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A SIMPLE METHOD FOR CALCULATION OF GLAUBER'S AMPLITUDE

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1. INTRODUCTION

At present a problem of building the theoretical methods of description of elastic nucleus-nucleus collisions is very actual in high energy nuclear physics because of the necessity of interpreting the existing experimental data and a dissatisfaction of the results of the known theoretical models, most of which contain considerable arbitrariness in their foundation. Thus hopes paid to the Glauber eikonal approximation are quite understandable. Below a key problem of eikonal approach - determination of differential cross sections of elastic nucleus-nucleus scattering processes will be considered.

In this approach amplitude is determined by the sum of $2^{AB}-1$ terms representing different rescattering processes, where A and B are mass numbers of interacting nuclei.

Among these terms there are many similar terms, that is why the amplitude is actially determined by a smaller number of essentially different terms. For example, if $A=B=3$, there are only 25 essentially different terms from the total of 511.

Thus reduction of similar terms in the scattering amplitude presents a complicated combinatorical problem.

For the first time this problem was solved by Czyz and Maximon [1]. But it was not widely used because of being cumbersome. For example in papers [2,3,4] only first several terms of Glauber's series were taken into account.

In papers [5,6,7] combinatorical coefficients are calculated by the usual method.

In the present work we treat a simple algorithm for calculation of combinatorical coefficients and differential cross sections of elastic scattering processes in case of the Gaussian parametrization.

In the following section we consider basic structure of Glauber's amplitude. The third section expounds the method of calculation of the scattering amplitude. In section 4 we expound the idea of general diagram which gives the possibility of computing combinatorical coefficients. In the last fifth section a more simple method of calculation of differential cross sections is presented.

2. CALCULATION OF THE AMPLITUDE OF ELASTIC SCATTERING

According to basic principles of Glauber's approximation the differential cross section of nucleus-nucleus scattering is determined by the following expression

$$\mathcal{F}(\vec{q}) = \frac{iP}{2\pi} \int d^2b\, e^{i\vec{q}\vec{b}} \langle \psi_{A_f} \psi_{B_f} | [1 - \prod_{i=1}^{A} \prod_{j=1}^{B} (1 - \gamma(\vec{b} - \vec{s}_i + \vec{r}_j))] | \psi_{B_i} \psi_{A_i} \rangle, \quad (1)$$

where P is the momentum of the prejectile nucleus; $\vec{q}$ is the transverse momentum; ψ_{A_i}, ψ_{B_i} and ψ_{A_f}, ψ_{B_f} are the wave functions of nuclei A and B in the initial and final states, respectively, $\gamma(\vec{b})$ is the amplitude of elastic NN scattering in the impact parameter representation. $\{s_A\}$, $\{r_B\}$ are the coordinates of nucleus of the nuclei within the plane of the impact parameter $\vec{b}$ (in the plane perpendicular to the momentum P).

It is obvious from (1) that the elastic scattering amplitude is determined by the sum of a large number of terms which have a form

$$\frac{iP}{2\pi} \int d^2b\, e^{i\vec{q}\vec{b}} \langle \psi_{A_f} \psi_{B_f} | \prod_{(i,j)\in M} \gamma_{ij} | \psi_{B_i} \psi_{A_i} \rangle, \quad \gamma_{ij} = \gamma(\vec{b} - \vec{s}_i + \vec{r}_j), \quad (2)$$

where $M \subset I_1' \otimes I_2'$ and $I_1' \subset I_1$, $I_2' \subset I_2$, while I_1 and I_2 are sets of integers with elements $I_1 = (1, \dots, A)$; $I_2 = (1, \dots, B)$. Since there are quite many such terms $(2^{AB} - 1)$, reduction of similar terms among those is a non-trivial problem which can be significantly simplified if we turn to scattering diagrams.

The scattering diagram which represents a general term of series (2) is plotted in the following way.

Draw m vertical lines and n horizontal lines crossing the former ones (m and n are cardinal numbers of sets I_1' and I_2', respectively), then points of intersections (nodes) of these lines represent the set $I_1' \otimes I_2'$. Subset M is shown by black points (Fig.1). Such representation of terms of series (1) permits one to determine easily similar terms.

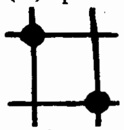

Fig. 1

In fact, we shall take, for example, the scattering diagram shown in Fig.1. It is clear, that such a diagram may be plotted in two ways. Besides m elements from I_1 may be selected by C_A^m manners and n elements from I_2 by C_B^n manners.

Consequently, the total number of similar terms represented by the diagram shown in Fig.1 in expression (1) equals $2C_A^2 C_B^2$. By analogy one can determine a number of isomorphik diagrams of an arbitrary form. Since such a problem is partially solved by Uzhinskii [8] we will not consider it here, but start calculating the integral in expression (2).

For this purpose, obviously, it is necessary to know the explicit form of functions ψ_{A_i}; ψ_{B_i} and ψ_{A_f}; ψ_{B_f}. In case of elastic scattering $|\psi_i\rangle = |\psi_f\rangle$ therefore it is necessary to have information only on the ground state of systems A and B.

The following approximations are most popular in the case of smaller A and B:

$$|\psi_A|^2 = C_1 \delta(\sum_{i=1}^{A} \vec{r}_i / A) \prod_{i=1}^{A} e^{-t \cdot \vec{r}_i^{\,2}}, \quad (3)$$

$$|\psi_B|^2 = C_2 \delta(\sum_{i=1}^{B} \vec{r}_i / B) \prod_{i=1}^{B} e^{-d \cdot \vec{r}_i^{\,2}}. \quad (4)$$

If we also assume that $\gamma(\vec{b}) = C_3 e^{-a\vec{b}^2}$, then the general term of series (2) can be represented in the form

$$N \frac{iP}{2\pi} \int d^2b \exp\{-X^T Q X - 2\vec{b}H^T X - b^2 C + i\vec{q}\vec{b}\} dX^{m+n}, \quad (5)$$

where $X^T = (\vec{s}_{i_1}, \vec{s}_{i_2}, \dots, \vec{s}_{i_n}, \vec{r}_{j_1}, \vec{r}_{j_2}, \dots, \vec{r}_{j_m})$, here i_k $(k = 1, 2, \dots, n)$ are elements of the set I_1' and j_k $(k = 1, 2, \dots, m)$ are elements of set I_2', Q an $(m+n)$-by-$(m+n)$ is a symmetrical matrix, H is a coefficient matrix of vector $\vec{s}_i$ and $\vec{r}_i$, C is a scalar, X^T, H^T are transposed matrices. Expression (5) is easily integrated:

$$N \frac{iP}{2\pi} \int \exp\{-X^T Q X - 2\vec{b}H^T X - 2\vec{b}^2 C + i\vec{q}\vec{b}\} d^{m+n} X d^2 b = \quad (6)$$

$$= \frac{\pi^{m+n+1}}{\mathrm{Det}W} \exp\{-\frac{\vec{q}^{\,2}}{4} \frac{|\mathrm{Det}Q|}{|\mathrm{Det}W|}\},$$

here Det Q and Det W are determinants of matrices Q and W. Matrix W is determined in the following way:

$$W = \begin{pmatrix} Q & H \\ H^T & C \end{pmatrix}. \quad (7)$$

Order of matrices Q and W depends upon concrete form of the scattering diagram.

For example, for the diagram with n vertical lines and m horizontal lines the corresponding matrix Q is $(m+n)$-by-$(m+n)$ matrix, and matrix W is $(m+n+1)$-by-$(m+n+1)$ matrix. Calculation of determinants of these matrices is a non-trivial problem. Therefore, the purpose of the following sections is to find an effective method for calculation of determinants of Det Q and Det W and reduction of similar terms in series (1).

3. DESCRIPTION OF THE METHOD

We shall take the diagram in Fig.1 for the sake of simplicity. The term of series (2) corresponding to such a diagram has the form:

$$\frac{iP}{2\pi}\left(\frac{td}{\pi^2}\right) C_3^2 \int \exp\{-(t+a)\vec{s}_1^{\,2} - (t+a)\vec{s}_2^{\,2} - (d+a)\vec{r}_1^{\,2} - (d+a)\vec{r}_2^{\,2} + 2a\vec{s}_1\vec{r}_1 + 2a\vec{s}_2\vec{r}_2 + 2a\vec{b}(\vec{s}_1+\vec{s}_2-\vec{r}_1-\vec{r}_2) - 2a\vec{b}^2 + i\vec{q}\vec{b}\} \times d^2b\, d^2s_1\, d^2s_2\, d^2r_1\, d^2r_2 . \qquad (8)$$

Consequently

$$Q = \begin{pmatrix} t+a & 0 & -a & 0 \\ 0 & t+a & 0 & -a \\ -a & 0 & d+a & 0 \\ 0 & -a & 0 & d+a \end{pmatrix} . \qquad (9)$$

Hence it is evident that matrix Q corresponding to the diagram shown in Fig.1 has the block structure

$$Q = \begin{pmatrix} T & \alpha \\ \alpha^T & D \end{pmatrix} , \qquad (10)$$

where T and D are diagonal matrices of the form $T = \begin{pmatrix} t+a & 0 \\ 0 & t+a \end{pmatrix}$; $D = \begin{pmatrix} d+a & 0 \\ 0 & d+a \end{pmatrix}$. Between the matrix α and the scattering diagram there exists a single valued connection; points of the diagram lying on intersection of the i-th horizontal and j-th vertical lines correspond to the element $\alpha_{ij} = -a$. Empty nodes of the diagrams correspond to elements of matrix equal to zero. α^T is a transposed matrix.

The structure of diagonal matrices T and D also depends upon the concrete form of the diagram. For example the matrix T element t_{ii} equals $t_{ii} = t + n_i a$, where n_i is sum of number of all points lying on the n-th horizontal line of the diagram. Similarly any element of the diagonal matrix D has the form

$d_{jj} = d + n_j a$, where n_j is a sum of all points lying on the j-th vertical line of the diagram.

Thus, for any diagram, for example, the diagram shown in Fig.2, there is a corresponding matrix

$$Q = \begin{pmatrix} t+4a & 0 & 0 & -a & -a & -a & -a \\ 0 & t+2a & 0 & -a & -0 & -a & -0 \\ 0 & 0 & t+a & 0 & 0 & -a & 0 \\ -a & -a & 0 & d+2a & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & d+a & 0 & 0 \\ -a & -a & -a & 0 & 0 & d+3a & 0 \\ -a & 0 & 0 & 0 & 0 & 0 & d+a \end{pmatrix} .$$

The matrix W in expression (6) is formed by bordering the matrix Q by the one-dimensional coefficient matrix H, elements of which are also connected with the concrete form of diagrams in the following way: H_i $(1 < i \leq n)$, an element of matrix H is equal to the sum of points on the i-th horizontal line multiplied by $-a$, but the element H_j $(n < j \leq n+m)$ is equal to the sum of points on the j-th vertical line multiplied by $+a$. Therefore, the matrix W corresponding to the diagram shown in Fig.1 has the form

$$W = \begin{pmatrix} t+a & 0 & -a & 0 & -a \\ 0 & t+a & 0 & -a & -a \\ -a & 0 & d+a & 0 & a \\ 0 & -a & 0 & d+a & a \\ -a & -a & a & d & a \end{pmatrix} .$$

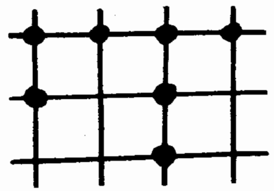

Fig. 2

Here angular element of the matrix, according to (5), corresponds to the scalar C.

Thus if the form of the diagram is known, then building of matrices Q and W is not difficult. The calculation of determinants of these matrices in the case of a large number of constituents is very awkward (i.e., for diagrams containing a large number of horizontal and vertical lines and representing scattering of higher multiplicity, matrices Q and W can be of a very high order).

But the matrix Q can always be presented in a form of product of two quasitriangular matrices. Since the matrix has a block structure $Q = \begin{pmatrix} T & a \\ a^T & D \end{pmatrix}$,

then

$$Q = \begin{pmatrix} T & a \\ a^T & D \end{pmatrix} = \begin{pmatrix} T & 0 \\ a^T & D \end{pmatrix} \begin{pmatrix} I & T' \\ 0 & Y \end{pmatrix}, \qquad (11)$$

where I is a unit matrix. To find the unknown blocks T' and Y there is a system of equations

$$\begin{cases} a = TT' \\ P = a^T T' + I , \end{cases} \qquad (12)$$

hence $T' = T^{-1} a$, $Y = -D^{-1} a^T T' + I$, where T^{-1} and D^{-1} are matrices inverse to the diagonal matrices T and D. They are diagonal and their diagonal elements are inverse to corresponding elements of the matrices T and D.

Thus the determinant of the matrix Q is equal to Det Q = $= (t + n_1 a) \dots (t + n_n a)(d + n_1 a) \dots (d + m_n a) \times \mathrm{Det}\, Y$. Hence, evidently, it is necessary to find only the determinant of matrix Y, the order of which is equal to the smallest of number from m and n.

Just in the same way we can simplify calculation of the determinants of the matrix W.

We would like to note that

$$W = \begin{pmatrix} T & a & aN_1 \\ a^T & D & -a'N_2 \\ N_1^T a^T & -N_2^T a & -N_1^T a N_2 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} U_1^1 \\ U_2^1 \\ \vdots \\ U_n^1 \end{pmatrix}; \quad N_2 = \begin{pmatrix} U_2^2 \\ U_2^2 \\ \vdots \\ U_m^2 \end{pmatrix}; \qquad \begin{matrix} U_1^1 = U_2^1 = \dots = U_n^1 = 1 \\ U_1^2 = U_2^2 = \dots = U_m^2 = 1 , \text{ hence} \end{matrix}$$

$$\mathrm{Det}\, W = \mathrm{Det} \begin{vmatrix} T & a & TN_2 + aN \\ a^T & D & 0 \\ N_1^T a^T + N_2^T T & 0 & N_1^T a N_2 + N_2^T T N_2 \end{vmatrix} .$$

The last determinant can be calculated in the same way as Det Q.

4. INTRODUCTION OF THE NOTION OF A GENERAL DIAGRAM

From the previous sections it is evident that for the calculation of the amplitude it is necessary to calculate the determinants Det Q and Det W corresponding to $2^{AB}-1$ terms of series (1).

For reduction of similar terms among $2^{AB}-1$ terms notion of a general diagram is introduced.

Case 1. Let now $B = 2$ and A is arbitrary. In this case we can represent a general diagram in the form of the diagram shown in Fig.3.

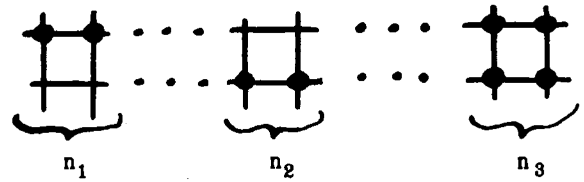

Fig. 3

Consequently,

$$\mathrm{Det}\, Q = (t + a)^{n_1 + n_2} (t + 2a)^{n_3} \times$$

$$\times \begin{vmatrix} -\left[\frac{n_1 a^2}{t + a} + \frac{n_3 a}{t + 2a}\right] + d + (n_1 + n_3) a & \frac{n_3 d^2}{t + 2a} \\ \frac{n_3 a^2}{t + 2a} - & -\left[\frac{n_2 a^2}{t + a} + \frac{n_3 a^2}{t + 2a}\right] + d + (n_2 + n_3) d \end{vmatrix} ,$$

$$\mathrm{Det}\, W(n_1, n_2, n_3) = (t + a)^{n_1 + n_2} (t + 2a)^{n_3} \times$$

$$\times \begin{vmatrix} -\left[\frac{n_1 a^2}{t + a} + \frac{n_3 a^2}{t + a}\right] + E_1 - \frac{n_3 a^2}{t + 2a} & \frac{n_1 at}{t + a} + \frac{n_3 at}{t + 2a} \\ -\frac{n_3 at}{t + 2a} - \left[\frac{n_2 a^2}{t + a} + \frac{n_3 a}{t + 2a}\right] + E_2 & \frac{n_2 at}{t + a} + \frac{n_3 at}{t + 2a} \\ \frac{n_1 at}{t + a} + \frac{n_3 at}{t + a} & \frac{n_2 at}{t + a} + \frac{n_3 at}{t + 2a} - \left[\frac{n_1 + n_2}{t + a} + \frac{n_3}{t + 2a}\right] + t^2 + t_3 \end{vmatrix}, \qquad (13)$$

where $E_1 = (n_1+n_3)d + d$; $E_2 = (n_2+n_3)a + d$; $E_3 = (n_1+n_2+n_3)t$. Knowing $\mathrm{Det}\, Q(n_1, n_2, n_3)$ and $\mathrm{Det}\, W(n_1, n_2, n_3)$ it is not difficult to determine the scattering amplitude

$$\mathcal{F}^{e\ell}_{2-A}(\vec{q}) = K_A(\vec{q}) K_B(\vec{q}) \frac{iP}{2\pi} \sum_{\substack{n_1, n_2, n_3 = 0 \\ (0 < n_1+n_2+n_3 \le A)}}^{A} C_A^{n_1+n_2+n_3} C_1^{n_1+n_2+n_3} \times \frac{(n_1+n_2+n_3)!}{n_1!\, n_2!\, n_3!} G_1 G_2 t^{n_1+n_2+n_3} (-1)^{1+n_1+n_2+2n_3} \frac{C_2^2}{\mathrm{Det}W} \times C_3^{n_1+n_2+2n_3} \exp\left\{ -\frac{\vec{q}^{\,2}}{4} \frac{|\mathrm{Det}\,Q(n_1,n_2,n_3)|}{|\mathrm{Det}\,W(n_1,n_2,n_3)|} \right\}, \qquad (14)$$

where

$$E_1 = \begin{cases} E_1, & \text{when } n_1+n_3 \neq 0 \\ 1, & \text{when } n_1+n_3 = 0 \end{cases}$$

$$E_2 = \begin{cases} E_2, & \text{when } n_2+n_3 \neq 0 \\ 1, & \text{when } n_2+n_3 = 0 \end{cases}$$

and

$$G_1 = \begin{cases} d, & \text{when } n_1+n_3 \neq 0 \\ 1, & \text{when } n_1+n_3 = 0, \end{cases}$$

$$G_2 = \begin{cases} d, & \text{when } n_2+n_3 \neq 0 \\ 1, & \text{when } n_2+n_3 = 0 \end{cases}$$

$K_A(q)$ and $K_B(q)$ are centre of mass correction factors.

Case 2. The general diagram for $B = 3$ and arbitrary A is given in Fig.4. The corresponding determinants Det Q and Det W may be obtained in the same way as in cases 1 and 2.

Similarly with the help of general diagrams we can obtain an expression for the scattering amplitude for any A and B.

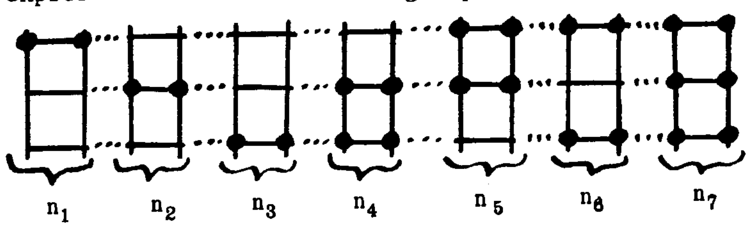

Fig. 4

5. A SIMPLER APPROACH

In this section we consider a simpler approach to calculation of the elastic scattering amplitude using the results obtained in section I of this paper.

Here we shall also use the connection between general diagrams and the structure of matrices a, T and D.

Let us introduce the matrix a in the following form:

$$a = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \qquad (15)$$

where a_{ij} has only two values zero and $-a$. In such a case matrices T and D have the form

$$T = \begin{pmatrix} t + (a_{11}+a_{12}+\dots+a_{1n}) & \dots & 0 \\ 0 & t + (a_{21}+a_{22}+\dots+a_{2n}) \dots & 0 \\ \dots & \dots & \dots \\ 0 \dots & t + (a_{n1}+a_{n2}+\dots a_{nn}) & \end{pmatrix}, \qquad (16)$$

$$D = \begin{pmatrix} d + (a_{11}+a_{21}+\dots+a_{n1}) & \dots & 0 \\ 0 & d + (a_{12}+a_{22}+\dots+a_{n2}) \dots & 0 \\ \dots & \dots & \dots \\ 0 \dots & d + (a_{1n}+a_{2n}+\dots+a_{nn}) & \end{pmatrix}. \qquad (17)$$

Matrix W can be written in the following form:

$$W = \begin{pmatrix} T & a & H_1 \\ a^T & D & H_2 \\ H_1^T & H_2^T & C \end{pmatrix}. \qquad (18)$$

Matrices H_1 and H_2 are determined in the following way:

$$H_1 = \begin{pmatrix} -(a_{11}+a_{12}+\dots+a_{1n}) \\ -(a_{21}+a_{22}+\dots+a_{2n}) \\ \dots \\ -(a_{n1}+a_{n2}+\dots+a_{nn}) \end{pmatrix}, \qquad (19)$$

$$H_2 = \begin{pmatrix} (a_{11} + a_{21} + \dots + a_{n1}) \\ (a_{12} + a_{22} + \dots + a_{n2}) \\ \dots\dots\dots\dots \\ (a_{1n} + a_{2n} + \dots + a_{nn}) \end{pmatrix} \qquad (20)$$

and

$$C = \sum_{m,n} a_{mn} . \qquad (21)$$

Thus in this approach matrices Q and W have the same block structure as in the previous approach. Therefore to calculate the determinants of matrices Q and W we can use the technique of resolution into two quasidiagonal matrices determined in (11). But in this case the problem of reduction of similar terms is solved automatically.

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Омбоо З.
Е2-83-775

Простой метод вычисления глауберовских амплитуд

Излагается метод вычисления членов глауберовских рядов для амплитуд упругого рассеяния составных систем. Привлечение понятия общей диаграммы рассеяния существенно упрощает процедуру вычисления. При этом легко решается трудная комбинаторная задача - приведение подобных членов глауберовского ряда. Детерминант, соответствующий различным членам ряда, уменьшается по крайней мере в два раза, если числа конституентов сталкивающихся систем равны. А если числа их не равны, то порядок этого детерминанта равен меньшему из них.

Работа выполнена в Лаборатории ядерных проблем ОИЯИ.

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A Simple Method for Calculation of Glauber's Amplitude

A method of calculating the terms of Glauber series expansions for elastic scattering of composed systems are presented. The inclusion of general scattering diagram simplifies essentially the calculation procedure. In this case the complicated combinatorical problem of reduction of similar terms in Glauber series is solved easily and order of determinant corresponding to various terms of the series decreases at least by a factor of two, if numbers of constituents of scattered systems are equal. If these numbers are not equal, the determinant order is equal to the smallest one.

The investigation has been performed at the Laboratory of Nuclear Problems, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1983