объединенный
институт
ядерных
исследований
дубна

E2-83-790

D.V.Shirkov

THE RENORMALIZATION GROUP METHOD AND FUNCTIONAL SELF-SIMILARITY IN PHYSICS

To be published in the Proceedings
of the II International Workshop
on Nonlinear and Turbulent Processes
in Physics (Kiev, 1983)

1983

INTRODUCTION

During the past decade, the renormalization group method (RGM) has rapidly spaced into various fields of theoretical physics. The RGM was first formulated [1] and successfully applied [2] in quantum field theory (QFT) almost 30 years ago. At the beginning of the seventies, the RGM was used in statistical mechanics for the analysis of critical phenomena.[4-6] Later, the RGM concept penetrated into nonquantum fields such as turbulence theory [7] and polymer physics.[8] Specific representations of RGM differ considerably in mathematical details, thus causing some confusion. Even in some review monographs,[9] there is no clear exposition of the mutual correspondence of these representations and even the origin of the term "renormalization group" sometimes remains obscure.[10]

The purpose of this paper is two-fold. First, we give the universal formulation for "different" renormalization groups (RG). This is done on the basis of the functional equations (FE). Second, we distinguish [11] the simple physical property underlying these FEs. We call it functional self-similarity. This property corresponds to transitivity with respect to the method by which the initial (or boundary) conditions for some characteristics of dynamic systems are prescribed. On the basis of this property, we obtain a simple recipe for the further promotion of RGM into new branches of physics.

FUNCTIONAL EQUATIONS OF THE RENORMALIZATION GROUP

The RG was discovered in QFT by Stueckelberg and Peterman [12] in 1953. One year later, Gell-Mann and Low [13] obtained an equivalent of RG FEs and used them for the general analysis of ultraviolet asymptotics in quantum electrodynamics (QED) without any reference to their group nature or any relation to reference [12]. In 1955, Bogolubov and Shirkov [1] established the connection between refs.[12, 13] and put forward the idea of using the differential group equations (DEqs) in combination with the results of perturbation theory. This essential step provided a simple regular procedure known now as the RGM. It was realized in QED for the first time in our subsequent papers.[2] The central role in the functional as well as in the differential formulation of RG is played by the invariant charge (or effective coupling) $\bar{g}(x,y,g)$ introduced in reference.[1] This function depends on three quantities: x, the main physical variable of the problem (in QFT, energy); y, the fixed parameter related to x (in QFT, the mass of the particle); and g, the coupling constant, and is involved in the RG transformation

$$\{x \to x/t,\ y \to y/t,\ g \to g_t = \bar{g}(t,y,g)\} \qquad T_t. \tag{1}$$

Since the parameter t is continuous, the set $\{T_t\}$ forms a Lie group. This group is the RG. The transformation law of the QFT functions (correlators, vertex functions, and matrix elements) looks like

$$s(x,y,g) = Z_s(t)\cdot s(x/t,\ y/t,\ \bar{g}(t,y,g)), \tag{2}$$

where Z_s is some constant depending on the transformation parameter. The invariant coupling $\bar{g}$ satisfies a rather simple FE

$$\bar{g}(x,y,g) = \bar{g}(x/t,\ y/t,\ \bar{g}(t,y,g)), \tag{3}$$

which is closured and consistent with the "normalization" condition $g(1,y,g) = g$. This is the central FE of RG.

The important generalization of (3) corresponds to QFT with several coupling constants. For the two-coupling case, $g \to (g,h)$, we have the system of FEs (first obtained in ref.[14])

$$\bar{g}(x,g,h) = \bar{g}(x/t,\ g_t,h_t),\quad g_t = \bar{g}(t,g,h),$$
$$h(x,g,h) = \bar{h}(x/t,\ g_t,h_t),\quad h_t = h(t,g,h). \tag{4}$$

A general solution of RG FEs was given by Ovsiannikov.[15] For (3), it has the form

$$\Phi(y,g) = \Phi(y/x,\ g(x,y,g)), \tag{5}$$

Φ being an arbitrary function of two arguments that is reversible with respect to the second argument.

The DE corresponding to (2,3) can be written in the form

$$\frac{\partial \bar{g}(x,y,g)}{\partial \ln x} = \beta\left[\frac{y}{x},\ g(x,y,g)\right], \tag{6}$$

$$\frac{\partial \ln s(x,y,g)}{\partial \ln x} = \gamma\left(\frac{y}{x},\ \bar{g}(x,y,g)\right], \tag{7}$$

where, e.g.,

$$\beta(y,g) = \frac{\partial}{\partial\xi}\bar{g}(\xi,y,g)\Big|_{\xi=1}. \tag{8}$$

Within the framework of a standard RGM in QFT, the functions β and γ are usually determined approximately through perturbation theory expansion.

RG AND CRITICAL PHENOMENA

In the statistical mechanics, the so-called "Wilson's RG" is based on two essential points: (a) Kadanoff's idea about the invariance of macroscopic observables under the appropriate change of microscopic scale and (b) Wilson's hypothesis on the scaling behaviour of physical quantities in the vicinity of the phase transition point.

For the moment, only point (a) is of interest to us. It was formulated for the problem of an interacting spin lattice with spacing a and coupling constant K. Kadanoff proposed[16] to consider the "equivalent" effective lattice with double spacing 2a and the effective coupling constant K_2. For the correlation length, ξ it follows (see, e.g., ref.[6]) from Kadanoff's arguments that $\xi(K_2) = \frac{1}{2}\xi(K)$, where K_2 is some unknown function of K. By changing the doubling parameter 2 to an arbitrary integer n, we obtain $\xi(K) = n\xi(K_n)$. Now let n be continuous and introduce the function $K = \bar{K}(1/n, K)$. This yields $\xi[\bar{K}(x,K)] = x\xi(K)$ $(x = 1/n)$.

It is now very simple to check that the effective coupling $\bar{K}(x,k)$ satisfies the FE (2) for $y = 0$.

Hence, for a large integer n, the Kadanoff transformation is approximately equivalent to the RG transformation (1). At the same time, under alternative formulations of the Kadanoff-Wilson RG that use the continuous analogue of the Kadanoff transformation (e.g., in turbulence theory), the equivalence with QFT RG is complete.

FUNCTIONAL SELF-SIMILARITY

Let us study the general structure of RG FEs. Restricting ourselves to FEqs for $\bar{g}$ only, we have

$$\bar{g}(x,y,g) = \bar{g}(x/t,\ y/t,\ \bar{g}(t,y,g)) \tag{9}$$

and its "zero-mass" counterpart (for $y = 0$)

$$\bar{g}(x,g) = \bar{g}(x/t,\ \bar{g}(t,g)). \tag{10}$$

It is convenient to change the notation and introduce the logarithmic variables $\xi = \ln x$, $\eta = \ln y$, and $\theta = \ln t$, and $\bar{g}(x,y,g) = G(\ln x, \ln y, g)$. We shall refer to the corresponding analogues of (9) and (10),

$$G(\xi, \eta, g) = G(\xi-\theta, \eta-\theta, G(\theta, \eta, g)), \tag{11}$$

$$G(\xi, g) = G(\xi-\theta, G(\theta, g)) \tag{12}$$

as the additive version of RG FEs.

Now consider several examples of systems from nonquantum physics, that can be described by (11,12).

ELASTIC ROD

Imagine an elastic rod with a fixed end (see point 0 of figure 1) that is bent due to some external forces (e.g., gravitational force). If the rod is longitudinally homogeneous and the external forces are uniform, then the angles g_i (figure 1) at the three different points 0, 1, and 2 are related by the single function $G(t,g)$, $g_1 = G(\tau, g_0)$, $g_2 = G(\tau+\theta, g_0) = G(\theta, g_1)$. Combining these relations, we get (12).

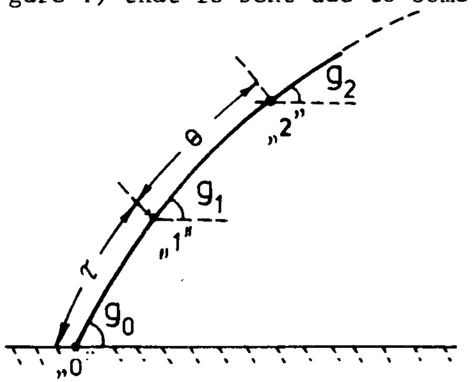

Fig.1. Elastic Rod.

HYDRODYNAMIC WAVE

Let us take the diverging (or converging) wave that propagates through a homogeneous gas or liquid under homogeneous external conditions. Consider the maximal amplitude $G(t)$ of the wave and consider its subsequent values at the moments t_0, t_1, t_2 (figure 2). These values will be related through equations similar to (10) and, again, we arrive at FE (12).

RADIATION TRANSFER [18]

Consider the two-dimensional problem of radiation transfer with a given flow g of the particles falling from the vacuum on the flat boundary of a region filled with homogeneous matter. Let us consider the number of particles moving from left to right inside the medium, e.g., at points 1 and 2 (figure 3). The values g, g_1, and g_2 can be related, with the help of a single function $G(t, g)$, through equations similar to (10). Again, we arrive at FEq.(12).

As is shown in detail in ref.[17], this problem allows us to make simple generalizations associated with the transition from the zero-mass FE (12) to the "massive" FEq (11) and to the two-coupling-constants case described by the additive version of the system (4).

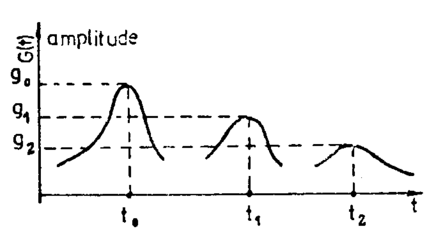

Fig.2. Hydrodynamical wave, t - radial variable.

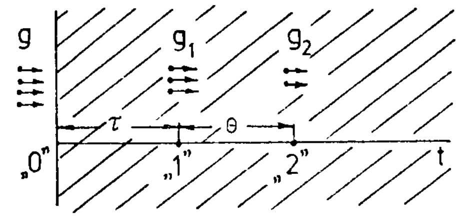

Fig.3. Radiation transfer.

Thus, we can conclude that the simple property underlying the RG FEqs is the property of transitivity of some physical characteristic G with respect to the way its initial or boundary value g is given. We refer to this property, which is reflected in FEs, as the functional self-similarity [14] (FSS) since it can be considered a generalization of the well-known concept of Sedov's self-similarity.

We have also shown that the FSS is typical of a very wide class of dynamic systems and that, for classical systems, it can be established in a considerably more simple way than for quantum systems. This remark opens the door to application of the so-called RG to problems of diverse branches of physics.

GUIDE FOR USE OF THE RG METHOD

Before formulating a set of conditions that should be fulfilled for successful applications of the RGM in new branches of physics, let us make several important remarks.

Remark 1

The group DEs (Lie equations) of the type (6,7) must not be physically "trivial", i.e., they should not be equivalent to equations that describe the physical nature of the problem. An example of such a "trivial" situation is provided by the elastic rod problem. Along with the quantum physics an example of nontrivial equations is provided by the radiation transfer problem, where the physical equations are linear integro-differential kinetic equations.

Remark 2

For the use of RGM, explicit expressions for the functions β and γ that appear on the right-hand side of DEs (6,7) are necessary. They represent the infinitesimal response of the unknown functions $\bar{g}$ and s and must be at least approximately determinable.

Remark 3

As a rule, the use of RGM is especially effective for the analysis of singularities of physical quantities. The point is that the approximate solutions (e.g., perturbation expansions) usually distort the structure of the singularity, which can be restored by imposing the FSS in the form of a regular RGM procedure.

Taking these remarks into account, we write the final recipe for successful application of RGM as follows: (a) Find a dynamic system with a physical quantity G obeying the FSS property and check that the DEs are not "trivial". (2) Consider the physical situation when G or a related physical quantity S behaves singularly. (3) On the basis of an approximate solution or some other type of analysis of physical equations, evaluate the response functions (β and γ). (4) Solve the FSS DEs and find the true singular behaviour.

REFERENCES

1. Bogolubov N.N., Shirkov D.V. Dokl. AN SSSR, 1955, 103, p.203.
2. Bogolubov N.N., Shirkov D.V. Dokl.AN SSSR, 1955, 103, p.391; Nuovo Cim., 1956,3, p.845; see also.[3]
3. Bogolubov N.N., Shirkov D.V. Introduction into the Theory of Quantized Fields. Wiley, N.Y., 1981, ch.IX of the 3rd ed.
4. Wilson K.G. Phys.Rev., 1971, B4, p.3174,3184. See also review monographs.[5,6]
5. Wilson K.G., Kogut J. Phys.Rep., 1974, 12C, p.75.
6. Ma Shang-Hen. Modern Theory of Critical Phenomena. Benjamin, New York, 1976.
7. See e.g., Pelletier G. J.Plasma Phys., 1980, 24, p.421; Moffat H.K. Repts.Progr.Phys., 1983, 46, p.621.
8. de Gennes P.-J. Scaling Concepts in Polymer Physics. Cornell Univ.Press, Ithaca, 1979.
9. Nash C. Relativistic Quantum Fields. Academic Press, New York, 1978, p.165.
10. See [1,2], Ch.V in the book.[6]

11. Shirkov D.V. Doklady AN SSSR, 1982, 263, p.63.
12. Stueckelberg E., Petermann A. Helv.Phys.Acta, 1953, 26, p.499.
13. Gell-Mann M., Low F. Phys.Rev., 1954, 95, p.1300.
14. Shirkov D.V. Doklady AN SSSR, 1955, 105, p.972.
15. Ovsiannikov L.V. Doklady AN SSSR, 1956, 109, p.1112; see [3] as well.
16. Kadanoff L.P. Physica, 1966, 2, p.263; see also.[5]
17. Mnatsakanian M.A. Doklady AN SSSR, 1982, 262, p.856.

Received by Publishing Department
on November 11, 1983.

Ширков Д.В. E2-83-790

Метод ренормализационной группы и функциональная автомодельность в физике

За последнее десятилетие метод ренормализационной группы из квантовой теории поля распространился в теорию критических явлений, теорию турбулентности и физику полимеров. На основе функциональных уравнений дана общая формулировка "различных" ренормализационных групп. Указано простое физическое свойство, лежащее в основе этих уравнений - свойство функциональной автомодельности. Оно оказывается присущим широкому классу динамических систем, вследствие чего дает простую основу для дальнейшего распространения ренормализационной группы в новые области физики.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1983

Shirkov D.V. E2-83-790

The Renormalization Group Method and Functional Self-Simularity in Physics

During the past decade the renormalization group method of quantum field theory extended into the theory of critical phenomena, turbulence theory, and polymer physics. Based on functional equations we give a universal formulation for "different" renormalization groups. We formulate a simple property underlying these equations, the property of functional self-similarity (functional automodelity). It turns out to be inherent for a wide class of dynamical systems and therefore provides a simple basis for a further promotion of the renormalization group method into new branches of physics.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1983