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# ALPHA-REPRESENTATION AND SPECTRAL PROPERTIES OF MULTIPARTON FUNCTIONS

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# I. MULTIPARTON FUNCTIONS

1.1. Parton Correlation Functions. To give a parton-model interpretation of power corrections (higher twists) in deep inelastic leptoproduction processes, one should introduce $k$-particle parton correlation functions $F^{(k)}(\{x\})$ /1-8/ which are (for $k \geq 3$) generalizations of the usual parton distribution functions related to the lowest twist (2-particle) composite operators. By definition, the multiple moments of the parton correlation functions are proportional to matrix elements of composite operators containing (for $k \geq 3$) more than 2 fundamental fields

$$\int_{-1}^{1} dx_1 \ldots \int_{-1}^{1} dx_k\, x_1^{N_1} \ldots x_k^{N_k}\, \mathcal{F}(x_1,\ldots,x_k) = \langle P|[(i\frac{n\partial}{nP})^{N_1}\phi_1]\ldots[(i\frac{n\partial}{nP})^{N_k}\phi_k]|P\rangle , \qquad (1)$$

where $n^\mu$ is an arbitrary lightlike vector (with $(nP)\neq 0$) introduced to pick out the symmetric-traceless part of the composite operator and $|P\rangle$ is a hadronic state with momentum P. The relevant fields are denoted schematically by $\phi_i$.

Depending on sign of the $x_j$-parameter one should attribute the corresponding parton either to the initial $(x_j > 0)$ or to the final $(x_j < 0)$ states. As a result, the function $\mathcal{F}^{(k)}(x_1,\ldots,x_k)$ is decomposed into a sum of the functions $\mathcal{F}^{(\ell,k-\ell)}(x_{i_1},\ldots,x_{i_\ell}; x_{i_{\ell+1}},\ldots,x_{i_k})$ describing a set of $\ell$ partons in the initial state and $k-\ell$ ones in the final state.

Such a parton interpretation is self-consistent only if

$$\sum_{i=1}^{k} x_i = 0 \qquad (2)$$

(energy-momentum conservation) and, moreover,

$$|\sum_{m=1}^{\ell} x_{i_m}| \leq 1 \qquad (3)$$

for any set $(i_1,\ldots,i_\ell)$. Eq. (3) means that the total longitudinal



momentum carried by each set of partons in the infinite momentum frame should not exceed that of the hadron. The fulfillment of eq. (3) justifies, in particular, the limits of the $x_j$-integrations in eq. (1).

1.2. Multiparton Wave Functions. Studying higher twist effects in hard exclusive processes one should intrdoce multiparton wave functions $\phi(x_1,\ldots,x_k)$ absorbing information about the long-distance dynamics. These functions are related to the matrix elements of the corresponding local operators

$$\int_0^1 dx_1 \ldots \int_0^1 dx_k\, \phi(x_1,\ldots,x_k)\, x_1^{N_1}\ldots x_k^{N_k} = \langle 0|[(i\frac{n\partial}{nP})^{N_1}\phi_1]\ldots[(i\frac{n\partial}{nP})^{N_k}\phi_k]|P\rangle . \qquad (4)$$

Physically, $\phi(x_1,\ldots,x_k)$ should be interpreted as a probability amplitude to find (in the infinite momentum frame) the initial hadron in the state where the partons $\phi_1,\ldots,\phi_k$ carry the fractions $x_1,\ldots,x_k$ of its longitudinal momentum. Such an interpretation makes sense only if

$$0 < x_j < 1 \qquad (5)$$

for all $x_j$ and, moreover,

$$\sum_{j=1}^{k} x_j = 1 . \qquad (6)$$

1.3. Spectral Properties of the Multiparton Functions and Perturbative Analyticity. The constraints (2), (6) related to energy-momentum conservation can be trivially proved in momentum representation, where one can write

$$\mathcal{F}^{(k)}(x_1,\ldots,x_k) = \int d^4\ell_1 \ldots \int d^4\ell_k\, \delta^4(\sum_{j=1}^{k}\ell_j) \times [\prod_{i=1}^{k}\delta(x_i - \frac{(n\ell_i)}{(nP)})]\, T^{(k)}(\ell_1,\ldots,\ell_k; P, -P) \qquad (7)$$

for the correlation function (see fig.1), and

$$\phi(x_1,\ldots,x_k) = \int d^4\ell_1 \ldots \int d^4\ell_k\, \delta^4(\sum_{j=1}^{k}\ell_j - P) \times$$

$$\times [\prod_{i=1}^{k} \delta(x_i - \frac{(n\ell_i)}{(nP)})] T^{(1)}(\ell_1, \ldots, \ell_h; P) \qquad (8)$$

for the wave function.

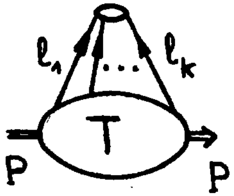

Fig.1

However, to prove the spectral constraints (3), (5) one should know the analyticity properties of the amplitudes $T^{(1)}(\{\ell\}; P)$, $T^{(2)}(\{\ell\}; P, -P)$. In an idealized situation one should extract necessary information concerning the analytic properties of the Green functions from the most general principles of the local quantum field theory, such as unitarity and causality [4]. In the real case, however, the promotion of such a program for amplitudes depending on large (and not fixed) number of momentum variables faces serious problems. In fact, the only way out is the analysis of the Green functions within the framework of perturbation theory (PT). In such an approach it is implied, of course, that if the constraint (3) (or (5)) is valid for any Feynman diagram contributing to $\mathcal{F}(\{x\})$ (or $\phi(\{x\})$, then the same constraint holds for the total function $\mathcal{F}(\{x\})$ ($\phi(\{x\})$).

In ref. [3] information about the perturbative analyticity of some N-point Green function $G_N(\{k\})$ (all momenta $k$ are treated as incoming, $\Sigma k_i = 0$ ) was extracted with the help of the Nakanishi representation [5]

$$G_N(\{k\}) = \int_0^\infty d\gamma [\int_0^1 \prod_j d\beta_j] \delta(1 - \sum_j \beta_j) \frac{\rho(\{\beta\}, \gamma)}{\sum_\ell \beta_\ell s_\ell - \gamma + i\epsilon} \qquad (9)$$

(proved in ref. [5] for an arbitrary Feynman diagram contributing into $G_N(\{k\})$). The parameters $\beta_\ell$, $s_\ell$ in eq. (9) correspond to all possible separations of the momenta $\{k_1, \ldots, k_N\}$ into two components $\{k_{i_1}, \ldots, k_{i_r}\}$, $\{k_{i_{r+1}}, \ldots, k_N\}$. By definition

$$s_r = (\sum_{\ell=1}^{r} k_{i_\ell})^2 . \qquad (10)$$

Existence of the representation (9) means that $G_N(\{k\})$ is an analytic function of the parameters $s_r$ (treated as independent variables) with singularities in the corresponding lower half-planes of the complex variables $s_r$.

Substituting the representation (9) for $T^{(2)}(\{\ell\}; P, -P)$ into eq. (7) one can calculate the $\ell_i$-integrals in the standard way by the Wick rotation. To do this it is convenient to represent the denominator of the integrand of eq. (9) in the following form

$$D(\ell) = \Sigma \beta_i s_i - \gamma + i\epsilon = \sum_{r,j} \ell_r A_{rj} \ell_j - 2 \sum_r B_r (P\ell_r) + C + i\epsilon \qquad (11)$$

and get rid of linear in $\ell_i$ terms by the shift $\ell_i = \hat{\ell}_i + r_i P$, where the vector $r_i(\alpha)$ is the solution of the linear equation

$$\sum_{k=1} A_{ik} r_k = B_i . \qquad (12)$$

As a result

$$F(x_1, \ldots, x_k) = \int_0^1 \prod_i d\beta_i \delta(1 - \Sigma\beta_j) [\prod_{\ell=1}^{k} \delta(x_\ell - r_\ell(\beta))] H(\beta), \qquad (13)$$

where H is some function of the $\beta$-parameters.

In each particular case one can solve eq. (12) and obtain $r_i(\beta)$ as a function of the $\beta$-parameters. In the simplest case, e.g., one has

$$D(k) = \beta_1 k^2 + \beta_2 (P-k)^2 + \beta_3 (P+k)^2 - \gamma + i\epsilon = k^2 - 2(\beta_2 - \beta_3)(Pk) - \tilde{\gamma} + i\epsilon \qquad (14)$$

whence it follows that $r_\alpha = \beta_2 - \beta_3$. Using now the fact that $0 \le \beta_k \le 1$ and $\beta_1 + \beta_2 + \beta_3 \le 1$ one gets the desired constraint $-1 \le x \le 1$.

However, in more complicated cases ($N \ge 3$) the use of the Nakanishi representation allows one to get the constraints much weaker than it is required in eqs. (3), (5). In particular, for the 3-parton correlation function $F(x_1, x_2, -x_1 - x_2)$ the authors of ref. [3] succeeded to derive only the inequalities $|x_1|$, $|x_2|$, $|x_1 + x_2| \le 2$. As it was emphasized in ref. [3], the failure of the approach is due to the fact that for a particular Feynman diagram there exist some correlations between the values of the $\beta_k$-parameters that are not taken into account in eq. (11). The existence of such correlations can be checked, e.g., by writing the contribution of the diagram considered in the Feynman parametric representation (in this case $\beta_k$ are functions of the Feynman parameters).

In the next section we shall analyze the spectral properties of the multiparton function directly within the framework of the exponential analog of the Feynman representation (alpha-representation, [4,6,7] ) for the corresponding PT diagrams. The alpha-representation is well known to be a very powerful tool to analyze the most general properties of the Feynman diagrams. It is especially effective to analyze scalar theories. In the latter case the complications are absent due to numerators of spinor propagators, derivatives in some vertices, etc. Note, that in the Nakanishi representation (9) the specifics of the field theory model considered is reflected by the $\rho(\gamma, \beta)$ -factor, which as we have seen, has no influence on the spectral properties of $F(\{x\})$ and $\phi(\{x\})$. This means that to analyze the spectral properties of the multiparton functions $\mathcal{F}(\{x\})$, $\phi(\{x\})$ it is sufficient to consider a scalar field theory*. This observation essentially simplifies the $\alpha$-representation analysis.

## II. ANALYSIS OF SPECTRAL PROPERTIES IN THE $\alpha$-REPRESENTATION

2.1. Alpha-Representation. To prove the validity of eqs. (3), (5) we shall incorporate the well-known parametric representation of Feynman amplitudes based on the following simple formula for the propagator [6]

$$(m^2 - p^2 - i\epsilon)^{-1} = i \int_0^\infty d\alpha \exp\{i\alpha p^2 - m^2 + i\epsilon)\} . \qquad (15)$$

After taking the Gaussian integrals over the virtual momenta one can write the contribution of any Feynman diagram as an integral over the $\alpha_\sigma$-parameters of all lines $\sigma$ belonging to the diagram considered. In particular, the matrix element (1) has the following $\alpha$-representation

$$\begin{aligned} &< P | [ (\frac{n\partial}{nP})^{N_1} \phi_1 ] \ldots [ (\frac{n\partial}{nP})^{N_k} \phi_k ] | P > = \\ &= \sum_{\text{diagr.}} \int_0^\infty \prod_\sigma d\alpha_\sigma [ \prod_{j=1}^{k} ( \frac{B_j^+(\alpha) - B_j^-(\alpha)}{D(\alpha)} ) ] \Phi(\alpha) , \end{aligned} \qquad (16)$$

* This observation is eventually due to the fact that any Feynman integral corresponding to some QCD diagram can be, in principle,represented as a sum of scalar Feynman integrals.To arrive at such a decomposition one should (after calculating the relevant trace) expand the scalar products of various momenta present in the numerator over the denominator structures neglecting on so doing the quantities that are odd in the integration momentum considered.This procedure, by the way, is the starting point of all most effective modern approaches to calculate multiloop Feynman diagrams in QCD (see, e.g., refs. [8, 9] ).



where $\Phi(\alpha)$ is some function of the $\alpha$-parameters, the explicit form of which is not essential for our purposes,

$$B_j^+(\alpha) = \sum_{\ell \neq j} b_{j\ell}(\alpha), \qquad (17)$$

$$B_j^-(\alpha) = \sum_{\ell \neq j} b_{\ell j}(\alpha), \qquad (18)$$

and $b_{j\ell}(\alpha) \geq 0$, $D(\alpha) \geq 0$ are also functions of the $\alpha$-parameters (to be specified below) satisfying the relation

$$D(\alpha) = \sum_{j=1}^{k} \sum_{\ell \neq j} b_{j\ell}(\alpha) + C(\alpha), \qquad (19)$$

where $C(\alpha) \geq 0$. Using eqs. (1) and (16) one can derive the $\alpha$-representation for $\mathcal{F}(x_1, \ldots, x_k)$:

$$F(x_1, \ldots, x_k) = \sum_{\text{diagr.}} \int_0^\infty \prod_\sigma d\alpha_\sigma [ \prod_{i=1}^{k} \delta(x_i - \frac{B_i^+(\alpha) - B_i^-(\alpha)}{D(\alpha)}) ] \Phi(\alpha). \qquad (20)$$

Now, incorporating eqs. (17)-(20) one can easily obtain the spectral inequality (3) and the energy-momentum conservation relation (2). Note, that in the $\alpha$-representation the latter is nontrivial, in contrast to the momentum representation.

Analogously, one can derive the $\alpha$-representation for the multiparton wave function:

$$\phi(x_1, \ldots, x_k) = \sum_{\text{diagr.}} \int_0^\infty \prod_\sigma d\alpha_\sigma \delta(x_1 - \frac{B_1(\alpha)}{D(\alpha)}) \ldots \delta(x_k - \frac{B_k(\alpha)}{D(\alpha)}) \Psi(\alpha), \qquad (21)$$

where $B_j(\alpha) \geq 0$, $D(\alpha) \geq 0$ are functions obeying the equality

$$D(\alpha) = \sum_{j=1}^{k} B_j(\alpha) \qquad (22)$$

from which one can easily derive eqs. (5), (6).

2.2. Alpha-Representation Functions and Topology of the Diagram. The functions $B_j^{\pm}(\alpha)$, $B_j(\alpha)$ , $D(\alpha)$ can be connected in a simple way with the topological properties of the corresponding Feynman graph. Recall that a $k$-tree of the graph $G$ is a subgraph of $G$ which (a) contains all its vertices; (b) has $k$ connected components, and (c) each component has a tree structure (i.e., has no loops). Any $k$-tree $G_k$ is determined in a unique way by the set of lines $\{\sigma\}$ which one should remove

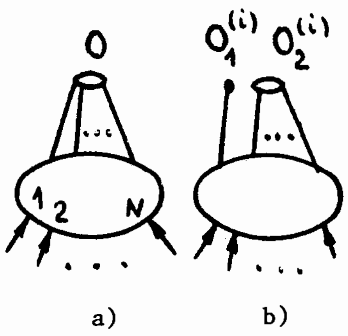


Fig.2

from the initial graph G to get $G_k$. The product of $\alpha_\sigma$-parameters related to these lines $\sigma$ will be referred to as an $(\alpha)-k-$tree. The function $D(\alpha)$ is the sum of all $(\alpha)-1-$trees (or, simply, $(\alpha)$-trees) of the graph. Next, denote by $B(i_1,\ldots,i_m|j_1,\ldots,j_n)$ the sum of all its $(\alpha)$-2-trees possessing the property that the vertices $i_1,\ldots,i_m$ belong to one component; $j_1,\ldots,j_n$ – to the other, while the vertices not indicated explicitly may belong to any component.

In the $\alpha$-representation each derivative $(n\partial)\phi_i$ related to the O-vertex (fig.2a) results in a factor $R_i(\alpha)$. To construct this factor, it is convenient to consider an auxiliary diagram (fig.2b) in which the O-vertex is splitted into two ones: $O \to O_1^{(i)} O_2^{(i)}$ so as to separate the line corresponding to the $\phi_i$-field. Then [10, 11]

$$R_i(\alpha) = \frac{1}{D(\alpha)} \sum_{j=1}^{N} B(O_1^{(i)}, j\,|\,O_2^{(i)})(np_j), \tag{23}$$

where j is the vertex into which the external momentum $p_j$ enters. A trivial but crucial observation is that the $(\alpha)$-2-trees of the graph 2b present in $B(O_1^{(i)}, j\,|\,O_2^{(i)})$ may be treated also as the $(\alpha)$-1-trees of the graph 2a.

We are interested only in the simplest cases $N=1$ and $N=2$.

a) $N=1,\ p_1=P$ (fig.3a). In this case (23) reduces to

$$\frac{1}{(nP)} R_i(\alpha) = \frac{1}{D(\alpha)} B(O_1^{(i)}, 1\,|\,O_2^{(i)}) \equiv \frac{B_i(\alpha)}{D(\alpha)}. \tag{24}$$

Consider an arbitrary tree of the graph 3a. According to the definition of a tree it should contain a continuous chain of lines joining the vertices 0 and 1. Furthermore such a chain is unique (otherwise we would have a loop graph rather than a tree). Now, if the line of this chain adjacent to the O-vertex corresponds to the $\phi_i$-field, then the relevant $(\alpha)$-tree is a part of $B_i(\alpha)$. Hence, each $(\alpha)$-tree present in $D(\alpha)$ has its unique counterpart in some of $B_i(\alpha)$'s. The reversed statement is also true: joining (by $O_1^{(i)} O_2^{(i)} \to O$ ) the components of a 2-tree contributing to some $B_i(\alpha)$ one obtains a tree contributing to $D(\alpha)$. This gives eq. (22).

b) $N=2,\ p_1=P,\ p_2=-P$ (fig.4a). In such a configuration

$$\frac{1}{(nP)} R_i(\alpha) = \frac{1}{D(\alpha)} [B(O_1^{(i)}, 1\,|\,O_2^{(i)}) - B(O_1^{(i)}, 2\,|\,O_2^{(i)})]. \tag{25}$$



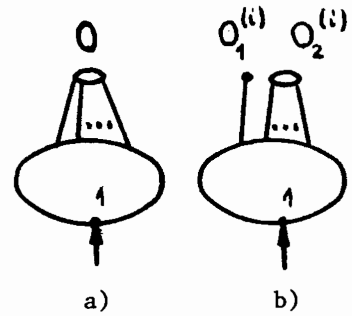


Fig.3

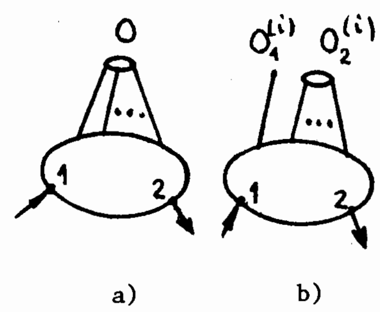


Fig.4

Note that according to the definition of $B(\ldots|\ldots)$ one can write

$$B(O_1^{(i)}, 1\,|\,O_2^{(i)}) = B(O_1^{(i)}, 1\,|\,O_2^{(i)}, 2) + B(O_1^{(i)}, 1,2\,|\,O_2^{(i)}) \equiv \equiv B_i^+(\alpha) + C_i(\alpha). \tag{26}$$

$$B(O_1^{(i)}, 2\,|\,O_2^{(i)}) = B(O_1^{(i)}, 2\,|\,O_2^{(i)}, 1) + B(O_1^{(i)}, 1,2\,|\,O_2^{(i)}) \equiv \equiv B_i^-(\alpha) + C_i(\alpha). \tag{27}$$

Hence the 2-trees for which the vertices 1,2 belong to the same component give the zero total contribution into $R_i(\alpha)$. However, they give a non-zero contribution

$$C(\alpha) = \Sigma\, C_i(\alpha)$$

into $D(\alpha)$.

To proceed further, consider a particular 2-tree of the graph 4b contributing into $B_i^+(\alpha)$. Due to absence of loops there exists only a single line (related, say to the $\phi_\ell$-field) going out of the $O_2^{(i)}$-vertex that starts a continuous chain joining $O_2^{(i)}$ with the vertex 2. Define now $b_{i\ell}$ to be the sum of all such $(\alpha)$-2-trees of the graph 3b. Furthermore, it is easy to realize that $b_{i\ell}$ may be treated also as the sum of all $(\alpha)$-trees of the graph 4a for which the continuous chain joining the vertices 1 and 2 looks like $1 \to \phi_i \to 0 \to \phi_\ell \to 2$. The trivial consequence of this observation are eqs. (18), (19).

## III. CONCLUSION AND ACKNOWLEDGEMENTS

Thus, using the $\alpha$-representation we established that the functions $F(x_1,\ldots,x_k)$ and $\phi(x_1,\ldots,x_k)$ defined by eqs.(1), (4) do possess the spectral properties necessary for their parton interpretation.

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Альфа-представление и спектральные свойства мультипартонных функций

С помощью параметрического представления для фейнмановских интегралов показано, что функции $\mathcal{F}(x_1,...,x_k)$, $\phi(x_1,...,x_k)$, обобщенные моменты которых пропорциональны редуцированным матричным элементам $k$-частичных составных операторов, обладают необходимыми для их партонной интерпретации спектральными свойствами.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Alpha-Representation and Spectral Properties of Multiparton Functions

Using a parametric representation for Feynman integrals it is demonstrated that functions $\mathcal{F}(x_1,...,x_k)$, $\phi(x_1,...,x_k)$, the multiple moments of which are proportional to reduced matrix elements of $k$-body composite operators, have the spectral properties necessary for their parton interpretation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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