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SOLITON-LIKE EXCITATIONS IN A ONE-DIMENSIONAL NUCLEAR MATTER

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I. INTRODUCTION

After more than ten years of explosive growth the interest to nonlinear models possessing soliton solutions and to their applications in various branches of physics still does not diminish. One of the most popular among them is clearly the nonlinear Schrödinger (NLS) equation. Its simplest cubic-nonlinearity form (S3), possessing U(1) symmetry, $i\psi_t + \psi_{xx} + g\psi|\psi|^2 = 0$ describes the quasiclassical gas of Bose particles with δ-function two-body interaction. In the case of attraction (g positive) the stable ground (vacuum) state of the system is trivial, i.e., ψ vanishes everywhere. Excitations over this vacuum are either linear waves (continuous spectrum) or drops, i.e., solitons. In the case of the repulsive potential the condensate constituted by an infinite amount of bosons can serve as a stable vacuum along with the trivial one. In their turn, excitations over the condensate are subdivided into the Bogolubov spectra and hole-like soliton modes [1]. The more complicated situation arises for the NLS equations possessing noncompact global invariance groups, where mixed hole-drop soliton modes become possible [2].

The other way to have both the types of stable ground state in the frame of the same, say, U(1)-invariant model implies increasing the degree of nonlinearity in Hamiltonian. The simplest instance of this is given by adding the next, ψ^6-term to the S3 Hamiltonian. Earlier this way obtained the so-called $\psi^4-\psi^6$ theory occured in context of several physical problems. Friedberg et al.[3] have derived this equation (in static form) studying soliton models for hadrons. It shed as well a new light on the description of heavy ion collisions, involving both the time-dependent Hartree-Fock approach [4] (for review see[5])and the nuclear hydrodynamics with Skyrme's forces [6]. The said model appeared to be also interesting from mathematical point of view, for it is the simplest nonlinear theory, possessing stable many-dimensional solitons [7,8] and admits rigorous existence theorems [9,10].

However, it is the soliton solutions over the trivial vacuum that have been considered in all mentioned papers. Here we look at this matter the other way round. Namely, we find a solution, describing a very heavy nucleus* such that in

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*Here and below we make use of the nuclear language mainly for convenience.

the limit of infinite number of nucleons it is transformed into the condensate, or "nuclear matter". Then we analyze the spectra of elementary excitations over this condensate, which thus correspond to internal excitations in heavy nuclei.

In this sense the present paper continues the study of condensate excitations in different NLS modifications /2,11/. From other point of view, the $\psi^4-\psi^6$ theory is now seriously treated as to describe the dynamical properties of heavy nuclei. Specifically, such remarkable phenomena as nuclear molecules and "fusion windows" (i.e., resonant in energy and angular momentum fusion regions of two collided nuclei) have been found by computer simulations in the frame of the above said equation /4/. The former is by now a well-established experimental fact, while the latter may throw a crucial light on the possibility of superheavy nuclei production in heavy ion collisions. Thus, the detailed study of various partial solutions of this non-integrable NLS version seems to be extremely interesting and even necessary.

Here we do not derive the $\psi^4-\psi^6$ NLS equation on the basis of nuclear theory concepts and give no arguments in its favour since all this has been done not far ago /5/.

The remainder of this paper is organized as follows. In Sec.II we give some connection between different versions of the $\psi^4-\psi^6$ NLS equation, find the soliton-like solution under the vanishing boundary conditions and demonstrate how it is transformed into the condensate. Sections III and IV are devoted to soliton-like excitations in the condensate (i.e., solutions under constant boundary conditions), which are found approximately in Sec.III and exactly in Sec.IV. The stability analysis is in the Vth Section followed by discussion of the results (Sec.VI).

II. GENERAL RELATIONS AND PASSAGE TO THE CONDENSATE

The evolution equation of the $\psi^4-\psi^6$ theory is of the form

$$i\psi_t + \Delta\psi \pm \psi + \psi|\psi|^2 + a\psi|\psi|^4 = 0,$$

where $a<0$. It describes, for instance, the Bose-gas with δ-function interaction potential, which is attractive for two-body collisions, and repulsive for three-body ones. In order to get solutions in a closed explicit form, we analyze the one-dimensional version of the above equation

$$i\psi_t + \psi_{xx} \pm \psi + \psi|\psi|^2 + a\psi|\psi|^4 = 0 \qquad (2.1)$$

(subscripts denote differentiation with respect to t and x, respectively). A simple scaling of the field and coordinates yields the equivalent form,

$$i\psi_t + \psi_{xx} + a\psi + \psi|\psi|^2 - \psi|\psi|^4 = 0, \qquad (2.2)$$

where a now can be both positive and negative. As soon as the nonlinearity $\psi^3-\psi^5$ appears "in a pure form" in eq.(2.2) and a may be completely arbitrary, we believe that eq. (2.2) is more useful and illuminating than (2.1). Another version of (2.1) and (2.2) is given by

$$i\phi_t + \phi_{xx} - r_0(2r_0 + A)\phi + 2(2r_0 + A)|\phi|^2\phi - 3|\phi|^4\phi = 0, \qquad (2.3)$$

where $r_0 > 0$, $2r_0+A>0$ and A/r_0 is a function of a:

$$A_{1,2} = \frac{r_0}{2}\{-4 - \frac{3}{2a} \pm \frac{3}{2|a|}\sqrt{1+4a}\}. \qquad (2.4)$$

Eq.(2.3) turns out to be the most convenient in finding solutions under constant boundary conditions, while one-parameter eq. (2.2) is important in classifying them. From eq.(2.4) we learn that the one-parameter version (2.2) is reducible to (2.3) only when $a \geq -1/4$.

Note that the $a\psi$ term in eq.(2.2) may be removed by the substitution

$$\psi(x,t) = \exp\{iat\} \times \tilde{\psi}(x,t) \qquad (2.5)$$

only in the case of vanishing boundary conditions, because otherwise (2.5) is inconsistent with

$$\psi(x,t) \to \psi_0 e^{i\theta_0^{\pm}} \quad \text{as} \quad x \to \pm\infty, \qquad (2.6)$$

i.e., boundary conditions corresponding to the condensate at rest.

Let us first consider the case of the zero conditions,

$$\psi(x,t) \to 0 \quad \text{as} \ |x| \to \infty. \qquad (2.7)$$

The first two integrals of motion for eq.(2.2), viz., energy and number of nucleons read

$$E = \int dx\{|\psi_x|^2 - 1/2|\psi|^4 + 1/3|\psi|^6\} \qquad (2.8)$$

and

$$N = \int |\psi|^2 dx. \qquad (2.9)$$

Under assumption (2.7) eq.(2.2) can be two times integrated to give a soliton ("nucleus") at rest*

$$\psi_n(x,t) = e^{i(\theta_0 + \beta t)} \times \left[\frac{4(\beta - a)}{1 + \sqrt{1 + (16/3)(a-\beta)}\cosh\{2\sqrt{\beta - a}\,(x - x_0)\}}\right]^{1/2}. \quad (2.10)$$

The travelling solution is obtained from (2.10) via the Galilean boost:

$$\psi_n(x,t) = e^{i(\frac{v}{2}x - \frac{v^2}{4}t + \beta t + \theta_0)} \times \left[\frac{4(\beta - a)}{1 + \sqrt{1 + \frac{16}{3}(a-\beta)}\cosh\{2\sqrt{\beta - a}(x - vt - x_0)\}}\right]^{1/2}. \quad (2.11)$$

This solution depends on four real parameters, x_0 , θ_0 , v and β. The first three of them are completely arbitrary, while the last one (β) lies within $[a, a + 3/16]$ interval. The corresponding conserved quantities (2.1), (2.8) are now

$$N_n(a-\beta) = \sqrt{3}\cosh^{-1}\left[\left(1 + \frac{16}{3}(a-\beta)\right)^{-1/2}\right], \quad (2.12)$$

$$E_n(a-\beta) = \left(v^2 - \frac{3}{4} + \beta - a\right) N_n(a-\beta) + \frac{3}{4}\sqrt{\beta - a}\,. \quad (2.13)$$

When $(a-\beta) \longrightarrow -3/16$, N_n increases infinitely along with the soliton's width, while the amplitude approaches the constant value of $\sqrt{3}/2$. This saturation is a crucial feature of nuclear many-body systems (see, e.g.,/5/). If $(a-\beta)$ is exactly (-3/16), (2.10) is none other than the condensate,

$$\psi_c(x,t) = \exp\{i(\theta_0 + \beta t)\} \times \sqrt{3}/2\,.$$

This fact implies that superheavy ($N_n \to \infty$) nucleus (2.10) may be regarded as the condensate state, i.e., as a transition to the "nuclear matter". Together with the saturation, this feature is believed to be one of the most seducing attributes of the discussed model.

The next step consists in examining of small oscillations about such a condensate. Here eq.(2.3) assumes its importance since it explicitly involves the condensate density r_0 , so we study its solution of the form $\phi(x,t) = \phi_0$, r_0 being merely $|\phi_0|^2$. Perturbing the condensate biharmonically,

$$\phi(x,t) = \phi_0 + \xi(x,t),$$

$$\xi(x,t) = \eta_1 \exp\{i(kx - \omega t)\} + \eta_2 \exp\{-i(kx - \omega t)\},$$

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*An equivalent solution has been independently found by Kartavenko /6/.

and equating coefficients of equal phases yields the set

$$\omega\eta_1 - k^2\eta_1 - 2r_0(r_0 - A)\eta_1 - 2\phi_0^2(r_0 - A)\eta_2^* = 0,$$

$$-\omega\eta_2^* - k^2\eta_2^* - 2r_0(r_0 - A)\eta_2^* + 2\phi_0^{*2}(A - r_0)\eta_1 = 0.$$

Putting its determinant equal to zero, we find the dispersion relation for the Bogolubov (phonon) spectra as

$$\omega^2 = k^2[k^2 + 4r_0(r_0 - A)]\,. \quad (2.14)$$

From (2.14) the vacuum stability condition is straightforward,

$$r_0 \geq A\,. \quad (2.15)$$

Relation (2.14) also defines the velocity of acoustic waves,

$$c = \lim_{k\to 0}\frac{\omega}{k} = 2[r_0(r_0 - A)]^{1/2}\,. \quad (2.16)$$

III. SMALL AMPLITUDE NONLINEAR WAVE APPROXIMATION

In order to obtain a weakly nonlinear solution to eq.(2.3) we make use of the polar decomposition of ϕ,

$$\phi(x,t) = \rho(x,t)\exp\{i\theta(x,t)\}; \quad \rho, \theta \in R \quad (3.1)$$

and arrive at

$$\rho_t = -\theta_{xx}\rho - 2\theta_x\rho_x\,, \quad (3.2)$$

$$\theta_t = \rho_{xx}\rho^{-1} - \theta_x^2 - (\rho^2 - r_0)(3\rho^2 - r_0 - 2A)\,. \quad (3.3)$$

It appears useful to introduce here the following new variables instead of ρ and θ : $r = \rho^2$; $w = \theta_x$. Then eq.(3.2) is equivalent to

$$r_t = -2(wr)_x\,, \quad (3.4)$$

while differentiating of eq.(3.3) leads to

$$w_t = \left(\frac{1}{2}\frac{r_{xx}}{r} - \frac{1}{4}\frac{r_x^2}{r^2}\right)_x - 2ww_x + [(r_0 - r)(3r - r_0 - 2A)]_x\,. \quad (3.5)$$

Let us pass to the "slow time" reference frame, travelling with the velocity of sound /12/, $\xi = \sqrt{\epsilon}(x - ct)$; $\tau = \epsilon^{3/2}t$. Then the nuclear hydrodynamics equations (3.4), (3.5) become

$$\epsilon r_\tau - c r_\xi + 2(wr)_\xi = 0, \tag{3.6}$$

$$w_\xi c - \epsilon w_\tau + \epsilon\left(\frac{r_{\xi\xi}}{2r} - \frac{r_\xi^2}{4r^2}\right)_\xi - 2ww_\xi + [(r_0 - r)(3r - r_0 - 2A)]_\xi = 0. \tag{3.7}$$

We expand now r and w in powers of small magnitude ϵ around the vacuum configuration $r = r_0$, $w = 0$,

$$r = r_0 + \epsilon r_1 + \epsilon^2 r_2 + \dots \qquad w = \epsilon w_1 + \epsilon^2 w_2 + \dots$$

and insert these expansions into eqs.(3.6), (3.7). Equating then the coefficients at first and second powers of ϵ, we obtain

$$c r_{1\xi} = 2r_0 w_{1\xi}, \tag{3.8}$$

$$c r_{2\xi} = r_{1\tau} + 2(w_1 r_1)_\xi + 2r_0 w_{2\xi}, \tag{3.9}$$

$$c w_{1\xi} = 2(r_0 - A) r_{1\xi}, \tag{3.10}$$

$$c w_{2\xi} = w_{1\tau} - \frac{r_{1\xi\xi\xi}}{2r_0} + 2w_1 w_{1\xi} + 2(r_0 - A) r_{2\xi} + 3(r_1^2)_\xi. \tag{3.11}$$

Integrating eq.(3.8) (or equivalently eq.(3.10)) gives

$$w_1 = (c/2r_0) r_1. \tag{3.12}$$

The set (3.9), (3.11) added by the relation (3.12) is easily reduced to the celebrated Korteveg-de Vries (KdV) equation,

$$2\sqrt{r_0(r_0 - A)}\, r_{1\tau} - \frac{1}{2} r_{1\xi\xi\xi} + 3(2r_0 - A)(r_1^2)_\xi = 0. \tag{3.13}$$

After the space and time scaling

$$\xi \to [2(2r_0 - A)]^{-1/2}\xi; \qquad \tau \to \frac{c}{\sqrt{2}}(2r_0 - A)^{-3/2}\tau$$

eq.(3.13) becomes KdV in the familiar form,

$$r_{1\tau} - r_{1\xi\xi\xi} + 3(r_1^2)_\xi = 0, \tag{3.14}$$

which possesses the well-known soliton solution:

$$r_1(\xi, \tau) = -\frac{g}{2}\operatorname{sech}^2\left[\frac{\sqrt{g}}{2}(\xi + g\tau - \xi_0)\right]. \tag{3.15}$$

Returning to the eq.(2.3), we find that the solution (3.15) is transformed into the following one:

$$\rho(x, t) = \left\{r_0 - \frac{a}{2r_0 - A} \times \operatorname{sech}^2\sqrt{a}\left(x - ct + \frac{2a}{c}t - x_0\right)\right\}^{1/2}, \tag{3.16}$$

with

$$a = \frac{\epsilon}{2} g(2r_0 - A) \geq 0, \tag{3.17}$$

and the inequality $2r_0 - A > 0$ being simply consequence of the stronger relation (2.15).

The solution (3.16) to eq.(2.3) (or, equivalently, eqs.(3.2)-(3.3)) describes a localized rarefaction domain, propagating with the velocity close but less than that of the acoustic waves. From KdV representation of our model it follows that localized stable nonlinear waves of compression, or humps, travelling through the condensate are not eigenmodes of the system. An initial excitation of that kind will decay into a dispersive succession of linear acoustic waves with periodically alternating rarefactions and compressions (the last compression may be rather strong but gradually damping shock). Indeed, the hump solitons are the solutions of that KdV version, which in the laboratory frame looks like

$$2\sqrt{r_0(r_0 - A)}\,(r_{1\tau} + c r_{1\xi}) + \frac{1}{2} r_{1\xi\xi\xi} + 3(2r_0 - A)(r_1^2)_\xi = 0, \tag{3.18}$$

while eq.(3.13) in this frame is

$$2\sqrt{r_0(r_0 - A)}\,(r_{1\tau} + c r_{1\xi}) - \frac{1}{2} r_{1\xi\xi\xi} + 3(2r_0 - A)(r_1^2)_\xi = 0. \tag{3.19}$$

The dispersion relations of eqs.(3.18) and (3.19) are, respectively

$$\omega = k\left(c - \frac{k^2}{2c}\right) \tag{3.20}$$

and

$$\omega = k\left(c + \frac{k^2}{2c}\right). \tag{3.21}$$

The dispersion (2.14) of Bogolubov's spectra $\omega = k(k^2 + c^2)^{1/2}$ for small k transforms into the formula (3.21), but by no means into (3.20).

The second consequence of the KdV representation consists in the fact that a localized rarefaction produced by decay of a hole-like initial condition "sends" ahead linear waves, "forerunners". Having information on the rarefaction's velocity (or, equivalently, on its amplitude) and on the separation distance between the soliton and the forerunners, one can estimate the propagation distance back to the source. This provides a means of estimating the location of the initial wave form.

IV. EXACT SOLUTIONS UNDER THE NONVANISHING BOUNDARY CONDITIONS

Let us impose the following boundary conditions on the field ϕ:

$$\phi(x, t) \to \rho_0 \exp\{i\theta_0^{\pm}\} \quad \text{as } x \to \pm\infty. \tag{4.1}$$

In order to assure the finiteness of energy and to obey the equation at $|x| = \infty$ simultaneously, $r_0 = \rho_0^2$ must be the double zero of polynomial part of the energy, viz.

$$E = \int dx\, \{|\phi_x|^2 + (|\phi|^2 - r_0)^2 (|\phi|^2 - A)\}. \tag{4.2}$$

Here A is a real constant. As is easily verified, the evolution equation related to the Hamiltonian (4.2) is exactly eq.(2.3). This observation reveals why we have chosen just the ϕ notation from the beginning of this Section. A is therefore given by formula (2.4). The number of nucleons for eq. (2.3) is

$$N = \int dx\, \{|\phi|^2 - r_0\}. \tag{4.3}$$

As soon as the motionless condensate conditions (4.1) violate the Galilean invariance (it's impossible to boost a solution keeping (4.1) unchanged), we search for the travelling solutions from the very beginning, i.e.,

$$\phi(x, t) = \phi(\zeta), \quad \zeta = x - vt. \tag{4.4}$$

The polar decomposition (3.1) of the field ϕ appears helpful again and the set (3.2), (3.3) with ansatz (4.4) is converted into

$$\rho'' + v\theta'\rho - (\theta')^2\rho - \rho(\rho^2 - \rho_0^2)(3\rho^2 - \rho_0^2 - 2A) = 0, \tag{4.5}$$

$$-v\rho' + 2\theta'\rho' + \theta''\rho = 0; \quad {}' \equiv d/d\zeta. \tag{4.6}$$

Upon multiplying (4.6) by ρ and integrating we arrive at

$$\rho^2\theta' + \frac{v}{2}(\rho_0^2 - \rho^2) = 0, \tag{4.7}$$

boundary conditions (3.1) having been used. Expressing θ' from (4.7), substituting it into (4.5) and integrating with the factor $2\rho'$ gives finally

$$\pm 2(\zeta - \zeta_0) = \int dr\, (r - r_0)^{-1} \times (r^2 - rA - \frac{v^2}{4})^{-1/2},$$

where $r = \rho^2$. Let $z = r - r_0$ and

$$a = r_0^2 - Ar_0 - \frac{v^2}{4} = (c^2 - v^2)/4, \tag{4.8}$$

then the integral becomes standard,

$$\pm 2(\zeta - \zeta_0) = \int dz \times z^{-1} \times (z^2 + (2r_0 - A)z + a)^{-1/2}. \tag{4.9}$$

Localized solutions $z(\zeta)$ emerge only if $a > 0$, i.e.,

$$v^2 < c^2. \tag{4.10}$$

The inequality (4.10) states that solitary wave solutions to eq.(2.3) under conditions (4.1) can travel not faster than acoustic waves. Two appearing solutions are defined by

$$z_{\pm}(\zeta) = \frac{\pm 2a}{\sqrt{A^2 + v^2}\cosh\{2\sqrt{a}(\zeta - \zeta_0)\} \pm (A - 2r_0)}.$$

The first one is however singular due to (4.10). The modulus of it looks like

$$\rho_s(\zeta) = \left\{ r_0 + \frac{2a}{\sqrt{A^2 + v^2}\cosh\{2\sqrt{a}(\zeta - \zeta_0)\} + A - 2r_0} \right\}^{1/2}. \tag{4.11}$$

Expression (4.11) is defined only outside the interval (ζ_1, ζ_2), where ζ_1, ζ_2 are two roots of equation

$$\sqrt{A^2 + v^2}\cosh\{2\sqrt{a}(\zeta - \zeta_0)\} = 2r_0 - A.$$

When $\zeta \to \zeta_1 - 0$ or $\zeta \to \zeta_2 + 0$, solution (4.11) becomes infinite. The condensate, in which $\phi_s(\zeta)$ propagates is stable under condition (2.15).

The second solution defined by z_- is obviously regular and approaches the stable vacuum at the spatial infinities, inequality (2.15) being the consequence of (4.10). Here we are to distinguish two cases.

The first one is $0 \le A \le r_0$, or in terms of the single-parameter eq.(1.2) $-1/4 \le \alpha \le -3/16$, when the solution in the rest frame looks like

$$\phi_h(x) = e^{i\theta_0}\sqrt{r_0}\cosh(\frac{c}{2}(x - x_0)) \times \{\frac{r_0}{A} + \sinh^2(\frac{c}{2}(x - x_0))\}^{-1/2}. \tag{4.12}$$

For (4.12) we observe that

$$\rho_h(x_0) = \sqrt{A} > 0; \quad \frac{d\phi_h}{dx}(x_0) = 0, \tag{4.13}$$

i.e., the solution is hole-shaped. We shall call it simply "hole".

A different situation arises for the second case, viz. $A \leq 0$ or, equivalently, $\alpha \geq -3/16$. Here for the rest frame form we have

$$\phi_k(x) = e^{i\theta_0}\sqrt{r_0}\tanh(\frac{c}{2}(x-x_0)) \times \{1+\frac{r_0}{|A|}[\cosh(\frac{c}{2}(x-x_0))]^{-2}\}^{-1/2} \tag{4.14}$$

and we notice that in contrast to (4.13)

$$\rho_k(x_0) = 0; \quad \frac{d\phi_k}{dx}(x_0) = \sqrt{-A}\, r_0^{5/2} > 0.$$

Hence at $\alpha \geq -3/16$ our model possesses a kink solution.

The phase of both the kink and the hole solutions is straight-forward from eq.(4.7). The result is the travelling form,

$$\phi_{k,h}(x,t) = e^{i\theta_0}\frac{\tanh\{\sqrt{\bar a}(\zeta-\zeta_0)\}\,\mathrm{tg}\frac{\mu}{2} + i}{1+\tanh^2\{\sqrt{\bar a}(\zeta-\zeta_0)\}\,\mathrm{tg}^2\frac{\mu}{2}} \times \tag{4.15}$$
$$\times\left\{\frac{Ar_0 + \frac{v^2}{2} + r_0\sqrt{A^2+v^2}\cosh\{2\sqrt{\bar a}(\zeta-\zeta_0)\}}{2r_0 - A + \sqrt{A^2+v^2}\cosh\{2\sqrt{\bar a}(\zeta-\zeta_0)\}}\right\}^{1/2}.$$

Here as before $\zeta = x - vt$, $\cosh\mu = \dfrac{Ar_0 + v^2/2}{r_0\sqrt{A^2+v^2}}$, the kink and the hole being distinguished by the sign of A. The related integrals of motion are

$$N_{k,h} = -\cosh^{-1}\left(\frac{2r_0 - A}{\sqrt{A^2+v^2}}\right), \tag{4.16}$$

$$E_{k,h} = \sqrt{\bar a}\,(\frac{A}{2}+r_0) + N_{k,h}\times(Ar_0 + \frac{v^2-A^2}{4}). \tag{4.17}$$

It is worth noticing that in spite of the fact that the kink and the hole are described formally via the same formulae (4.15)-(4.17), they are essentially different solutions.

At small amplitudes a and at v only slightly differing from the sound velocity, the modulus of (4.15)

$$\rho_-(\zeta) = \rho_{k,h}(x,t) = \left\{r_0 - \frac{2a}{\sqrt{A^2+v^2}\cosh\{2\sqrt{\bar a}(\zeta-\zeta_0)\} + 2r_0 - A}\right\}^{1/2}$$

coincides with the approximate solution (3.16).

Finally, the last possibility of boundary conditions to be compatible with finiteness of energy is the following

$$\phi(x,t) \to 0 \text{ as } x \to -\infty, \quad \phi(x,t) \to \sqrt{r_0}\, e^{i\theta_0} \text{ as } x \to +\infty, \tag{4.18}$$

or vice versa. The mixed conditions are permissible for $A = 0$, or $\alpha = -3/16$. In this case we obtain two pairs of solutions in the rest frame: singular ones, for which

$$r_\pm(x) = r_0[1 - 2r_0\exp\{\pm 2r_0(x-x_0)\}]^{-1} \tag{4.19}$$

and a regular pair ("precipice" solutions)

$$\phi_p(x) = e^{i\theta_0}\sqrt{r_0}[1 + 2r_0\exp\{\pm 2r_0(x-x_0)\}]^{-1/2}. \tag{4.20}$$

Regularization cannot be performed at calculating the number of nucleons for (4.20). Of course we can count off some intermediate value of r, say $r_0/2$. However, since N defined in this way as $N_p \equiv \int(r_p - \frac{r_0}{2})dx$ vanishes, the definition turns out to be useless. At the same time, (4.20) are finite-energy solutions,

$$E_p = r_0^2/2. \tag{4.21}$$

Travelling precipices do not exist under the motionless condensate boundary conditions (4.18). Indeed, the moving solution with different asymptotics in $|\phi|^2$ would be obviously changing the area under the graph of $|\phi|^2$. The latter, however, is forbidden by the nucleon number conservation. Nevertheless, the precipice interacting with some other solution can be made moving by permanent compensation of the mentioned area changing. For example, assume that a plane wave falls on the precipice from one of the infinities. In this case the precipice can be shown to travel with a constant velocity against this flow of nucleons.

The class of solutions under the constant conditions is certainly not exhausted by (4.11), (4.15), (4.19) and (4.20). There is a lot of solutions localized weaker than exponentially, for example, at $A = r_0$ ($\alpha = -1/4$) along with the "pure" condensate (4.15) eq.(2.3) admits rational singular solution,

$$\phi_r(x) = e^{i\theta_0}\sqrt{r_0}\times(x-x_0)\times[(x-x_0)^2 - r_0^{-1}]^{-1/2} \qquad (4.22)$$

(rest frame). It exists outside the region $(-1/\sqrt{r_0}+x_0, 1/\sqrt{r_0}+x_0)$.

V. STABILITY UNDER SMALL PERTURBATIONS

Let us examine first the stability of the nucleus solution (2.11) to eq.(2.2). Making use of the Galilean and U(1)-invariances, we pass to the $v = x_0 = 0$ frame of reference and choose $\theta_0 = 0$, denoting $\Phi(x)\exp\{i\beta t\}$ $(\Phi = \Phi^*)$ solution obtained in this way. The perturbed solution is chosen to be $\psi(x,t) = (\Phi(x) + \eta(x,t)) \times \exp\{i\beta t\}$. Assuming $\eta(x,t)/\Phi(x)$ to be small everywhere (at least initially), we linearize eq.(2.2) with respect to (η/Φ) and suppose the disturbance to possess the usual time behaviour: $\eta(x,t) = \{f(x) + ig(x)\}\exp(\nu t)$; $f, g, \nu \in R$. Then the linearized eq.(2.2) is reduced to

$$\hat{U}f = (-d^2/dx^2 + \beta - a - 3\Phi^2 + 5\Phi^4)f = -\nu g,$$
$$\hat{L}g = (-d^2/dx^2 + \beta - a - \Phi^2 + \Phi^4)g = \nu f. \qquad (5.1)$$

Following the Q-stability idea [13] we demand the fluctuation not to perturb the initial number of nucleons N, i.e.,

$$N[\Phi] = N[\Phi + \eta]. \qquad (5.2)$$

For sufficiently small (η/Φ) this implies

$$\int \Phi(x) f(x) dx = 0. \qquad (5.3)$$

The $\hat{L}$ operator is positive definite on the space of functions orthogonal to Φ as soon as $\Phi(x)$ is its nodeless zero mode, related to U(1)-symmetry. On the other hand, the translational $d\Phi/dx$ zero mode to the upper operator in (5.1) possesses a single node, hence U has a single negative eigenvalue. The corresponding eigenfunction vanishes nowhere and cannot be orthogonal to everywhere positive $\Phi(x)$. This is so the case of the Q-theorem applicability [13,14], which guarantees Q-stability of solution (2.11) if

$$dN_n/da < 0. \qquad (5.4)$$

By means of the explicit formula (2.12) we make sure that (5.4) indeed holds.

Now let us turn to the hole solution (4.15). We examine its stability at rest. Denoting $\Phi(x)$ the $x_0 = \theta_0 = 0$ form

of $\phi_h(x)$ (4.12) $(\Phi = \Phi^*)$, and choosing the disturbed solution eq.(2.3) as $\phi(x,t) = \Phi(x) + (f(x) + ig(x))\exp\{\nu t\}$; $f, g, \nu \in R$, we arrive at the following symplectic eigenvalue problem,

$$\hat{U}f = (-d^2/dx^2 + r_0(2r_0 + A) - 6(2r_0 + A)\Phi^2 + 15\Phi^4)f = -\nu g$$
$$\hat{L}g = (-d^2/dx^2 + r_0(2r_0 + A) - 2(2r_0 + A)\Phi^2 + 3\Phi^4)g = \nu f \qquad (5.5)$$

As before, the condition (5.3) is imposed. The lower operator in (5.5) is again positive definite on the space defined by (5.3), because $L\Phi = 0$ is exactly the equation (2.3) for static Φ. Since $U\Phi = 0$ is again simply d/dx(2.3), the eigenfunction related to the single negative eigenvalue has no nodes. Q-theorem is thus again applicable and the hole solution (4.12) is stable when

$$dN_h/d\gamma < 0, \qquad (5.6)$$

with $\gamma = -r_0(A + 2r_0)$. For $v = 0$ eq.(4.16) is just

$$-N_h = \cosh^{-1}\left(-\frac{2}{2+\gamma r_0^{-2}} - 1\right),$$

whereby (5.6) is established.

Let us note that in the case of nonvanishing boundary conditions (4.1) stability of the travelling solutions does not ensue from their stability at rest since the presence of medium violates the Galilean invariance. Whether the hole (4.15) is stable or not for $v \neq 0$ still remains the open question.

Another open question is stability of the kink (3.14) (at least at the rest frame). In this case $\hat{L}$ is not positively defined, since its zero mode is nothing but the kink itself. On the other hand, $\hat{U}^{-1}$ does not exist even on the space defined by (4.3). The reason is that U's translational zero mode obeys (4.3).

However, the two mentioned problems are trivially solved when the kink or the hole travel near the speed of sound. In Sec.II we have shown that in this case both of them are governed by the KdV equation. The KdV soliton moving with arbitrary velocity is well-known to be stable and this fact proves the stability of the kink and the hole.

The remained unexplored regular solution is the precipice (4.20). In this case both the upper and lower operators (5.5) possess no negative eigenvalues at all since both the zero modes are nowhere vanishing. In addition, the $\hat{L}^{-1}$ operator is well defined on the Φ's orthogonal supplement. Eq. (5.5) is thus equivalent to

$$\hat{U} f = -\nu^2 \hat{L}^{-1} f. \qquad (5.7)$$

The variational principle is applicable to the eigenvalue problems of this type[15]. It states that the lowest eigenvalue to (5.7) is

$$-(\nu_0)^2 = \min \frac{\int f(x) \hat{U} f(x) dx}{\int f(x) \hat{L}^{-1} f(x) dx},$$

which is obviously non-negative. This implies $\nu^2 \leq 0$, and the precipice is Q-stable.

VI. DISCUSSION

In the present paper we have considered the one-dimensional version of the nuclear matter theory,which describes the dynamics of the so-called slabs[16]. In spite of the fact that the picture simplified in this way is rather far from reality,the one-dimensional description provides substantial insight into qualitative behaviour of possible solutions in higher dimensions. For instance, now it is clear from the explicit "nucleus" solution given above that such a crucial feature of nuclear many-body systems as nuclear saturation[5] is an attribute of the $\psi^4-\psi^6$ model. However, from our point of view, the main result of the present work is that the $\psi^4-\psi^6$ theory not only describes the motion of a nucleus as a whole, but also may be used to analyze its internal excitations. We anticipate this property to survive under transition to higher dimensions. From the mathematical point of view, this means that in contrast to all other previously analyzed NLS versions, solutions both under vanishing and constant boundary conditions are expected to exist in two and three space-dimensional versions of the $\psi^4-\psi^6$ model.

Despite the availability of several explicit solutions, we have considered it useful to derive the simpler, approximate equation in addition to the initial one. The said simplification is the KdV equation, which has been shown to describe small-amplitude rarefactions, propagating in a constant-density medium with the velocity close but less than that of sound. The reason in studying KdV rather than $\psi^4-\psi^6$ NLS consists in that the former is exactly solvable and hence characterizes the dynamics completely. For instance, stability of the near-sound solitons is the trivial consequence of such an approximation. The passage to KdV allows one to obtain the N-soliton solutions, i.e., to describe interactions between the rarefaction bubbles. This is impossible in the frame of $\psi^4-\psi^6$ NLS equation due to the fundamental obstacles ensuing from its nonintegrability.

In conclusion let us summarize briefly our results concerning exact solutions. Solution (2.11) to eq.(2.2), vanishing at infinities, describes a "one-dimensional nucleus", or slab. It is stable according to the Q-theorem (Sec.V). When the corresponding number of nucleons tends to infinity, the slab approaches the constant density condensate. The regular localized excitations of the condensate are divided into holes and kinks, which appear in different parameter domains. The hole solution to eq.(2.3) exists for $0 \leq A \leq r_0$ (or for $-1/4 \leq a \leq -3/16$ in terms of eqs.(2.1), (2.2)) and is given by eq.(4.15). When $A \leq 0$ ($a \geq -3/16$) eq.(2.3) admits the kink solution,which is given by (4.15) as well. The motionless hole is Q-stable(Sec.V), while both the hole and the kink are stable when travelling with the velocity close to that of sound. This fact naturally leads to the assumption that the two solutions are stable at arbitrary speed. At $A = 0$ ($a = -3/16$) eq.(2.3) admits the stable static "precipice" solution (4.19) under the mixed boundary conditions (4.18). In addition to the listed ones, the model possesses a number of localized singular solutions (4.11), (4.19), (4.22), and rich spectrum of cnoidal waves. The latter can be explicitly found in terms of elliptic functions from eqs.(4.5), (4.6), yet we do not discuss them here.

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Солитоноподобные возбуждения в одномерной ядерной материи

Рассматривается нелинейное уравнение Шредингера /НУШ/ с $\psi^3-\psi^5$ нелинейностью, описывающее статические и динамические свойства тяжелых ионов. Насыщающее солитоноподобное решение, интерпретируемое как ядро, найдено явно при нулевых граничных условиях. Показано, что этот вариант НУШ можно использовать не только для описания ядра как целого, но и исследовать внутренние возбуждения в ядерной материи. В такой материи распространяются только дырочные, но не выпуклые возбуждения. Распространяясь со скоростью, близкой к скорости звука, они описываются приближенно уравнением КдВ. Найдены явные выражения для них и некоторых других локализованных решений. Проведен анализ устойчивости найденных солитоноподобных конфигураций.

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Soliton-Like Excitations in a One-Dimensional Nuclear Matter

We consider the nonlinear Schrödinger equation with $\psi^3-\psi^5$ nonlinearity describing static and dynamic properties of heavy ions. Saturating soliton-like solution which can be interpreted as a nucleus is explicitly found under the vanishing boundary conditions. We demonstrate that the said equation may be used not only to describe the motion of a nucleus as a whole, but also to investigate internal excitations in a nuclear matter. Only hole-like, but in no way hump-like stable localized solutions can travel in the nuclear condensate. Those solutions when propagating near the velocity of sound are shown to be approximately governed by the KdV equation. We also give exact expressions for them and for some other localized solutions. Stability analysis of the found soliton-like configurations is presented.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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