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PHASE SPACE REPRESENTATIONS AND PATH INTEGRALS

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1. Distribution functions in phase space representations (PSR's) can be calculated using approaches similar to those for amplitudes: the Schrödinger picture, T-exponential (interaction picture), Green function, path integral methods. Here the latter approach is treated, namely, a derivation is given of a path integral form for the Wigner distribution function $\rho_2(xpt,x_0p_0t_0)$ (the density "matrix" in the Wigner representaion (PSR-2)), which in the Schrödinger picture may be written as follows

$$\rho_2(xpt,x_0p_0t_0) = \Lambda^{-1}\Lambda_0^{-1}|\langle xp|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|x_0p_0\rangle|^2 = \tag{1.a}$$

$$= e^{-\mathcal{L}(t-t_0)}\rho_2(xpt_0,x_0p_0t_0) = \tag{1.b}$$

$$= e^{\mathcal{L}^0(t-t_0)}\rho_2(xpt_0,x_0p_0t_0)\,. \tag{1.c}$$

Obviously it is a solution of the Liouville equation

$$\frac{\partial}{\partial t}\rho_2(xpt,x_0p_0t_0) = -\mathcal{L}\rho_2(xpt,x_0p_0t_0)\,, \tag{2.a}$$

$$\frac{\partial}{\partial t}\rho_2(xpt,x_0p_0t_0) = \mathcal{L}^0\rho_2(xpt,x_0p_0t_0)\,. \tag{2.b}$$

Here $\mathcal{L}$ is the (generalized) Liouville operator (Liouvillian)

$$\mathcal{L} = i\hbar^{-1}(H^{l} - H^{r})\,. \tag{3}$$

The Liouvillian $\mathcal{L}$ together with the left and right representatives H^l and H^r of a Hamiltonian $\hat{H}$ are linear partial differential operators, acting on the variables x and p. $\mathcal{L}^0$ is the same operator, acting on x_0 and p_0. We suppose that initial and final states are described on equal footing by the "density matrices"

$$\hat{\rho}_{2i}(x_0,p_0) = \Lambda_0^{-1}|x_0p_0\rangle\langle x_0p_0|\,, \quad \hat{\rho}_{2f}(x,p) = \Lambda^{-1}|xp\rangle\langle xp|\,, \tag{4}$$

and therefore the initial condition

$$\rho_2(xpt_0,x_0p_0t_0) = \mathrm{Tr}(\hat{\rho}_{2f}\hat{\rho}_{2i}) = \Lambda^{-1}\Lambda_0^{-1}|\langle xp|x_0p_0\rangle|^2 = (2\pi\hbar)^n\delta(x-x_0)\delta(p-p_0) \tag{5}$$

[illegible]

is assumed x). We define $\hat{\rho}_{2i}$ and $\hat{\rho}_{2f}$ via the coherent states and the inverse Gauss transformation Λ^{-1}. One can show, however, that/22/

$$\Lambda^{-1}|xp\rangle\langle xp| = \tag{6.a}$$

$$= \hbar^n \int da\, e^{-ip\alpha} |x-\tfrac{\hbar}{2}\alpha\rangle\langle x+\tfrac{\hbar}{2}\alpha| = \tag{6.b}$$

$$= \hbar^n \int d\beta\, e^{ix\beta} \;|p-\tfrac{\hbar}{2}\beta\rangle\langle p+\tfrac{\hbar}{2}\beta|, \tag{6.c}$$

where $|x\pm\frac{\hbar}{2}\alpha\rangle$ and $|p\pm\frac{\hbar}{2}\beta\rangle$ are eigenvectors of the coordinate and momentum operators, respectively. In fact, eqs. (6.b) and (6.c) correspond to the Wigner definition/1/.

At the first stage we shall consider a particular Hamiltonian of the form

$$\hat{H} = \sum \frac{\hat{p}^2}{2m} + V(\hat{x}), \qquad H^{\overset{l}{r}} = \sum \frac{p^{\overset{l}{r}2}}{2m} + V(x^{\overset{l}{r}}) . \tag{7}$$

In PSR-2

$$x^{\overset{l}{r}}_k = x_k \pm \frac{i\hbar}{2}\frac{\partial}{\partial p_k}, \quad p^{\overset{l}{r}}_k = p_k \mp \frac{i\hbar}{2}\frac{\partial}{\partial x_k} . \tag{8}$$

Note that eqs. (4) are not true density matrices, as they are not positive definite (what is easily seen taking their expectation values in oscillator states, see Appendix A). Hence $\rho_2(xpt,x_0p_0t_0)$ is in general not positive definite and cannot be interpreted as a probability density. However in all the linear cases, i.e., when equations of motion for coordinates are linear, and therefore, Hamiltonians are at most bilinear in x and p, the distribution functions $\rho_2(xpt,x_0p_0t_0)$ are identical with the corresponding phase space densities in the classics ($\hbar$ falls outxx)). Such are the cases of free particle, motion under a constant force, any harmonic oscillators, etc. In these cases ρ_2 may be easily calculated, e.g., using eq. (1.b) or (1.c) to obtain

$$\rho_2(xpt,x_0p_0t_0) = (2\pi\hbar)^n\,\delta(x-x_0-p_0(t-t_0))\,\delta(p-p_0), \tag{9}$$

$$\rho_2(xpt,x_0p_0t_0) = (2\pi\hbar)^n\delta(x-x_0-\tfrac{p_0}{m}(t-t_0)-\tfrac{F}{2m}(t-t_0)^2)\,\delta(p-p_0-F(t-t_0))$$

$$=(2\pi\hbar)^n\,\delta(x-\tfrac{p}{m}(t-t_0)+\tfrac{F}{2m}(t-t_0)^2-x_0)\,\delta(p-F(t-t_0)-p_0) \tag{10}$$

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x) Any number n of degrees of freedom is implied (any number of particles and dimensions of space), so that the δ -functions in eq. (5) are n-dimensional, $\delta(x-x_0)\equiv\delta^n(x-x_0)$, and $d\alpha \equiv d^n\alpha$, $d\beta \equiv d^n\beta$ in eq.(6)and in what follows.For identical particles symmetrization (antisymmetrization) must be taken into account.

xx) Since it enters only via $i\hbar^{-1}(H^l - H^r)$. Of course, not all quantities are governed by the latter. For instance, the Planck formula is defined by the Hamiltonian itself (either $\hat{H}$, or H^l, or H^r), and $\hbar$ does not fall out (see Appendix B).

$$\rho_2(xpt,x_0p_0t_0) = (2\pi\hbar)^n\delta(x-x_0\cos\omega\tau - \tfrac{p_0}{m\omega}\sin\omega\tau)\,\delta(p-p_0\cos\omega\tau+m\omega x_0\sin\omega\tau)$$

$$=(2\pi\hbar)^n\,\delta(x\cos\omega\tau-\tfrac{p}{m\omega}\sin\omega\tau-x_0)\,\delta(p\cos\omega\tau+m\omega\sin\omega\tau-p_0). \tag{11}$$

$(\tau = t - t_0)$ and in general

$$\rho_2(xpt,x_0p_0t_0) = (2\pi\hbar)^n\delta(x-\mathbb{x}(x_0,p_0,t-t_0))\,\delta(p-\mathbb{p}(x_0,p_0,t-t_0))=$$

$$=(2\pi\hbar)^n\,\delta(\mathbb{x}(x\;,p\;,t_0-t)-x_0)\,\delta(\mathbb{p}(x,p,t_0-t)-p_0), \tag{12}$$

where $\mathbb{x}$ and $\mathbb{p}$ are obtained by solving the Newton or Hamilton equations.

Being not positive definite in general, ρ_2 leads to all the usual positive definite probability densities of quantum mechanics, including the distribution function in PSR-1 (based on the coherent state representation)

$$\rho_1(xpt,x_0p_0t_0)=\Lambda\Lambda_0\rho_2(xpt,x_0p_0t_0)=|\langle xp|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|x_0p_0\rangle|^2 \tag{13}$$

as a result of the Gauss transformations Λ and Λ_0 , i.e., of the convolution with a given normal distribution (see, e.g., ref. /22/, eq.(110.b)). The PSR-1 distribution function (13) is positive definite and may be interpreted as a probability density, but between non-orthogonal states

$$\hat{\rho}_{1i}(x_0,p_0)=|x_0p_0\rangle\langle x_0\,p_0|, \qquad \hat{\rho}_{1f}(x,p)=|xp\rangle\langle xp|, \tag{14}$$

i.e., with

$$\rho_1(xpt_0,x_0p_0t_0)=Tr(\hat{\rho}_{1f}(xp)\hat{\rho}_{1i}(x_0p_0))=|\langle xp|x_0p_0\rangle|^2 . \tag{15}$$

ρ_1 is also subjected to the evolution in form (1.b), (1.c) and equations of motion (2.a) and (2.b), but with other $\mathbb{L}$ and $\overset{\circ}{\mathbb{L}}$, because now representatives $x^{1,r}$ and $p^{1,r}$ differ from eq.(8) (see ref. /22/, eqs. (104) and (105)). Both ρ_1 and ρ_2 are real functions, and contain, however, the same total information, as a corresponding complex wave function (amplitude). Note an advantage of the distribution functions: they admit transition to the classical limit ($\hbar \to 0$) unlike amplitudes (S-matrix).

Give the relations

$$\frac{1}{(2\pi\hbar)^n}\int dq\, e^{-iq\beta}\Lambda^{-1}|qp\rangle\langle qp| = |p+\tfrac{\hbar}{2}\beta\rangle\langle p-\tfrac{\hbar}{2}\beta| \equiv |k'\rangle\langle k|, \tag{16}$$

$$\frac{1}{(2\pi\hbar)^n}\int dp\, e^{ip\alpha}\Lambda^{-1}|qp\rangle\langle qp| = |q+\tfrac{\hbar}{2}\alpha\rangle\langle q-\tfrac{\hbar}{2}\alpha| \equiv |x'\rangle\langle x| , \tag{17}$$

$$\frac{1}{(2\pi\hbar)^n}\int dp\,\Lambda^{-1}|xp\rangle\langle xp| = |x\rangle\langle x| , \tag{18}$$

$$\frac{1}{(2\pi\hbar)^n}\int dx\,\Lambda^{-1}|xp\rangle\langle xp| = |p\rangle\langle p|\ , \tag{19}$$

$$\frac{1}{(2\pi\hbar)^n}\int dp\,\Lambda^{-1}\langle x'|xp\rangle\langle xp|x''\rangle = \delta(x'-x)\,\delta(x-x''), \tag{20}$$

$$\frac{1}{(2\pi\hbar)^n}\int dx\,\Lambda^{-1}\langle p'|xp\rangle\langle xp|p''\rangle = \delta(p'-p)\,\delta(p-p''), \tag{21}$$

where on LHS's x(q) and p(k) have the meaning of "expectation values" ($\Lambda^{-1}\langle xp|\hat x|xp\rangle = x,\ \Lambda^{-1}\langle xp|\hat p|xp\rangle = p$) and on RHS's are eigenvalues of the coordinate and momentum operators. Hence the theorem follows on transition probabilities (cross sections): in PSR-2 distributions over these "expectation values" of momenta (coordinates) coincide with those over eigenvalues of the momentum (coordinate) operators:

$$w(p_f,p_i) = \frac{1}{(2\pi\hbar)^{n_f+n_i}}\int dx_f\,dx_i\,\rho_2(x_f p_f t, x_i p_i t_0) = |\langle p_f|e^{-i\hbar^{-1}\hat H\tau}|p_i\rangle|^2 \tag{22}$$

$$w(x_f,x_i) = \frac{1}{(2\pi\hbar)^{n_f+n_i}}\int dp_f\,dp_i\,\rho_2(x_f p_f t, x_i p_i t_0) = |\langle x_f|e^{-i\hbar^{-1}\hat H\tau}|x_i\rangle|^2 \tag{23}$$

$$w(x_f,p_i) = \frac{1}{(2\pi\hbar)^{n_f+n_i}}\int dp_f\,dx_i\,\rho_2(x_f p_f t, x_i p_i t_0) = |\langle x_f|e^{-i\hbar^{-1}\hat H\tau}|p_i\rangle|^2 \tag{24}$$

$$w(p_f,x_i) = \frac{1}{(2\pi\hbar)^{n_f+n_i}}\int dx_f\,dp_i\,\rho_2(x_f p_f t, x_i p_i t_0) = |\langle p_f|e^{-i\hbar^{-1}\hat H\tau}|x_i\rangle|^2, \tag{25}$$

where x_i, x_f, p_i, p_f stand for sets of all initial and final coordinates and momenta. In particular,

$$w(\vec p_4\vec p_3\vec p_2\vec p_1) = \frac{1}{(2\pi\hbar)^{12}}\int d^3\vec x_1\,d^3\vec x_2\,d^3\vec x_3\,d^3\vec x_4\,\rho_2(\vec x_4\vec p_4\vec x_3\vec p_3 t, \vec x_2\vec p_2\vec x_1\vec p_1 t_0) =$$
$$= |\langle\vec p_4\vec p_3|e^{-i\hbar^{-1}\hat H\tau}|\vec p_2\vec p_1\rangle|^2. \tag{26}$$

One can easily check this using eq. (21) after the substitution

$$\rho_2(\vec x_4\vec p_4\vec x_3\vec p_3 t, \vec x_2\vec p_2\vec x_1\vec p_1 t_0) = \Lambda_1^{-1}\Lambda_2^{-1}\Lambda_3^{-1}\Lambda_4^{-1}|\langle\vec x_4\vec p_4\vec x_3\vec p_3|e^{-i\hbar^{-1}\hat H(t-t_0)}|\vec x_2\vec p_2\vec x_1\vec p_1\rangle|^2$$
$$\langle\vec x_4\vec p_4\vec x_3\vec p_3|e^{-i\hbar^{-1}\hat H(t-t_0)}|\vec x_2\vec p_2\vec x_1\vec p_1\rangle = \int d^3p_1'\,d^3p_2'\,d^3p_3'\,d^3p_4'\,\langle\vec x_4\vec p_4|\vec p_4'\rangle\langle\vec x_3\vec p_3|\vec p_3'\rangle\cdot$$
$$\langle\vec p_4{}'\vec p_3{}'|e^{-i\hbar^{-1}\hat H(t-t_0)}|\vec p_2{}'\vec p_1{}'\rangle\langle\vec p_2{}'|\vec x_2\vec p_2\rangle\langle\vec p_1{}'|\vec x_1\vec p_1\rangle. \tag{27}$$

The densities (22)-(25) are obtained simply integrating ρ_2 over variables of no interest[x], just as if we calculated them in classics. Thus, eqs. (22) correspond to problems on beams (ensembles) of particles, momenta of which are measured, but coordinates are not (in particular, so are the Rutherford and Compton scatterings). This concerns all transition probabilities and cross sections of quantum physics. While ρ_2 is in principle convenient for obtaining $w(p_f,p_i)$, $w(x_f,x_i)$, $w(x_f,p_i)$, and $w(p_f,x_i)$, the distribution function ρ_1 is a generating function for transition probabilities between oscillator states:

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[x] One can say that, e.g., $w(p_f,p_i)$ is calculated by "averaging of ρ_2 over initial and summing over final coordinates" (like for spins).

$$|\langle n|e^{-i\hbar^{-1}\hat H(t-t_0)}|m\rangle|^2 =$$
$$= \frac{1}{m!n!}\left(\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}\right)^n\left(\frac{\partial}{\partial a_0}\frac{\partial}{\partial a_0^*}\right)^m \frac{|\langle xp|e^{-i\hbar^{-1}\hat H(t-t_0)}|x_0p_0\rangle|^2}{|\langle xp|0\rangle|^2\ |\langle x_0p_0|0\rangle|^2}\Bigg|_{\substack{a=a^*=0\\ a_0=a_0^*=0}} \tag{28}$$

(for the notation see Appendix A).

Path integrals for amplitudes in phase space terms have been proposed by Feynman /3/, Tobocman /5/, Schweber /10/, Faddeev /17,20/ and others. In refs. /10,20/ the coherent state representation was used together with the completeness relation

$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,|xp\rangle\langle xp| = \mathbb{1}\ . \tag{29}$$

2. Let us proceed to the path integral solution of the Liouville eq. (2). When considering the phase space distribution functions another completeness relation is needed (see, e.g., ref. /22/)[x]:

$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\Lambda^{-1}|xp\rangle\langle xp|\otimes\Lambda^{-1}\|xp\rangle\langle xp\| = |\mathbb{1}\|\otimes\|\mathbb{1}|\ . \tag{30}$$

Four state vectors enter into it (unlike eq. (29)). Eq. (30) permits to conclude that the distribution function ρ_2 satisfies the Markovian property

$$\frac{1}{(2\pi\hbar)^n}\int dx_1dp_1\,\Lambda^{-1}\Lambda_1^{-1}|\langle xp|e^{-i\hbar^{-1}\hat H(t-t_1)}|x_1p_1\rangle|^2\,\Lambda_1^{-1}\Lambda_0^{-1}|\langle x_1p_1|e^{-i\hbar^{-1}\hat H(t_1-t_0)}|x_0p_0\rangle|^2 =$$
$$= \Lambda^{-1}\Lambda_0^{-1}|\langle xp|e^{-i\hbar^{-1}\hat H(t-t_0)}|x_0p_0\rangle|^2. \tag{31}$$

Using this property we can subdivide the interval between initial and final times t_0 and t into a large number N of small intervals, and represent evolution of the distribution function as follows

$$\rho_2(xpt, x_0p_0t_0) = \Lambda^{-1}\Lambda_0^{-1}|\langle xp|e^{-i\hbar^{-1}\hat H(t-t_0)}|x_0p_0\rangle|^2 =$$
$$= \frac{1}{(2\pi\hbar)^{(N-1)n}}\int dx_1dp_1dx_2dp_2\dots dx_{N-1}dp_{N-1}\,\Lambda^{-1}\Lambda_{N-1}^{-1}|\langle xp|e^{-i\hbar^{-1}\hat H(t-t_{N-1})}|x_{N-1}p_{N-1}\rangle|^2\cdot$$
$$\dots\Lambda_2^{-1}\Lambda_1^{-1}|\langle x_2p_2|e^{-i\hbar^{-1}\hat H(t_2-t_1)}|x_1p_1\rangle|^2\,\Lambda_1^{-1}\Lambda_0^{-1}|\langle x_1p_1|e^{-i\hbar^{-1}\hat H(t_1-t_0)}|x_0p_0\rangle|^2, \tag{32}$$

conceiving this subdivision to be carried out infinitely ($N\to\infty$). When Δt is small, each factor can be represented approximately as follows

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[x] Single and double bars are used for the "matrix" notation. In similar terms, e.g., the completeness relation for the matrices γ takes the form $\sum_{A=1\cdots16}|\gamma_A|\otimes\|\gamma_A\| = 4\,|\mathbb{1}\|\otimes\|\mathbb{1}|$.

$$\Lambda_2^{-1}\Lambda_1^{-1}|\langle x_2p_2|e^{-i\hbar^{-1}\Delta t\hat{H}}|x_1p_1\rangle|^2=$$

$$=e^{\Delta t L_{(1)}}\Lambda_2^{-1}\Lambda_1^{-1}|\langle x_2p_2|x_1p_1\rangle|^2=e^{\Delta t L_{(1)}}(2\pi\hbar)^n\delta(x_2-x_1)\delta(p_2-p_1)\approx$$

$$\approx e^{\Delta t L_{o(1)}}\,e^{\Delta t L_{I(1)}}(2\pi\hbar)^n\delta(x_2-x_1)\delta(p_2-p_1)=$$

$$=e^{\Delta t\sum\frac{p_1}{m}\frac{\partial}{\partial x_1}}\,e^{i\hbar^{-1}\Delta t\left[V\left(x_1+\frac{i\hbar}{2}\frac{\partial}{\partial p_1}\right)-V\left(x_1-\frac{i\hbar}{2}\frac{\partial}{\partial p_1}\right)\right]}(2\pi\hbar)^n\delta(x_2-x_1)\delta(p_2-p_1)=$$

$$=\hbar^n\delta\left(x_2-x_1-\frac{p_1}{m}\Delta t\right)\int da\,e^{-i(p_2-p_1)a}\,e^{-i\hbar^{-1}\Delta t\left[V(x_2+\frac{\hbar}{2}a)-V(x_2-\frac{\hbar}{2}a)\right]}$$

$$\left(\approx\hbar^n\delta\left(x_2-x_1-\frac{p_1}{m}\Delta t\right)\int da\,e^{-i(p_2-p_1)a}\,e^{-i\hbar^{-1}\Delta t\left[V(x_1+\frac{\hbar}{2}a)-V(x_1-\frac{\hbar}{2}a)\right]}\right), \qquad (33)$$

where $L_{(1)}=i\hbar^{-1}(H^l-H^r)_{(1)}$, $L_{o(1)}=i\hbar^{-1}(H_o^l-H_o^r)_{(1)}$ and $L_{I(1)}=i\hbar^{-1}(H_I^l-H_I^r)_{(1)}$ are total, free and interaction Liouvillians, acting on x_1 and p_1. Expression (33) follows directly from eq. (1) and is valid only up to the first order in Δt. At the last step the representation of the δ - function $\delta(p_2-p_1)$ via the Fourier integral is used. With eq. (33) we can write eq. (32) as follows

$$\rho_2(xpt,x_op_ot_o)\approx\frac{\hbar^{\cdot\cdot}}{(2\pi)^{(N-1)n}}\int dx_1dp_1\dots dx_{N-1}dp_{N-1}\,da_1\dots da_N$$

$$\delta\left(x-x_{N-1}-\frac{p_{N-1}}{m}(t-t_{N-1})\right)\dots\delta\left(x_2-x_1-\frac{p_1}{m}(t_2-t_1)\right)\delta\left(x_1-x_o-\frac{p_o}{m}(t_1-t_o)\right)$$

$$e^{-i\sum_{k=1}^{N}\left\{(p_k-p_{k-1})a_k+\hbar^{-1}(t_k-t_{k-1})\left[V(x_k+\frac{\hbar}{2}a_k)-V(x_k-\frac{\hbar}{2}a_k)\right]\right\}}. \qquad (34)$$

After performing the integration over $p_1\dots p_{N-1}$ with the help of the δ -functions

$$\rho_2(xpt,x_op_ot_o)\approx\hbar^nC_1\int dx_1\dots dx_{N-1}\,da_1\dots da_N\,e^{-ipa_N}$$

$$e^{i\sum_{k=2}^{N}\left\{m\frac{x_k-x_{k-1}}{t_k-t_{k-1}}(a_k-a_{k-1})-\hbar^{-1}(t_k-t_{k-1})\left[V(x_k+\frac{\hbar}{2}a_k)-V(x_k-\frac{\hbar}{2}a_k)\right]\right\}}$$

$$e^{ip_oa_1}\delta\left(x_1-x_o-\frac{p_o}{m}(t_1-t_o)\right)e^{-i\hbar^{-1}(t_1-t_o)\left[V(x_1+\frac{\hbar}{2}a_1)-V(x_1-\frac{\hbar}{2}a_1)\right]} \qquad (35)$$

and the substitution

$$e^{ip_oa_1}\delta\left(x_1-x_o-\frac{p_o}{m}(t_1-t_o)\right)=\left(\frac{m}{2\pi(t_1-t_o)}\right)^n e^{im\frac{x_1-x_o}{t_1-t_o}a_1}\int da_o\,e^{-i\left(m\frac{x_1-x_o}{t_1-t_o}-p_o\right)a_o} \qquad (36)$$

we obtain

$$\rho_2(xpt,x_op_ot_o)\approx\hbar^nC\int dx_1\dots dx_{N-1}\int da_o\,da_1\dots da_N\,e^{-ipa_N+ip_oa_o}$$

$$e^{i\sum_{k=1}^{N}\left\{m\frac{x_k-x_{k-1}}{t_k-t_{k-1}}(a_k-a_{k-1})-\hbar^{-1}(t_k-t_{k-1})\left[V(x_k+\frac{\hbar}{2}a_k)-V(x_k-\frac{\hbar}{2}a_k)\right]\right\}} \qquad (37)$$

In eqs. (34), (35) and (37) $t_N=t$, $x_N=x$, $a_N=a$,

$$C=\left(\frac{m}{2\pi(t-t_{N-1})}\right)^n\left(\frac{m}{2\pi(t_{N-1}-t_{N-2})}\right)^n\dots\left(\frac{m}{2\pi(t_1-t_o)}\right)^n=C_1\left(\frac{m}{2\pi(t_1-t_o)}\right)^n. \qquad (38)$$

Taking the limit $N\to\infty$ and interpreting

$$\frac{x_k-x_{k-1}}{t_k-t_{k-1}}(a_k-a_{k-1})\longrightarrow\dot{x}(t)\dot{a}(t)dt \qquad (39)$$

we obtain the desired functional integral representation of ρ_2

$$\rho_2(xpt,x_op_ot_o)=\hbar^{2n}\int da_o\,da\,e^{-ipa+ip_oa_o}$$

$$\int\mathcal{D}x\,\mathcal{D}(\hbar a)\exp i\int_{t_o}^{t}dt\left\{m\dot{x}(t)\dot{a}(t)-\hbar^{-1}\left[V\left(x(t)+\frac{\hbar}{2}a(t)\right)-V\left(x(t)-\frac{\hbar}{2}a(t)\right)\right]\right\},$$

$$\mathcal{D}x\,\mathcal{D}(\hbar a)\approx\left(\frac{m}{2\pi\hbar(t-t_{N-1})}\cdots\frac{m}{2\pi\hbar(t_2-t_1)}\,\frac{m}{2\pi\hbar(t_1-t_o)}\right)^n dx_1\dots dx_{N-1}\,d(\hbar a_1)\dots d(\hbar a_{N-1}). \qquad (40)$$

The change of the integration variables

$$x^{(1)}(t)=x(t)+\frac{\hbar}{2}a(t),\qquad x^{(2)}(t)=x(t)-\frac{\hbar}{2}a(t) \qquad (41)$$

leads us to the expressions[x)]

$$\rho_2(xpt,x_op_ot_o)=\hbar^{2n}\int da_o\,da\,e^{-ipa+ip_oa_o}\int\mathcal{D}x^{(1)}\mathcal{D}x^{(2)}e^{i\hbar^{-1}\int_{t_o}^{t}dt\left[L^{(1)}(t)-L^{(2)}(t)\right]}= \qquad (42.a)$$

$$=\hbar^{2n}\int da_o\,da\,e^{-ipa+ip_oa_o}\langle x+\frac{\hbar}{2}a|e^{-i\hbar^{-1}\hat{H}(t-t_o)}|x_o+\frac{\hbar}{2}a_o\rangle$$

$$\langle x_o-\frac{\hbar}{2}a_o|e^{i\hbar^{-1}\hat{H}(t-t_o)}|x-\frac{\hbar}{2}a\rangle, \qquad (42.b)$$

where $L^{(j)}(t)$ are the Lagrangians

$$L^{(j)}(t)=\frac{m}{2}\dot{x}^{(j)2}(t)-V(x^{(j)}(t))\qquad(j=1,2). \qquad (43)$$

Thus, the distribution function ρ_2 is expressed via the product of the two Feynman path integrals for the two amplitudes

$$\langle x+\frac{\hbar}{2}a|e^{-i\hbar^{-1}\hat{H}(t-t_o)}|x_o+\frac{\hbar}{2}a_o\rangle\quad\text{and}\quad\langle x_o-\frac{\hbar}{2}a_o|e^{i\hbar^{-1}\hat{H}(t-t_o)}|x-\frac{\hbar}{2}a\rangle.$$

In fact, the amplitudes appear as a result of the Fourier expansion in eq. (33). Linear cases do not require such expansions, and one can do without amplitudes. In the classical limit ($\hbar=0$) one gets

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[x)] Cf. an influence functional of Feynman and Vernon /11/.

$$\rho_c(xpt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^n}\rho_2(xpt,x_0p_0t_0)|_{\hbar=0}=$$

$$=\int\prod_t{}'dx(t)dp(t)\prod_t\left(\frac{m}{(\Delta t)^2}\right)^n\delta(m\dot{x}(t)-p(t))\delta(\dot{p}(t)-F(x(t)))=$$

$$=\int\prod_t{}'dx(t)dp(t)\prod_t\delta\left(dx(t)-\frac{p(t)}{m}dt\right)\delta\left(dp(t)-F(x(t))dt\right)=$$

$$=\int\prod_t{}'dx(t)dp(t)\prod_t\frac{1}{(2\pi)^{2n}}da(t)d\theta(t)\exp i\int_{t_0}^{t}dt\{\theta(t)(\dot{x}(t)-\frac{p(t)}{m})+\alpha(t)(\dot{p}(t)-F(x(t))\}$$

$$\approx\int dx_1dp_1\dots dx_{N-1}dp_{N-1}\,\delta\left(x-x_{N-1}-\frac{p_{N-1}}{m}(t-t_{N-1})\right)\delta\left(p-p_{N-1}-F(x_{N-1})(t-t_{N-1})\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\delta\left(x_2-x_1-\frac{p_1}{m}(t_2-t_1)\right)\delta\left(p_2-p_1-F(x_1)(t_2-t_1)\right)$$

$$\delta\left(x_1-x_0-\frac{p_0}{m}(t_1-t_0)\right)\delta\left(p_1-p_0-F(x_0)(t_1-t_0)\right), \qquad (44)$$

where $F_i(x)=-\frac{\partial V}{\partial x_i}(x)$, and $\prod$ and $\prod'$ may be interpreted in difference terms as follows

$$\prod_t\frac{1}{(2\pi)^{2n}}da(t)d\theta(t)\approx\frac{1}{(2\pi)^{2Nn}}da_1d\theta_1\dots da_Nd\theta_N, \qquad (45)$$

$$\prod_t{}'dx(t)dp(t)\approx dx_1dp_1\dots dx_{N-1}dp_{N-1}. \qquad (46)$$

The δ-functions in eq. (34) show that the "momentum" and "velocity" are connected like in the classical mechanics ($p(t)=m\dot{x}(t)$). The δ-functions appear due to a bilinear form of the kinetic energy. That is also why in quantum mechanics in PSR-2 the first set of the Hamilton equations

$$x_i(t)=\frac{\partial H}{\partial p_i(t)}=\frac{p_i(t)}{m} \qquad (47)$$

is satisfied for such Hamiltonians (for c-number quantities, but not for operators). However, in general case the potential V(x) does not define momenta uniquely and the second set of the Hamilton equations is not satisfied (unlike the classics). In any linear case, i.e., when V(x) also is at most bilinear, $\hbar$ falls out of the exponent of eq. (34) and integration over the variables α gives δ-functions too, and ρ_2 is expressed via a classical trajectory, a solution of the classical Hamilton equations (see eqs. (9)-(12) above)/23/x).

3. **General Hamiltonians.** The Liouville equation in PSR-2 can be written also as follows /23/

$$\frac{\partial}{\partial t}\rho(xpt)=-i\hbar^{-1}\left\{H^{ord}\left(x+\frac{i\hbar}{2}\frac{\partial}{\partial p},p-\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)-H^{ord}\left(x-\frac{i\hbar}{2}\frac{\partial}{\partial p},p+\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)\right\}\rho(xpt) \qquad (48.a)$$

or

$$\frac{\partial}{\partial t}\rho(xpt)=-i\hbar^{-1}(H^{l\,ord}-H^{r\,ord})\rho(xpt), \qquad (48.b)$$

where ord (ordered) means that the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial p}$ stand to the right of all x and p in $H^{l\,ord}$ and $H^{r\,ord}$. The function H(x,p) is defined as follows

$$H(x,p)=\Lambda^{-1}\langle xp|\hat{H}|xp\rangle. \qquad (49)$$

Now we consider the Hamiltonian $\hat{H}$ of an arbitrary form. It follows from eq. (48) up to the first order in Δt that

$$\Lambda_2^{-1}\Lambda_1^{-1}|\langle x_2p_2|e^{-i\hbar^{-1}\Delta t\hat{H}}|x_1p_1\rangle|^2=$$

$$=e^{-\Delta t\mathcal{L}_{(2)}}\Lambda_2^{-1}\Lambda_1^{-1}|\langle x_2p_2|x_1p_1\rangle|^2=e^{-\Delta t\mathcal{L}_{(2)}}(2\pi\hbar)^n\delta(x_2-x_1)\delta(p_2-p_1)=$$

$$=e^{-\Delta t\mathcal{L}_{(2)}}\frac{\hbar^n}{(2\pi)^n}\int d\alpha d\theta\,e^{-i\alpha(p_2-p_1)-i\theta(x_2-x_1)}\approx$$

$$\approx\frac{\hbar^n}{(2\pi)^n}\int d\alpha d\theta\,e^{-i\alpha(p_2-p_1)-i\theta(x_2-x_1)-i\hbar^{-1}\Delta t[H(x_2+\frac{\hbar}{2}\alpha,p_2-\frac{\hbar}{2}\theta)-H(x_2-\frac{\hbar}{2}\alpha,p_2+\frac{\hbar}{2}\theta)]}. \qquad (50)$$

Inserting this into eq. (32), we obtain the desired difference expression

$$\rho_2(xpt,x_0p_0t_0)=\Lambda^{-1}\Lambda_0^{-1}|\langle xp|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|x_0p_0\rangle|^2\approx$$

$$\approx\frac{\hbar^n}{(2\pi)^{(2N-1)n}}\int dx_1dp_1\dots dx_{N-1}dp_{N-1}d\alpha_1\dots d\alpha_Nd\theta_1\dots d\theta_N$$

$$\exp i\sum_{k=1}^{N}\{-\alpha_k(p_k-p_{k-1})-\hat{\theta}_k(x_k-x_{k-1})-$$

$$-\hbar^{-1}(t_k-t_{k-1})[H(x_k+\frac{\hbar}{2}\alpha_k,p_k-\frac{\hbar}{2}\theta_k)-H(x_k-\frac{\hbar}{2}\alpha_k,p_k+\frac{\hbar}{2}\theta_k)]\} \qquad (51)$$

and finally its continuous limit

$$\rho_2(xpt,x_0p_0t_0)=\int\prod_t{}'\frac{1}{(2\pi\hbar)^n}dx(t)dp(t)\prod_t\left(\frac{\hbar}{2\pi}\right)^n d\alpha(t)d\theta(t)$$

$$\exp i\int_{t_0}^{t}dt\{-\dot{p}(t)\alpha(t)-\dot{x}(t)\theta(t)-$$

$$-\hbar^{-1}[H(x(t)+\frac{\hbar}{2}\alpha(t),p(t)-\frac{\hbar}{2}\theta(t))-H(x(t)-\frac{\hbar}{2}\alpha(t),p(t)+\frac{\hbar}{2}\theta(t))]\}. \qquad (52)$$

Note that the coordinates and momenta enter eqs. (51) and (52) (and eq. (67) below) on equal footing.

Expression (52) can be reduced to amplitudes in a path integral form by the following ways:

a) integrating the first term in the exponent by parts

$$\int_{t_0}^{t}dt\,\dot{p}(t)\alpha(t)=p\alpha-p_0\alpha_0-\int_{t_0}^{t}dt\,p(t)\dot{\alpha}(t), \qquad (53)$$

and extracting integrations over a_0 and a;

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x) Cf. the Chernikov solution of the Schrödinger equation for such systems/14/ and ref./21/.

b) integrating the second term in the exponent by parts

$$\int_{t_0}^{t} dt\,\dot{x}(t)b(t) = xb - x_0b_0 - \int_{t_0}^{t} dt\,x(t)\dot{b}(t), \qquad (54)$$

and extracting integrations over b_0 and b.

These ways lead to

$$\text{a)}\ \rho_2(xpt,x_0p_0t_0)=$$

$$=\hbar^{2n}\int_{-\infty}^{\infty} da_0\,da\,e^{-ipa+ip_0a_0}\int\prod_t{}' dx(t)\,d(\hbar a(t))\prod_t\frac{1}{(2\pi\hbar)^{2n}}dp(t)\,d(\hbar b(t))$$

$$\exp i\int_{t_0}^{t}dt\{p(t)\dot{a}(t)-\dot{x}(t)b(t)-$$

$$-\hbar^{-1}\left[H(x(t)+\tfrac{\hbar}{2}a(t),p(t)-\tfrac{\hbar}{2}b(t))-H(x(t)-\tfrac{\hbar}{2}a(t),p(t)+\tfrac{\hbar}{2}b(t))\right]\}=$$

$$=\hbar^{2n}\int_{-\infty}^{\infty} da_0\,da\,e^{-ipa+ip_0a_0}\int\prod_t{}' dx^{(1)}(t)\,dx^{(2)}(t)\prod_t\frac{1}{(2\pi\hbar)^{2n}}dp^{(1)}(t)\,dp^{(2)}(t)$$

$$\exp i\hbar^{-1}\int_{t_0}^{t}dt\{p^{(1)}(t)\dot{x}^{(1)}(t)-p^{(2)}(t)\dot{x}^{(2)}(t)-H(x^{(1)}(t),p^{(1)}(t))+H(x^{(2)}(t),p^{(2)}(t))\}=$$

$$=\hbar^{2n}\int da_0\,da\,e^{-ipa+ip_0a_0}$$

$$\langle x+\tfrac{\hbar}{2}a|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|x_0+\tfrac{\hbar}{2}a_0\rangle\langle x_0-\tfrac{\hbar}{2}a_0|e^{i\hbar^{-1}\hat{H}(t-t_0)}|x-\tfrac{\hbar}{2}a\rangle. \qquad (55)$$

$$\text{b)}\ \rho_2(xpt,x_0p_0t_0)=$$

$$=\hbar^{2n}\int_{-\infty}^{\infty} db_0\,db\,e^{-ixb+ix_0b_0}\int\prod_t{}' dp(t)\,d(\hbar b(t))\prod_t\frac{1}{(2\pi\hbar)^{2n}}dx(t)\,d(\hbar a(t))$$

$$\exp i\int_{t_0}^{t}dt\{-\dot{p}(t)a(t)+x(t)\dot{b}(t)-$$

$$-\hbar^{-1}\left[H(x(t)+\tfrac{\hbar}{2}a(t),p(t)-\tfrac{\hbar}{2}b(t))-H(x(t)-\tfrac{\hbar}{2}a(t),p(t)+\tfrac{\hbar}{2}b(t))\right]\}=$$

$$=\hbar^{2n}\int_{-\infty}^{\infty} db_0\,db\,e^{-ixb+ix_0b_0}\int\prod_t{}' dp^{(1)}(t)\,dp^{(2)}(t)\prod_t\frac{1}{(2\pi\hbar)^{2n}}dx^{(1)}(t)\,dx^{(2)}(t)$$

$$\exp i\hbar^{-1}\int_{t_0}^{t}dt\{-x^{(1)}(t)\dot{p}^{(1)}(t)+x^{(2)}(t)\dot{p}^{(2)}(t)-H(x^{(1)}(t),p^{(1)}(t))+H(x^{(2)}(t),p^{(2)}(t))\}=$$

$$=\hbar^{2n}\int db_0\,db\,e^{-ixb+ix_0b_0}$$

$$\langle p-\tfrac{\hbar}{2}b|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|p_0-\tfrac{\hbar}{2}b_0\rangle\langle p_0+\tfrac{\hbar}{2}b_0|e^{i\hbar^{-1}\hat{H}(t-t_0)}|p+\tfrac{\hbar}{2}b\rangle. \qquad (56)$$

The last but one expressions (55) and (56) are obtained by the change of variables

$$x^{(1)}(t)=x(t)+\tfrac{\hbar}{2}a(t),\quad x^{(2)}(t)=x(t)-\tfrac{\hbar}{2}a(t), \qquad (57)$$
$$p^{(1)}(t)=p(t)-\tfrac{\hbar}{2}b(t),\quad p^{(2)}(t)=p(t)+\tfrac{\hbar}{2}b(t).$$

Let us also trace the above transformations in difference terms (in the converse order)

$$\text{a)}\ \rho_2(xpt,x_0p_0t_0)=\hbar^{2n}\int da_0\,da\ e^{-ipa\ +ip_0a_0}$$

$$\langle x+\tfrac{\hbar}{2}a|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|x_0+\tfrac{\hbar}{2}a_0\rangle\langle x_0-\tfrac{\hbar}{2}a_0|e^{i\hbar^{-1}\hat{H}(t-t_0)}|x-\tfrac{\hbar}{2}a\rangle\approx$$

$$\approx\hbar^{2n}\int da_0\,da_N\,e^{-ipa_N+ip_0a_0}\frac{1}{(2\pi\hbar)^{2Nn}}\int dx_1^{(1)}dp_1^{(1)}\ldots dx_{N-1}^{(1)}dp_{N-1}^{(1)}dp_N^{(1)}dx_1^{(2)}dp_1^{(2)}\ldots dx_{N-1}^{(2)}dp_{N-1}^{(2)}dp_N^{(2)}$$

$$\exp i\hbar^{-1}\sum_{k=1}^{N}\{p_k^{(1)}(x_k^{(1)}-x_{k-1}^{(1)})-p_k^{(2)}(x_k^{(2)}-x_{k-1}^{(2)})-(t_k-t_{k-1})[H(x_k^{(1)},p_k^{(1)})-H(x_k^{(2)},p_k^{(2)})]\}=$$

$$=\frac{\hbar^n}{(2\pi)^{2Nn}}\int da_0\,da_N\,e^{-ipa_N+ip_0a_0}\int dx_1dp_1\ldots dx_{N-1}dp_{N-1}dp_N\,da_1\ldots da_{N-1}db_1\ldots db_N$$

$$\exp i\sum_{k=1}^{N}\{p_k(a_k-a_{k-1})-b_k(x_k-x_{k-1})-$$

$$-\hbar^{-1}(t_k-t_{k-1})[H(x_k+\tfrac{\hbar}{2}a_k,p_k-\tfrac{\hbar}{2}b_k)-H(x_k-\tfrac{\hbar}{2}a_k,p_k+\tfrac{\hbar}{2}b_k)]\}=$$

$$=\frac{\hbar^{n}}{(2\pi)^{2Nn}}\int dx_1dp_1\cdots dx_{N-1}dp_{N-1}dp_N\,da_0da_1\ldots da_N\,db_1\ldots db_N\,e^{-i(p-p_N)a}$$

$$\exp i\sum_{k=1}^{N}\{-a_{k-1}(p_k-p_{k-1})-b_k(x_k-x_{k-1})-$$

$$-\hbar^{-1}(t_k-t_{k-1})[H(x_k+\tfrac{\hbar}{2}a_k,p_k-\tfrac{\hbar}{2}b_k)-H(x_k-\tfrac{\hbar}{2}a_k,p_k+\tfrac{\hbar}{2}b_k)]\}=$$

$$=\frac{\hbar^n}{(2\pi)^{(2N-1)n}}\int dx_1dp_1\ldots dx_{N-1}dp_{N-1}da_1\ldots da_N\,db_1\ldots db_N$$

$$\exp i\sum_{k=1}^{N}\{-a_k(p_k-p_{k-1})-b_k(x_k-x_{k-1})-$$

$$-\hbar^{-1}(t_k-t_{k-1})[H(x_k+\tfrac{\hbar}{2}a_k,p_k-\tfrac{\hbar}{2}b_k)-H(x_k-\tfrac{\hbar}{2}a_k,p_k+\tfrac{\hbar}{2}b_k)]\}, \qquad (58)$$

$$\text{b)}\quad \rho_2(xpt,x_0p_0t_0)=\hbar^{2n}\int db_0\,db\ \ e^{-ixb\ +ix_0b_0}.$$

$$\langle p-\tfrac{\hbar}{2}b|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|p_0-\tfrac{\hbar}{2}b_0\rangle\langle p_0+\tfrac{\hbar}{2}b_0|e^{i\hbar^{-1}\hat{H}(t-t_0)}|p+\tfrac{\hbar}{2}b\rangle\approx$$

$$\approx \hbar^{2n}\int db_0 db_N\, e^{-ixb_N+ix_0b_0}\frac{1}{(2\pi\hbar)^{2Nn}}\int dx_1^{(1)}dp_1^{(1)}\dots dx_{N-1}^{(1)}dp_{N-1}^{(1)}dx_N^{(1)}dx_1^{(2)}dp_1^{(2)}\dots dx_{N-1}^{(2)}dp_{N-1}^{(2)}dx_N^{(2)}$$

$$\exp i\hbar^{-1}\sum_{k=1}^{N}\{-x_k^{(1)}(p_k^{(1)}-p_{k-1}^{(1)})+x_k^{(2)}(p_k^{(2)}-p_{k-1}^{(2)})-(t_k-t_{k-1})[H(x_k^{(1)},p_k^{(1)})-H(x_k^{(2)},p_k^{(2)})]\}=$$

$$=\frac{\hbar^n}{(2\pi)^{2Nn}}\int db_0 db_N\, e^{-ixb_N+ix_0b_0}\int dx_1dp_1\dots dx_{N-1}dp_{N-1}dx_N\, da_1\dots da_N\, db_1\dots db_{N-1}$$

$$\exp i\sum_{k=1}^{N}\{x_k(b_k-b_{k-1})-a_k(p_k-p_{k-1})-$$

$$-\hbar^{-1}(t_k-t_{k-1})[H(x_k+\tfrac{\hbar}{2}a_k,p_k-\tfrac{\hbar}{2}b_k)-H(x_k-\tfrac{\hbar}{2}a_k,p_k+\tfrac{\hbar}{2}b_k)]\}=$$

$$=\frac{\hbar^n}{(2\pi)^{2Nn}}\int dx_1dp_1\dots dx_{N-1}dp_{N-1}dx_N\, da_1\dots da_N\, db_0db_1\dots db_N\, e^{-i(x-x_N)b_N}$$

$$\exp i\sum_{k=1}^{N}\{-b_{k-1}(x_k-x_{k-1})-a_k(p_k-p_{k-1})-$$

$$-\hbar^{-1}(t_k-t_{k-1})[H(x_k+\tfrac{\hbar}{2}a_k,p_k-\tfrac{\hbar}{2}b_k)-H(x_k-\tfrac{\hbar}{2}a_k,p_k+\tfrac{\hbar}{2}b_k)]\}=$$

$$=\frac{\hbar^n}{(2\pi)^{(2N-1)n}}\int dx_1dp_1\dots dx_{N-1}dp_{N-1}da_1\dots da_N\, db_1\dots db_N$$

$$\exp i\sum_{k=1}^{N}\{-\hat{b}_k(x_k-x_{k-1})-a_k(p_k-p_{k-1})-$$

$$-\hbar^{-1}(t_k-t_{k-1})[H(x_k+\tfrac{\hbar}{2}a_k,p_k-\tfrac{\hbar}{2}b_k)-H(x_k-\tfrac{\hbar}{2}a_k,p_k+\tfrac{\hbar}{2}b_k)]\}. \quad (59)$$

Here the following transformations were performed:

i) change of variables

$$x_k^{(1)}=x_k+\tfrac{\hbar}{2}a_k,\quad x_k^{(2)}=x_k-\tfrac{\hbar}{2}a_k, \quad (60)$$
$$p_k^{(1)}=p_k-\tfrac{\hbar}{2}b_k,\quad p_k^{(2)}=p_k+\tfrac{\hbar}{2}b_k;$$

ii) summations by parts

$$\sum_{k=1}^{N}p_k(a_k-a_{k-1})=p_Na_N-p_0a_0-\sum_{k=1}^{N}a_{k-1}(p_k-p_{k-1}),$$
$$\sum_{k=1}^{N}x_k(b_k-b_{k-1})=x_Nb_N-x_0b_0-\sum_{k=1}^{N}b_{k-1}(x_k-x_{k-1}), \quad (61)$$

in eqs. (58) and (59), respectively;

iii) integrations over a_0, giving $\delta(p_1-p_0)$, in eq. (58) and over b_0, giving $\delta(x_1-x_0)$, in eq. (59), and further integrations over p_1 and x_1, respectively;

iiii) renumberings $p_k\rightarrow p_{k-1}$ and $x_k\rightarrow x_{k-1}$ in eqs. (58) and (59), respectively, (k=2,3,...N) throughout except for the Hamiltonians, and $p_0\rightarrow p_1$ and $x_0\rightarrow x_1$ in the latters.

A formal classical limit of eqs. (51) and (52) can be written as

$$\rho_c(xpt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^n}\rho_2(xpt,x_0p_0t_0)|_{\hbar=0}=$$

$$=\int\prod_t{}'dx(t)dp(t)\prod_t(\Delta t)^{-2n}\delta(\dot{x}(t)-\frac{\partial H}{\partial p}(x(t),p(t)))\delta(\dot{p}(t)+\frac{\partial H}{\partial x}(x(t),p(t)))=$$

$$=\int\prod_t{}'dx(t)dp(t)\prod_t\delta(dx(t)-\frac{\partial H}{\partial p}(x(t),p(t))dt)\delta(dp(t)+\frac{\partial H}{\partial x}(x(t),p(t))dt)=$$

$$=\int\prod_t{}'dx(t)dp(t)\prod_t\frac{1}{(2\pi)^{2n}}da(t)db(t)\exp i\int_{t_0}^{t}dt\{b(t)[\dot{x}(t)-\frac{\partial H}{\partial p}(x(t),p(t))]+$$
$$+a(t)[\dot{p}(t)+\frac{\partial H}{\partial x}(x(t),p(t))]\}=$$

$$\approx\int dx_1dp_1\dots dx_{N-1}dp_{N-1}$$

$$\delta(x-x_{N-1}-\frac{\partial H}{\partial p}(x_{N-1},p_{N-1})(t-t_{N-1}))\delta(p-p_{N-1}+\frac{\partial H}{\partial x}(x_{N-1},p_{N-1})(t-t_{N-1}))$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\delta(x_2-x_1-\frac{\partial H}{\partial p}(x_1,p_1)(t_2-t_1))\delta(p_2-p_1+\frac{\partial H}{\partial x}(x_1,p_1)(t_2-t_1))$$
$$\delta(x_1-x_0-\frac{\partial H}{\partial p}(x_0,p_0)(t_1-t_0))\delta(p_1-p_0+\frac{\partial H}{\partial x}(x_0,p_0)(t_1-t_0)). \quad (62)$$

4. Now we turn to quantum field theory and represent a functional ρ_2, which is a solution of the corresponding generalized Liouville equation[x)], in a path-integral form. Scalar field theory will be considered as a pattern. The completeness relation

$$\int\delta^2\varphi\,\Lambda^{-1}|\varphi\pi\rangle\langle\varphi\pi|\otimes\Lambda^{-1}\|\varphi\pi\rangle\langle\varphi\pi\|=|\mathbb{1}\|\otimes\|\mathbb{1}|, \quad (63)$$
$$\delta^2\varphi=\prod_{\vec{x}}(2\pi\hbar)^{-1}d\varphi(\vec{x})d\pi(\vec{x}),$$

leads to the Markovian property

$$\int\delta^2\varphi_1\Lambda^{-1}\Lambda_1^{-1}|\langle\varphi\pi|e^{-i\hbar^{-1}\hat{H}(t-t_1)}|\varphi_1\pi_1\rangle|^2\Lambda_1^{-1}\Lambda_0^{-1}|\langle\varphi_1\pi_1|e^{-i\hbar^{-1}\hat{H}(t_1-t_0)}|\varphi_0\pi_0\rangle|^2=$$
$$=\Lambda^{-1}\Lambda_0^{-1}|\langle\varphi\pi|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|\varphi_0\pi_0\rangle|^2 \quad (64)$$

and to the subdivision of evolution into $N\rightarrow\infty$ small time intervals

$$\rho_2(\varphi\pi t,\varphi_0\pi_0t_0)=\Lambda^{-1}\Lambda_0^{-1}|\langle\varphi\pi|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|\varphi_0\pi_0\rangle|^2=$$
$$=\int\delta^2\varphi_1\delta^2\varphi_2\dots\delta^2\varphi_{N-1}\Lambda^{-1}\Lambda_{N-1}^{-1}|\langle\varphi\pi|e^{-i\hbar^{-1}\hat{H}(t-t_{N-1})}|\varphi_{N-1}\pi_{N-1}\rangle\cdot$$
$$\cdots\Lambda_2^{-1}\Lambda_1^{-1}|\langle\varphi_2\pi_2|e^{-i\hbar^{-1}\hat{H}(t_2-t_1)}|\varphi_1\pi_1\rangle|^2\Lambda_1^{-1}\Lambda_0^{-1}|\langle\varphi_1\pi_1|e^{-i\hbar^{-1}\hat{H}(t_1-t_0)}|\varphi_0\pi_0\rangle|^2. \quad (65)$$

For small Δt one can represent the factors as follows

[x)] For details see ref. /24/. However here $|\varphi\pi\rangle$ stands for coherent states, denoted by $|\varphi\rangle$ in this reference.

$$\Lambda_2^{-1}\Lambda_1^{-1}|\langle\varphi_2\pi_2|e^{-i\hbar^{-1}\Delta t\hat{H}}|\varphi_1\pi_1\rangle|^2=e^{-\Delta t\mathbb{L}_{(2)}}\Lambda_2^{-1}\Lambda_1^{-1}|\langle\varphi_2\pi_2|\varphi_1\pi_1\rangle|^2=$$
$$=e^{-i\hbar^{-1}\Delta t(H^l-H^r)_{(2)}}\prod_{\vec{x}}2\pi\hbar\,\delta(\varphi_2(\vec{x})-\varphi_1(\vec{x}))\delta(\pi_2(\vec{x})-\pi_1(\vec{x}))\approx$$
$$\approx\int\prod_{\vec{x}}\frac{\hbar}{2\pi}da(\vec{x})db(\vec{x})\exp i\int d^3x\{-a(\vec{x})(\pi_2(\vec{x})-\pi_1(\vec{x}))-b(\vec{x})(\varphi_2(\vec{x})-\varphi_1(\vec{x}))-$$
$$-\hbar^{-1}(t_2-t_1)[\mathcal{H}(\varphi_2(\vec{x})+\frac{\hbar}{2}a(\vec{x}),\pi_2(\vec{x})-\frac{\hbar}{2}b(\vec{x}))-\mathcal{H}(\varphi_2(\vec{x})-\frac{\hbar}{2}a(\vec{x}),\pi_2(\vec{x})+\frac{\hbar}{2}b(\vec{x}))]\}. \qquad (66)$$

Thus, the desired functional integral form of ρ_2 is

$$\rho_2(\varphi\pi t,\varphi_0\pi_0t_0)=\Lambda^{-1}\Lambda_0^{-1}|\langle\varphi\pi|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|\varphi_0\pi_0\rangle|^2=$$
$$=\int\prod_t{}'\delta^2\varphi(t)\prod_{\vec{x},t}\frac{\hbar}{2\pi}da(x)db(x)\exp i\int_{t_0}^{t}d^4x\{-a(x)\dot{\pi}(x)-b(x)\dot{\varphi}(x)-$$
$$-\hbar^{-1}[\mathcal{H}(\varphi(x)+\frac{\hbar}{2}a(x),\pi(x)-\frac{\hbar}{2}b(x))-\mathcal{H}(\varphi(x)-\frac{\hbar}{2}a(x),\pi(x)+\frac{\hbar}{2}b(x))]\}\approx$$
$$\approx\int\delta^2\varphi_1\dots\delta^2\varphi_{N-1}\prod_{\vec{x}}\left(\frac{\hbar}{2\pi}\right)^N da_1(\vec{x})db_1(\vec{x})\dots da_N(\vec{x})db_N(\vec{x})$$
$$\exp i\int d^3x\sum_{k=1}^{N}\{-a_k(\vec{x})(\pi_k(\vec{x})-\pi_{k-1}(\vec{x}))-b_k(\vec{x})(\varphi_k(\vec{x})-\varphi_{k-1}(\vec{x}))- \qquad (67)$$
$$-\hbar^{-1}(t_k-t_{k-1})[\mathcal{H}(\varphi_k(\vec{x})+\frac{\hbar}{2}a_k(\vec{x}),\pi_k(\vec{x})-\frac{\hbar}{2}b_k(\vec{x}))-\mathcal{H}(\varphi_k(\vec{x})-\frac{\hbar}{2}a_k(\vec{x}),\pi_k(\vec{x})+\frac{\hbar}{2}b_k(\vec{x}))]\},$$

where $x\equiv(\vec{x},t)$. This expression can be reduced to amplitudes as follows

$$\text{a)}\ \rho_2(\varphi\pi t,\varphi_0\pi_0t_0)=\int\prod_{\vec{x}}\hbar^2da_0(\vec{x})da(\vec{x})\,e^{-i\int d^3x(\pi(\vec{x})a(\vec{x})-\pi_0(\vec{x})a_0(\vec{x}))}$$
$$\int\prod_t{}'\prod_{\vec{x}}d\varphi^{(1)}(x)d\varphi^{(2)}(x)\prod_{\vec{x},t}\frac{1}{(2\pi\hbar)^2}d\pi^{(1)}(x)d\pi^{(2)}(x)$$
$$\exp i\hbar^{-1}\int_{t_0}^{t}d^4x\{\pi^{(1)}(x)\dot{\varphi}^{(1)}(x)-\pi^{(2)}(x)\dot{\varphi}^{(2)}(x)-\mathcal{H}(\varphi^{(1)}(x),\pi^{(1)}(x))+\mathcal{H}(\varphi^{(2)}(x),\pi^{(2)}(x))\}=$$
$$=\int\prod_{\vec{x}}\hbar^2da_0(\vec{x})da(\vec{x})\,e^{-i\int d^3x(\pi(\vec{x})a(\vec{x})-\pi_0(\vec{x})a_0(\vec{x}))}$$
$$\langle\varphi+\frac{\hbar}{2}a|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|\varphi_0+\frac{\hbar}{2}a_0\rangle\langle\varphi_0-\frac{\hbar}{2}a_0|e^{i\hbar^{-1}\hat{H}(t-t_0)}|\varphi-\frac{\hbar}{2}a\rangle, \qquad (68)$$

$$\text{b)}\ \rho_2(\varphi\pi t,\varphi_0\pi_0t_0)=\int\prod_{\vec{x}}\hbar^2db_0(\vec{x})db(\vec{x})\,e^{-i\int d^3x(\varphi(\vec{x})b(\vec{x})-\varphi_0(\vec{x})b_0(\vec{x}))}$$
$$\int\prod_t{}'\prod_{\vec{x}}d\pi^{(1)}(x)d\pi^{(2)}(x)\prod_{\vec{x},t}\frac{1}{(2\pi\hbar)^2}d\varphi^{(1)}(x)d\varphi^{(2)}(x)$$
$$\exp i\hbar^{-1}\int_{t_0}^{t}d^4x\{-\varphi^{(1)}(x)\dot{\pi}^{(1)}(x)+\varphi^{(2)}(x)\dot{\pi}^{(2)}(x)-\mathcal{H}(\varphi^{(1)}(x),\pi^{(1)}(x))+\mathcal{H}(\varphi^{(2)}(x),\pi^{(2)}(x))\}=$$
$$=\int\prod_{\vec{x}}\hbar^2db_0(\vec{x})db(\vec{x})\,e^{-i\int d^3x(\varphi(\vec{x})b(\vec{x})-\varphi_0(\vec{x})b_0(\vec{x}))}$$
$$\langle\pi-\frac{\hbar}{2}b|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|\pi_0-\frac{\hbar}{2}b_0\rangle\langle\pi_0+\frac{\hbar}{2}b_0|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|\pi+\frac{\hbar}{2}b\rangle, \qquad (69)$$

where the variables are connected as follows

$$\varphi^{(1)}(x)=\varphi(x)+\frac{\hbar}{2}a(x),\qquad\varphi^{(2)}(x)=\varphi(x)-\frac{\hbar}{2}a(x), \qquad (70)$$
$$\pi^{(1)}(x)=\pi(x)-\frac{\hbar}{2}b(x),\qquad\pi^{(2)}(x)=\pi(x)+\frac{\hbar}{2}b(x).$$

It follows from eq. (67) that for the particular Hamiltonians $\mathcal{H}=$ $=\frac{1}{2}\pi^2+\frac{1}{2}\partial_k\varphi\partial_k\varphi+\frac{m^2}{2}\varphi^2+\mathcal{H}_I(\varphi(x))$ $(\pi(x)=\dot{\varphi}(x))$

$$\rho_2(\varphi\pi t,\varphi_0\pi_0t_0)=\int\prod_{\vec{x}}\hbar^2da_0(\vec{x})da(\vec{x})\,e^{-i\int d^3x(\pi(\vec{x})a(\vec{x})-\pi_0(\vec{x})a_0(\vec{x}))}$$
$$\int\mathcal{D}\varphi^{(1)}\mathcal{D}\varphi^{(2)}\exp i\hbar^{-1}\int_{t_0}^{t}d^4x\,[\mathcal{L}^{(1)}(x)-\mathcal{L}^{(2)}(x)], \qquad (71)$$

where

$$\mathcal{D}\varphi^{(1)}\mathcal{D}\varphi^{(2)}\approx\prod_{\vec{x}}\frac{1}{2\pi\hbar(t-t_{N-1})}\cdots\frac{1}{2\pi\hbar(t_2-t_1)}\frac{1}{2\pi\hbar(t_1-t_0)}d\varphi_1^{(1)}(\vec{x})\dots d\varphi_{N-1}^{(1)}(\vec{x})d\varphi_1^{(2)}(\vec{x})\dots d\varphi_{N-1}^{(2)}(\vec{x}), \qquad (72)$$
$$\mathcal{L}^{(i)}(x)=-\frac{1}{2}\partial_\mu\varphi^{(i)}(x)\partial_\mu\varphi^{(i)}(x)-\frac{m^2}{2}\varphi^{(i)}(x)\varphi^{(i)}(x)-\mathcal{H}_I(\varphi^{(i)}(x)). \qquad (73)$$

Independent derivation of eq. (71) may proceed as follows

$$\Lambda_2^{-1}\Lambda_1^{-1}|\langle\varphi_2\pi_2|e^{-i\hbar^{-1}\Delta t\hat{H}}|\varphi_1\pi_1\rangle|^2=e^{\Delta t\mathbb{L}_{(1)}}\Lambda_2^{-1}\Lambda_1^{-1}|\langle\varphi_2\pi_2|\varphi_1\pi_1\rangle|^2=$$
$$=e^{\Delta t\mathbb{L}_{(1)}}\prod_{\vec{x}}2\pi\hbar\,\delta(\varphi_2(\vec{x})-\varphi_1(\vec{x}))\delta(\pi_2(\vec{x})-\pi_1(\vec{x}))\approx$$
$$\approx e^{\Delta t\mathbb{L}_{0(1)}}e^{\Delta t\mathbb{L}_{I(1)}}\prod_{\vec{x}}2\pi\hbar\,\delta(\varphi_2(\vec{x})-\varphi_1(\vec{x}))\delta(\pi_2(\vec{x})-\pi_1(\vec{x}))=$$
$$=\prod_{\vec{x}}2\pi\hbar\delta(\varphi_2(\vec{x})-\varphi_1(\vec{x},t_2))e^{\Delta t\mathbb{L}_{0(1)}}\exp i\hbar^{-1}\Delta t\int d^3x'[\mathcal{H}_I(\varphi_1(\vec{x}')+\frac{i\hbar}{2}\frac{\delta}{\delta\pi_1(\vec{x}')})-$$
$$-\mathcal{H}_I(\varphi_1(\vec{x}')-\frac{i\hbar}{2}\frac{\delta}{\delta\pi_1(\vec{x}')})]\delta(\pi_2(\vec{x})-\pi_1(\vec{x}))=$$
$$=\prod_{\vec{x}}\hbar\delta(\varphi_2(\vec{x})-\varphi_1(\vec{x},t_2))\int\prod_{\vec{x}}da(\vec{x})\exp\int d^3x\{-ia(\vec{x})(\pi_2(\vec{x})-\pi_1(\vec{x},t_2))-$$
$$-i\hbar^{-1}(t_2-t_1)[\mathcal{H}_I(\varphi_2(\vec{x})+\frac{\hbar}{2}a(\vec{x}))-\mathcal{H}_I(\varphi_2(\vec{x})-\frac{\hbar}{2}a(\vec{x}))]\}, \qquad (74)$$

$$\rho_2(\varphi\pi t,\varphi_0\pi_0t_0)\approx\int\delta^2\varphi_1\dots\delta^2\varphi_{N-1}\prod_{\vec{x}}da_1(\vec{x})\dots da_N(\vec{x})$$
$$\prod_{\vec{x}}\hbar\delta(\varphi(\vec{x})-\varphi_{N-1}(\vec{x},t))\cdots\prod_{\vec{x}}\hbar\delta(\varphi_2(\vec{x})-\varphi_1(\vec{x},t_2))\prod_{\vec{x}}\hbar\delta(\varphi_1(\vec{x})-\varphi_0(\vec{x},t_1))$$
$$\exp\sum_{k=1}^{N}\int d^3x\{-ia_k(\vec{x})(\pi_k(\vec{x})-\pi_{k-1}(\vec{x},t_k))-i\hbar^{-1}(t_k-t_{k-1})[\mathcal{H}_I(\varphi_k(\vec{x})+\frac{\hbar}{2}a_k(\vec{x}))-$$
$$-\mathcal{H}_I(\varphi_k(\vec{x})-\frac{\hbar}{2}a_k(\vec{x}))]\}, \qquad (75)$$

where $\varphi_{k-1}(\vec{x},t_k)$ and $\pi_{k-1}(\vec{x},t_k)$ satisfy the classical Hamilton equations for the free field with the initial conditions $\varphi_{k-1}(\vec{x})$ and $\pi_{k-1}(\vec{x})$ at t_{k-1}.

After integrating over the momenta $\pi_k(\vec{x})$ in eq. (75) we will obtain eq. (71). The easiest way to do this is to assume $\varphi_{k-1}(\vec{x},t_k)$

to be a solution of the equation $\ddot{\varphi}_{k-1}(\vec{x},t)=0$, i.e.,

$$\varphi_{k-1}(\vec{x},t_k)=\varphi_{k-1}(\vec{x})+(t_k-t_{k-1})\pi_{k-1}(\vec{x}),\quad \pi_{k-1}(\vec{x},t_k)=\pi_k(\vec{x}),$$

and substitute $\tilde{\mathcal{H}}_I=\frac{1}{2}\partial_k\varphi\,\partial_k\varphi+\frac{m^2}{2}\varphi^2+\mathcal{H}_I$ for $\mathcal{H}_I$ in eq. (75); thus easily obtaining eq. (71) (like above in quantum mechanics). Another way is to assume $\varphi_{k-1}(\vec{x},t_k)$ to be a solution of the Klein-Gordon equation $(\Box-\varkappa^2)\varphi_{k-1}(\vec{x},t)=0$, $\mathcal{H}_I$ being the true interaction Hamiltonian. This way is more complicated, giving however the same result (see also Appendix C).

Appendix A. Expectation values of the density matrices

$$\hat{\rho}=|x\rangle\langle x|,\quad |p\rangle\langle p|,\quad |xp\rangle\langle xp|,\quad \Lambda^{-1}|xp\rangle\langle xp|$$

are positive definite in the x- and p-representations and in PSR-1 PSR-2, in particular,[x]

$$\langle x''|xp\rangle\langle xp|x'\rangle=e^{i\hbar^{-1}p(x''-x')}\,e^{-(4\hbar)^{-1}A(x''-x')^2}(\pi\hbar)^{-\frac{3n}{2}}(\det A)^{\frac{n}{2}}e^{-\hbar^{-1}A\left(x-\frac{x'+x''}{2}\right)^2}, \tag{A.1}$$

$$\Lambda^{-1}\langle x''|xp\rangle\langle xp|x'\rangle=e^{i\hbar^{-1}p(x''-x')}\,\delta\left(x-\frac{x''+x'}{2}\right), \tag{A.2}$$

$$\langle p''|xp\rangle\langle xp|p'\rangle=e^{-i\hbar^{-1}x(p''-p')}\,e^{-(4\hbar)^{-1}A^{-1}(p''-p')^2}(\pi\hbar)^{-\frac{3n}{2}}(\det A)^{-\frac{n}{2}}e^{-\hbar^{-1}A^{-1}\left(p-\frac{p'+p''}{2}\right)^2}, \tag{A.3}$$

$$\Lambda^{-1}\langle p''|xp\rangle\langle xp|p'\rangle=e^{-i\hbar^{-1}x(p''-p')}\,\delta\left(p-\frac{p''+p'}{2}\right), \tag{A.4}$$

$$|\langle x'p'|xp\rangle|^2=e^{-(2\hbar)^{-1}\left(A(x-x')^2+A^{-1}(p-p')^2\right)}=e^{-|\alpha-\alpha'|^2}, \tag{A.5}$$

$$\Lambda^{-1}|\langle x'p'|xp\rangle|^2=\Lambda'^{-1}|\langle x'p'|xp\rangle|^2=2^n e^{-\hbar^{-1}\left(A(x-x')^2+A^{-1}(p-p')^2\right)}, \tag{A.6}$$

$$\Lambda^{-1}\Lambda'^{-1}|\langle x'p'|xp\rangle|^2=(2\pi\hbar)^n\delta(x-x')\delta(p-p'). \tag{A.7}$$

However, eigenstates $|n\rangle\langle n|$ of the oscillator Hamiltonian or $\hat{N}=\hat{a}^+\hat{a}$ are positive definite in PSR-1, but, in general, not positive definite in PSR-2:

$$|\langle xp|n\rangle|^2=\frac{(aa^*)^n}{n!}|\langle xp|0\rangle|^2=\frac{(aa^*)^n}{n!}e^{-a^*a}=\frac{(Ax^2+A^{-1}p^2)^n}{2^n\hbar^n n!}e^{-(2\hbar)^{-1}(Ax^2+A^{-1}p^2)}, \tag{A.8}$$

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[x] In a many-dimensional case $A=\|a_{\mu\nu}\|$ is a matrix, $Ax''x'=a_{\mu\nu}x''_\mu x'_\nu$,

$$\frac{\hbar}{2}a_{\mu\nu}=\langle xp|(\hat{p}_\mu-p_\mu)(\hat{p}_\nu-p_\nu)|xp\rangle\equiv\overline{\Delta p_\mu\Delta p_\nu},\quad \frac{\hbar}{2}c_{\mu\nu}=\langle xp|(\hat{x}_\mu-x_\mu)(\hat{x}_\nu-x_\nu)|xp\rangle\equiv\overline{\Delta x_\mu\Delta x_\nu},$$

$\overline{\Delta p_\mu\Delta p_\nu}\,\overline{\Delta x_\nu\Delta x_\lambda}=\frac{\hbar^2}{4}\delta_{\mu\lambda}$ $(AC=1)$ (minimal uncertainty relation),

$$\hat{a}=\sqrt{\frac{A}{2\hbar}}\hat{x}+\frac{i}{\sqrt{2\hbar A}}\hat{p},\quad \hat{a}^+=\sqrt{\frac{A}{2\hbar}}\hat{x}-\frac{i}{\sqrt{2\hbar A}}\hat{p},\quad a=\sqrt{\frac{A}{2\hbar}}x+\frac{i}{\sqrt{2\hbar A}}p,\quad a^*=\sqrt{\frac{A}{2\hbar}}x-\frac{i}{\sqrt{2\hbar A}}p,$$

$(a=\langle xp|\hat{a}|xp\rangle,\ a^*=\langle xp|\hat{a}^+|xp\rangle)$.

$$\Lambda^{-1}|\langle xp|0\rangle|^2=2\,e^{-\hbar^{-1}(Ax^2+A^{-1}p^2)}, \tag{A.9}$$

$$\Lambda^{-1}|\langle xp|1\rangle|^2=\left[-2+4\hbar^{-1}(Ax^2+A^{-1}p^2)\right]e^{-\hbar^{-1}(Ax^2+A^{-1}p^2)}. \tag{A.10}$$

One can obtain $\hat{\rho}=|n\rangle\langle n|$ in PSR-1 and PSR-2 as follows

$$1)\ |\langle xp|n\rangle|^2=\frac{(-1)^n}{n!}\left(\frac{d}{d\alpha}\right)^n g(xp\alpha)_{|\alpha=1},\quad g(xp\alpha)=e^{-\alpha(2\hbar)^{-1}(Ax^2+A^{-1}p^2)} \tag{A.11}$$

$$\Lambda^{-1}|\langle xp|n\rangle|^2=\frac{(-1)^n}{n!}\left(\frac{d}{d\alpha}\right)^n\Lambda^{-1}g(xp\alpha)_{|\alpha=1},\quad \Lambda^{-1}g(xp\alpha)=\frac{2}{2-\alpha}e^{-\frac{\alpha}{2-\alpha}\hbar^{-1}(Ax^2+A^{-1}p^2)} \tag{A.12}$$

$$\Lambda^{-1}|\langle xp|n\rangle|^2=\frac{(-1)^n}{n!}2L_n(u)e^{-\frac{u}{2}},\quad u=2\hbar^{-1}(Ax^2+A^{-1}p^2), \tag{A.13}$$

where $L_n(u)$ are the Laguerre polynomials.

$$2)\ \langle n|xp\rangle\langle xp|m\rangle=\frac{1}{\sqrt{m!n!}}a^{*m}a^n|\langle xp|0\rangle|^2 \tag{A.14}$$

$$\Lambda^{-1}\langle n|xp\rangle\langle xp|m\rangle=\frac{1}{\sqrt{m!n!}}\left(a^*-\frac{1}{2}\frac{\partial}{\partial a}\right)^m\left(a-\frac{1}{2}\frac{\partial}{\partial a^*}\right)^n\Lambda^{-1}|\langle xp|0\rangle|^2=$$

$$=\frac{1}{\sqrt{m!n!}}a^{*m}a^n\left[1-\frac{1}{4r}\left(\frac{\partial}{\partial r}+\frac{2n}{r}\right)\right]^m\left[1-\frac{1}{4r}\frac{\partial}{\partial r}\right]^n\Lambda^{-1}|\langle xp|0\rangle|^2. \tag{A.15}$$

where in the last expression the action-angle variables

$$a=re^{i\varphi},\qquad \frac{\partial}{\partial a}=\frac{a^*}{2r}\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}\right). \tag{A.16}$$

are used. In the many-dimensional case

$$|\langle xp|\mu_1\ldots\mu_n\rangle|^2=|a_{\mu_1}|^2|a_{\mu_2}|^2\ldots|a_{\mu_n}|^2e^{-a^*_\mu a_\mu} \tag{A.17}$$

$$g(xp\alpha)=\exp(-a^*_\mu\alpha_{\mu\nu}a_\nu)=\exp\left[-(2\hbar)^{-1}(\sqrt{A}\,\alpha\sqrt{A}\,x^2+\sqrt{A^{-1}}\,\alpha\,\sqrt{A^{-1}}\,p^2\right) \tag{A.18}$$

$$\Lambda^{-1}g(xp\alpha)=\left[\det\left(1-\frac{\alpha}{2}\right)\right]^{-1}\exp\left\{-(2\hbar)^{-1}\left[\sqrt{A}\left(\alpha^{-1}-\frac{1}{2}\right)^{-1}\sqrt{A}\,x^2+\sqrt{A^{-1}}\left(\alpha^{-1}-\frac{1}{2}\right)^{-1}\sqrt{A^{-1}}\,p^2\right]\right\}, \tag{A.19}$$

where α is a diagonal matrix.

$$3)\ \langle n|xp\rangle\langle xp|m\rangle=\frac{1}{\sqrt{m!n!}}\left(\frac{\partial}{\partial a'^*}\right)^n\left(\frac{\partial}{\partial a'}\right)^m\frac{|\langle xp|x'p'\rangle|^2}{|\langle x'p'|0\rangle|^2}\Big|_{x'=p'=0}. \tag{A.20}$$

An arbitrary operator can be represented as follows

$$\hat{\mathbb{F}}=:\mathbb{F}_1(\hat{x},\hat{p}):=\mathrm{sym}\,\mathbb{F}_2(\hat{x},\hat{p})=\vdots\,\mathbb{F}_3(\hat{x},\hat{p})\,\vdots=$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\mathbb{F}_1(x,p)\Lambda^{-2}|xp\rangle\langle xp|=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\mathbb{F}_2(x,p)\Lambda^{-1}|xp\rangle\langle xp|=$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\mathbb{F}_3(x,p)|xp\rangle\langle xp|, \tag{A.21}$$

$$\mathbb{F}_1(x,p)=\langle xp|\hat{\mathbb{F}}|xp\rangle,\ \mathbb{F}_2(x,p)=\Lambda^{-1}\langle xp|\hat{\mathbb{F}}|xp\rangle,\ \mathbb{F}_3(x,p)=\Lambda^{-2}\langle xp|\hat{\mathbb{F}}|xp\rangle. \tag{A.22}$$

Three last expressions (A.21) result from completeness relations (eq. (30) and similar ones given, e.g., in ref. /22/). The last expression

(A.21) may also be obtained in the following way

$$:\mathbb{F}_3(\hat{x},\hat{p}): =\sum K\,\hat{a}\cdots\hat{a}\,\hat{a}^+\cdots\hat{a}^+ =\sum K\,\hat{a}\cdots\hat{a}\frac{1}{(2\pi\hbar)^n}\int dx'dp'|x'p'\rangle\langle x'p'|\hat{a}^+\cdots\hat{a}^+=$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,|x'p'\rangle\langle x'p'|\sum K\,a'\cdots a'a'^*\cdots a'^*=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,\mathbb{F}_3(x',p')|x'p'\rangle\langle x'p'|. \qquad (A.23)$$

Using eqs. (A.21) one can obtain the relations

$$\langle xp|\hat{\mathbb{F}}|xp\rangle=\langle xp|:\mathbb{F}_1(\hat{x},\hat{p}):|xp\rangle=\Lambda^{-1}\langle xp|\mathrm{sym}\,\mathbb{F}_1(\hat{x},\hat{p})|xp\rangle=\Lambda^{-2}\langle xp|:\mathbb{F}_1(\hat{x},\hat{p}):|xp\rangle$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,\mathbb{F}_1(x',p')\underbrace{\Lambda^{-2}\,|\langle xp|x'p'\rangle|^2}_{(2\pi\hbar)^n\delta(x-x')\delta(p-p')}=\mathbb{F}_1(x,p)\,, \qquad (A.24)$$

$$\langle xp|\hat{\mathbb{F}}|xp\rangle=\langle xp|\mathrm{sym}\,\mathbb{F}_2(\hat{x},\hat{p})|xp\rangle=\Lambda^{-1}\langle xp|:\mathbb{F}_2(\hat{x},\hat{p}):|xp\rangle=$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,\mathbb{F}_2(x',p')\Lambda^{-1}|\langle xp|x'p'\rangle|^2, \qquad (A.25)$$

$$\langle xp|\hat{\mathbb{F}}|xp\rangle=\langle xp|:\mathbb{F}_3(\hat{x},\hat{p}):|xp\rangle=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,\mathbb{F}_3(x',p')|\langle xp|x'p'\rangle|^2, \qquad (A.26)$$

$$\langle 00|\hat{\mathbb{F}}|00\rangle=\langle 0|:\mathbb{F}_1(\hat{x},\hat{p}):|0\rangle=\mathbb{F}_1(0,0), \qquad (|00\rangle=|0\rangle) \qquad (A.27)$$

$$\langle 00|\hat{\mathbb{F}}|00\rangle=\langle 0|\mathrm{sym}\,\mathbb{F}_2(\hat{x},\hat{p})|0\rangle=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,\mathbb{F}_2(x',p')\Lambda^{-1}|\langle 0|x'p'\rangle|^2, \qquad (A.28)$$

$$\langle 00|\hat{\mathbb{F}}|00\rangle=\langle 0|:\mathbb{F}_3(\hat{x},\hat{p}):|0\rangle=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,\mathbb{F}_3(x',p')|\langle 0|x'p'\rangle|^2, \qquad (A.29)$$

$$\mathrm{Tr}(\hat{\mathbb{F}}\hat{\mathbb{G}})=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\mathbb{F}_1(xp)\mathbb{G}_3(xp)-\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\mathbb{F}_2(xp)\mathbb{G}_2(xp)=$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\mathbb{F}_3(xp)\mathbb{G}_1(xp)\,. \qquad (A.30)$$

Appendix B. Calculation of partition functions for linear systems in terms of PSR's.

1) The free particle $\left(\hat{H}=\frac{\hat{p}^2}{2m}\right)$.

$$\Lambda^{-1}\langle xp|e^{-\beta\hat{H}}|xp\rangle=e^{-\beta H^{\ell}}\cdot 1=e^{-\beta\frac{\left(p-\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)^2}{2m}}\cdot 1=e^{-\beta\frac{p^2}{2m}} \qquad (PSR\text{-}2), (B.1)$$

$$Z=\frac{1}{(2\pi\hbar)^3}\int d^3x\,d^3p\,\Lambda^{-1}\langle xp|e^{-\beta\hat{H}}|xp\rangle=V\left(\frac{\sqrt{2\pi mkT}}{2\pi\hbar}\right)^3. \qquad (B.2)$$

All looks like in the classics.

2) The oscillator $\left(\hat{H}=\frac{\omega}{2}(A^{-1}\hat{p}^2+A\hat{x}^2)=\hbar\omega\left(\hat{a}^+\hat{a}+\frac{1}{2}\right)\right)$. Here also the above way is possible. However, it is simpler to use the well-known formula of N-ordering

$$e^{x\hat{a}^+\hat{a}}=:e^{(e^x-1)\hat{a}^+\hat{a}}: \qquad (B.3)$$

Then

$$e^{-\beta\hat{H}}=e^{-\beta\hbar\omega\left(\hat{a}^+\hat{a}+\frac{1}{2}\right)}=:e^{-(1-e^{-\beta\hbar\omega})\hat{a}^+\hat{a}}:e^{-\beta\frac{\hbar\omega}{2}}, \qquad (B.4)$$

$$\langle xp|e^{-\beta\hat{H}}|xp\rangle=e^{-(1-e^{-\beta\hbar\omega})\frac{H}{\hbar\omega}-\beta\frac{\hbar\omega}{2}}, \qquad (PSR\text{-}1) \qquad (B.5)$$

$$H=\Lambda^{-1}\langle xp|\hat{H}|xp\rangle=\frac{\omega}{2}(A^{-1}p^2+Ax^2)=\hbar\omega a^+a\,, \qquad (B.6)$$

$$\Lambda^{-1}\langle xp|e^{-\beta\hat{H}}|xp\rangle=\frac{1}{\mathrm{ch}\left(\beta\frac{\hbar\omega}{2}\right)}e^{-\mathrm{th}\left(\beta\frac{\hbar\omega}{2}\right)\frac{H}{\hbar\omega/2}}, \qquad (PSR\text{-}2) \qquad (B.7)$$

$$Z=\frac{1}{2\pi\hbar}\int dx\,dp\,\langle xp|e^{-\beta\hat{H}}|xp\rangle=\frac{1}{2\pi\hbar}\int dx\,dp\,\Lambda^{-1}\langle xp|e^{-\beta\hat{H}}|xp\rangle=\frac{1}{2\mathrm{sh}\left(\beta\frac{\hbar\omega}{2}\right)}. \qquad (B.8)$$

Both the ways (PSR-1 and PSR-2) give the same result as $Z=\sum_n\langle n|e^{-\beta\hat{H}}|n\rangle$.

Appendix C. Expression (34) results from dividing $e^{-\Delta t\mathcal{L}}\approx e^{-\Delta t\mathcal{L}_0}e^{-\Delta t\mathcal{L}_I}$; $\mathcal{L}_0$ being the free Liouvillian (identical with the classical one), $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I$. One can also divide $\mathcal{L}$ into an arbitrary classical part $\mathcal{L}_a$ and all the rest $\mathcal{L}_b$, $\mathcal{L}=\mathcal{L}_a+\mathcal{L}_b$ and split $e^{-\Delta t\mathcal{L}}\approx e^{-\Delta t\mathcal{L}_a}e^{-\Delta t\mathcal{L}_b}$. Then, instead of eq. (34), one gets

$$\rho_2(xpt,x_0p_0t_0)\approx\frac{\hbar^n}{(2\pi)^{(N-1)n}}\int dx_1dp_1\ldots dx_{N-1}dp_{N-1}\,da_1\ldots da_N$$

$$\delta(x-\mathbf{x}(x_{N-1},p_{N-1},t-t_{N-1}))\ldots\delta(x_2-\mathbf{x}(x_1,p_1,t_2-t_1))\,\delta(x_1-\mathbf{x}(x_0,p_0,t_1-t_0))$$

$$e^{-i\sum_{k=1}^{N}\{[p_k-\mathbf{p}(x_{k-1},p_{k-1},t_k-t_{k-1})]a_k+\hbar^{-1}(t_k-t_{k-1})[U(x_k+\frac{\hbar}{2}a_k)-U(x_k-\frac{\hbar}{2}a_k)]\}}, \qquad (C.1)$$

where $\mathbf{x}\,(x,p,t-t')$ and $\mathbf{p}\,(x,p,t-t')$ describe a classical trajectory generated by $\mathcal{L}_a$. In particular, eq. (C.1) may correspond to splitting $\mathcal{L}=\mathcal{L}_c+\mathcal{L}_q$, where $\mathcal{L}_c$ is the total classical part, and $\mathcal{L}_q$ is the pure quantum one (multiple of $\hbar$). Let $\mathcal{L}_a$ correspond to an oscillator

$$\mathbf{x}(x,p,t-t')=x\cos\omega(t-t')+\frac{p}{m\omega}\sin\omega(t-t'),\quad \mathbf{p}(x,p,t-t')=m\dot{\mathbf{x}}(x,p,t-t') \qquad (C.2)$$

Now we can also easily perform the integration over p_k using δ-functions. However, expanding an exponent of the exponential function up to Δt generates eq. (37) with $V(x)=\frac{1}{2}m\omega^2x^2+U(x)$ again. The same with any other $\mathcal{L}_a$. Equation (75) in quantum field theory is an analog of eq. (C.1). The functions $\varphi_{k-1}(\vec{x},t)$ and $\pi_{k-1}(\vec{x},t)$ may correspond to any classical motion governed by $\mathcal{L}_a$, and integration using the δ-functions leads to eq. (71), like above.

When φ_{k-1} satisfy the Klein-Gordon equation,

$$\varphi_{k-1}(\vec{x},t_k)=\int d^3y\,\left[\frac{\partial}{\partial y_0}D(x-y)\varphi_{k-1}(\vec{y})-D(x-y)\pi_{k-1}(\vec{y})\right]_{x_0=t_k,\,y_0=t_{k-1}}. \qquad (C.3)$$

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Представления фазового пространства и интегралы по путям

В рамках представлений фазового пространства в квантовой механике и в квантовой теории поля рассматриваются функции распределения Вигнера, которые являются решениями соответствующих обобщенных уравнений Лиувилля. Дается вывод представления этих функций распределения /решение уравнений Лиувилля/ в форме континуальных интегралов /подход аналогичен методу интегралов по путям Фейнмана для амплитуд/. Этим путем достигается полное равноправие относительно координат и импульсов. Найденные выражения сведены также к амплитудам в форме фейнмановских интегралов по путям. Рассмотрен формальный переход к классическому пределу /h = 0/. Дан обзор некоторых соотношений теории представлений фазового пространства.

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Phase Space Representations and Path Integrals

Within frameworks of phase space representations in quantum mechanics and quantum field theory the Wigner distribution functions are considered, which are solutions of corresponding generalized Liouville equations. A derivation is given for representation of these distributions / solving of the Liouville equation/ in the form of functional integrals /method is anglogous to the Feynman path integral one for amplitudes/. This way gives a full equality of coordinates and momenta. The expressions found are also reduced to amplitudes in the form of path integrals. A formal transition to the classical limit /h = 0/ is considered. Some relations of the theory of phase representations are reviewed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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