-\alpha_4)^{-1}\cdot$$

$$\alpha_3\left(\frac{(D/2-\alpha_5)(D-\alpha_1-\alpha_4-\alpha_5)}{(D/2-\alpha_3-1)(\alpha_1+\alpha_2+\alpha_5-1-\frac{D}{2})}\cdot f(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4,\alpha_5-1) - \frac{(D/2-\alpha_2)(\alpha_3+\alpha_4+\alpha_5-D/2)}{(D/2-\alpha_3-1)(\alpha_1+\alpha_2+\alpha_5-1-\frac{D}{2})}\cdot f(\alpha_1,\alpha_2-1,\alpha_3+1,\alpha_4,\alpha_5)\right)+ \qquad (3.14)$$

$$+\alpha_4\left(\frac{(D/2-\alpha_5)(D-\alpha_2-\alpha_3-\alpha_5)}{(D/2-\alpha_4-1)(\alpha_1+\alpha_2+\alpha_5-1-\frac{D}{2})}\cdot f(\alpha_1,\alpha_2,\alpha_3,\alpha_4+1,\alpha_5-1) - \frac{(D/2-\alpha_1)(\alpha_3+\alpha_4+\alpha_5-D/2)}{(D/2-\alpha_4-1)(\alpha_1+\alpha_2+\alpha_5-1-\frac{D}{2})}\cdot f(\alpha_1-1,\alpha_2,\alpha_3,\alpha_4+1,\alpha_5)\right)\Bigg].$$

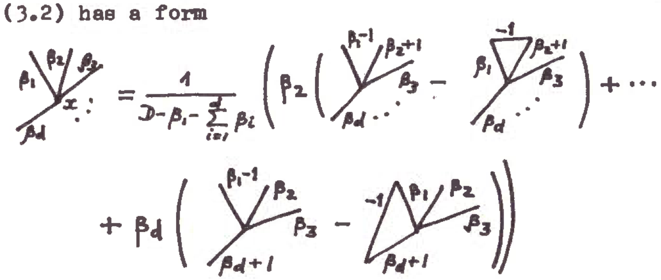
Unfortunately, in spite of so rich information, the general solution of this system is not found yet. The discussion of a special case of this system can be found in /10/.

4. Thus, in the present paper we have suggested a generalization of the identity $\int d^D x\,(x^2)^{-\alpha} = 0,\ \alpha \neq D/2$, see (2.3). With the help of this generalization all sets of the dimensionally (and analytically) regularized P-integrals can be divided into classes of equal integrals with rather different structures. As a consequence, we calculated some new classes of integrals not evaluated earlier.

It has been shown, using the example of two-loop integrals, that the dimensionally (and analytically) regularized integrals possess nontrivial symmetry properties which naturally follow from the found relation of the P-integrals to V-integrals. The symmetries found are used for deriving the complete set of the functional equations for the coefficient functions of two-loop P-integrals. These equations follow from the formulae of "uniqueness" and integration by parts.

We draw the reader's attention to some possible generalizations of our results. A part of results of the third section of the paper can be generalized to the case of diagrams with the number $d+1\,(>4)$ of vertices. Really, linking all $d+1$ vertices of this propagator diagram by lines (indices of which can be equal to zero), we get a vacuum diagram. One can imagine this diagram as a d-simplex in d-dimensional space (in the case of $d+1=4$, we deal with a tetrahedron in the 3-dimensional space). In that case we have $\sum_i \alpha_i = \frac{dD}{2}$ for indices α_i of its lines. The simplex coefficient function is

symmetric with respect to all permutations of the indices α_i corresponding to any permutations of vertices (or to unproper rotations). This symmetry results in d equations analogous to the equations (3.5a). Further, a generalization of the integration by parts formula (3.2) has a form



and leads to an equation like (3.7). Unfortunately, we do not know transformation like (3.6) of the d-simplex although one can obtain special cases of this transformation if he puts zero some of the indices or makes inversion of the corresponding integral with the base in one of the vertices.

The coefficient function of the d-simplex is a meromorphic function with simple poles at the points, in which sum of the indices of any its k-dimensional subsimplices $(k<d)$ is $\frac{kD}{2}+n$ $(n \geq 0$ -integer). As it was emphasized above, this statement is in agreement with the general theorems of the papers /13/.

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Appendix

Unlike momentum space, the procedure of decomposing vertices in position space does not reduce the number of integrations. Indeed, due to translational invariance we can consider the decomposed vertex as an "external" one. As a result of cutting the vertex, a dimensionless propagator diagram is obtained. Such a diagram does not depend on external coordinates (see eq. (2.4)), thus, without changing its value we could set both the external vertices to zero. Then, the initial vacuum generic graph is restored and we could conclude that the corresponding integrals are equal. However, this argument is a formal one, because of the divergence of all dimensionless integrals. When the dimension becomes zero, coefficient functions of P-integrals acquire pole singularities whereas V-integrals exhibit the behaviour like $\delta(\beta)$. The problem of relating the coefficients at these singularities arises. It can be solved as follows.

Consider a $\mathcal{V}$ -integral (2.5) and set $\sigma = D/2 - \xi$. Further, multiply both sides of (2.5) by ξ and take the limit $\xi \to 0$. Then, using the relation

$$\lim_{\xi \to 0} \xi \frac{1}{(x^2)^{D/2-\xi}} = \frac{\pi^{D/2}}{\Gamma(D/2)} \delta^D(x), \tag{A.1}$$

we get

$$\lim_{\xi\to 0} \xi \left\{ \right\} = \lim_{\xi \to 0} \xi X(\vec{\alpha}) \delta(\beta - \xi) \pi i \Omega_D =$$

$$= \frac{\pi^{D/2}}{\Gamma(D/2)} \left\{ \right\} \equiv \frac{\pi^{D/2}}{\Gamma(D/2)} \left\{ Y(\vec{\alpha}) \delta(\beta) \right\} \pi i \Omega_D, \tag{A.2}$$

where $Y(\vec{\alpha})$ is the coefficient function of the $\mathcal{V}$ -integral arising after the verices x and 0 are shrunk to one point. Assuming $X(\vec{\alpha})$ at $\beta \to 0$ to be

$$X(\vec{\alpha}) \underset{\beta \to 0}{\longrightarrow} \frac{\mathrm{res}\, X(\vec{\alpha})}{\beta} + o\left(\frac{1}{\beta}\right), \tag{A.3}$$

where $\mathrm{res}\, X(\vec{\alpha})$ is the residue of $X(\vec{\alpha})$ at the point $\beta = 0$, we obtain the following relation:

$$\frac{\pi^{D/2}}{\Gamma(D/2)} Y(\vec{\alpha}) = \mathrm{res}\, X(\vec{\alpha}). \tag{A.4}$$

It only remains to note that $Y(\vec{\alpha})$ is connected via (2.8) with the coefficient function of the p -integral arising after decomposing a vertex in p -space.

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Горишний С.Г., Исаев А.П. E2-84-192

Об одном подходе к вычислению многопетлевых безмассовых фейнмановских интегралов

Предложено обобщение тождества размерной регуляризации $\int d^Dk(k^2)^{-\alpha}=0$, $\alpha\neq D/2$. Обобщение использовано для разбиения всего множества размерно /и аналитически/ регуляризованных фейнмановских интегралов с одним внешним импульсом на классы равных интегралов, а также для вычисления некоторых из них. Вскрыта нетривиальная симметрия пропагаторных интегралов, на основе которой записана полная система функциональных уравнений для определения двухпетлевых интегралов. Обсуждаются возможные обобщения этих уравнений.

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Gorishny S.G., Isaev A.P. E2-84-192

On an Approach to the Calculation of Multiloop Massless Feynman Integrals

A generalization of the dimensional regularization identity $\int d^Dk(k^2)^{-\alpha} = 0$, $\alpha\neq D/2$. is suggested, which is used for grouping dimensionally /and analytically/ regularized massless Feynman integrals with one external momentum into classes of equal ones, and also for calculating some of them. A nontrivial symmetry of propagator-type integrals is revealed and used for deriving a complete set of functional equations for two-loop integrals. Possible generalizations of these equations are discussed.

The investigation has been performed at the Laboratory of Theoretocal Physics, JINR.

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