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S.B.Il'yn, E.A.Tagirov

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S.B.Il'yn,* E.A.Tagirov

NOTE ON LOCAL PROPERTIES OF THE GREEN FUNCTIONS OF SCALAR FIELD IN GENERAL RELATIVITY

*Research Institute for Mechanics and Physics of Saratov State University.

Various authors have lately used two different general-relativistic generalizations of the special-relativistic Klein-Gordon-Fock equation describing the interaction of a scalar field with external (non-quantized) gravitation. The first one, traditional *

$$\square\phi + m^2\phi = 0, \qquad \square = \frac{1}{\sqrt{-g}}\,\partial_\alpha(\sqrt{-g}\; g^{\alpha\beta}\,\partial_\beta) \tag{1}$$

is often called the simplest and the corresponding interaction - a minimal one. Another generalization

$$\square\phi + \frac{R}{6}\phi + m^2\phi = 0 \tag{2}$$

is conformal-covariant when $m=0$ and in this sense has a higher symmetry than eq. (1). Penrose /1/ was the first who noticed this property of eq. (2). The most interesting consequences of the theory based on eq. (2) have been obtained in paper /2/ where it has specifically been shown, in some sense, that just eq. (2) should be considered as the general-relativistic wave equation of a spinless particle.

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* Notations: $g_{\alpha\beta}(x)$ is the metric tensor (with the signature - 2) of the Riemann space-time $V_{1,3}$; $g = \det \| g_{\alpha\beta} \|$; $\alpha,\beta,\gamma,\delta = 0,1,2,3$; $\partial_\alpha = \partial/\partial x^\alpha$, ∇_α is the operator of covariant differentiation. The sign of the curvature tensor $R^\alpha_{\ \beta\gamma\delta}$ is chosen according to the rule $\nabla_\beta\nabla_\gamma A_\alpha - \nabla_\gamma\nabla_\beta A_\alpha = R^\delta_{\ \alpha\beta\gamma} A_\delta$; $R = g^{\beta\gamma} R^\alpha_{\ \beta\alpha\gamma}$ is the scalar curvature. The units are used for which $c = \hbar = 1$.

It is important to note that in general relativity the conformal covariant (when all rest masses vanish) interactions are just the minimal interactions /4/. In view of this fact it is more reasonable to consider as minimal the interaction of a scalar field with external gravitation corresponding to eq. (2) rather than to eq. (1).

In the present paper we would like to point out a fact which evidences that eq. (2) is simpler than eq. (1). The assertion is that in the limit $x \to x'$ the classical Green function $\bar{G}(x,x')$ for eq. (2) (but not for eq. (1)) in an arbitrary $V_{1,3}$ coincides with $\bar{G}(x,x')$ in the Minkowski space-time $E_{1,3}$. In other words, $\bar{G}(x,x')$ in $V_{1,3}$ coincides with $\bar{G}(x,x')$ in the space, which is tangent to $V_{1,3}$ at the point x'. One can easily perceive this property of the classic Green function of the scalar field in $V_{1,3}$ from results of paper /5/ by DeWitt. However, it has not been mentioned clearly anywhere and seems to be unknown. Therefore here we shall briefly outline its proof.

DeWitt's results suggest the following ansatz for $G(x,x')$:

$$\bar{G}(x,x') = \frac{\Delta^{1/2}}{4\pi}\{a_0\delta(\sigma) + \frac{1}{2}\theta(\sigma)\sum_{n=0}^{\infty} a_n \left(\frac{\sigma}{2m^2}\right)^{\frac{n-1}{2}} J_{n-1}(m\sqrt{2\sigma})\}. \quad (3)$$

Here $\sigma \equiv \sigma(x,x')$ is the geodesic interval * for $x,x' \subset V_{1,3}$; J_n is the ordinary Bessel function and

$$\Delta \equiv \Delta(x,x') = \sqrt{-g(x)}\,[\,\det \| \partial_{\alpha'}\, \partial_{\beta'}\, \sigma(x,x') \|\,]\sqrt{-g(x')},$$

where the primed indices refer to the point x'. After substitution of expression (3) into eq. (2) one obtaines the following recurrent equations for symmetric functions $a_n = a_n(x,x') = a_n(x',x)$

$$\partial^{\alpha}\sigma\partial_{\alpha}a_0 = 0, \quad (4)$$

$$\partial^{\alpha}\sigma\partial_{\alpha}a_{n+1} + (n+1)a_{n+1} + \Delta^{-1/2}\Box(a_n\Delta^{1/2}) + \frac{1}{6}R(x)a_n = 0. \quad (5)$$

Since $\partial_{\alpha}\sigma(x,x')$ is a tangent vector at the point x to the geodesic, connecting x and x' equations (4) and (5) are ordinary differential equations of the first order along the geodesic. The latter is unique if x and x' are sufficiently close to each other and we are just interested in $\bar{G}(x,x')$ for $x \to x'$.

Under the boundary condition $\lim_{x \to x'} a_0(x,x') = 1$ from eq. (4) one obtains

$$a_0(x,x') = 1. \quad (6)$$

It is easily to be convinced that a_0 having been fixed recurrent equations (5) all together define coefficient functions $a_n(x,x')$ uniquely. Taking into account eq. (6) $\bar{G}(x,x')$ can be expressed in the form

$$\bar{G}(x,x') = \Delta^{1/2}\{\bar{G}_E(\sigma) + \frac{\theta(\sigma)}{8\pi}a_1 J_0(m\sqrt{2\sigma}) +$$

$$+ \sum_{n=1}^{\infty} a_{n+1}\left(\frac{\sigma}{2m^2}\right)^{\frac{n}{2}} J_n(m\sqrt{2\sigma})\},$$

where

$$\bar{G}_E(\sigma) = \frac{1}{2\pi}\delta(\sigma) - \frac{\theta(\sigma)}{4\pi\sqrt{2\sigma}} J_1(m\sqrt{2\sigma}). \quad (7)$$

Evidently $\bar{G}_E(\sigma)$ is the classic Green function in $E_{1,3}$ with the argument $\sigma(x,x')$ instead of $\sigma_E(x,x')$ and $\epsilon(x^0 - x^{0\prime})\bar{G}_E(\sigma_E)$ is the Pauli-Jordan function.

Proceeding to calculation of $\lim_{x \to x'} \bar{G}(x,x')$ we note first that

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* The geodesic interval is one half the square of the geodesic distance; it is equal in $E_{1,3}$ to

$$\sigma_E(x,x') = \frac{1}{2}[(x^0 - x^{0\prime})^2 - (x^1 - x^{1\prime})^2 - (x^2 - x^{2\prime}) - (x^3 - x^{3\prime})^2],$$

x^{α} being Cartesian coordinates.

$$\sigma^{\frac{n}{2}} J_n(m\sqrt{2\sigma}) = O(\sigma^n), \quad n \geq 0$$

and from [5] we know

$$\lim_{x\to x'} \Delta^{1/2}(x,x') = 1.$$

Therefore

$$\lim_{x\to x'} \bar{G}(x,x') = \lim_{x\to x'} \bar{G}_E[\sigma(x,x')] + \frac{\theta(\sigma)}{8\pi} \lim_{x\to x'} a_1(x,x')$$

DeWitt [5] had considered eq. (1) and obtained that

$\lim_{x\to x'} a_1(x,x') = -\frac{1}{6} R(x')$. Using the same relations one can be convinced that for eq. (2) (i.e., for $a_1(x,x')$ satisfying eq. (5))

$$\lim_{x\to x'} a_1(x,x') = 0$$

and consequently

$$\lim_{x\to x'} \bar{G}(x,x') = \lim_{x\to x'} \bar{G}_E[\sigma(x,x')]. \qquad (8)$$

So the classic Green function for eq. (2) does not sense the curvature in small regions of the space-time. This fact is likely connected with validity of the equivalence principle for point sources of the scalar field (scalar charges) moving in an external gravitational field. However, to prove this strictly it is necessary to accomplish a study analogous to that on the radiation damping of point electric charge in the external gravitational field made by DeWitt and Brehme [6].

We have confined ourselves to consideration of the classic Green function $\bar{G}(x,x')$ for it is uniquely determined in $V_{1,3}$ in contrast to the causal Green function (the Feynman propagator) $D_c(x,x')$ which can be determined in general only with some arbitrariness [5,6]. However, the singular part of $D_c(x,x')$ is not influenced by this arbitrariness and therefore we have an assertion analogous to eq. (8) for this part. In addition, the difference between the limiting forms of $\bar{G}(x,x')$ for eq.(1) and eq. (2) is singular on the light conoid as $\theta(\sigma)$ and such a difference for $D_c(x,x')$ has a more strong logarothmic singularity.

Note at last, that there is no equality of the form (8) for the local limits of the classic Green functions of the spinor and electromagnetic fields. This is natural, in our opinion, due to that the corresponding particles have a non-zero spin.

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