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ON THE DECAY OF A BOUND STATE OF A $\mu^+\mu^-$ PAIR INTO AN e^+e^- DALITZ PAIR AND A γ-QUANTUM

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1. INTRODUCTION

In the literature devoted to spectroscopy of new (η_c, J/ψ, Υ) and old (π, ρ, ϕ) mesons a good deal of attention is paid to the decays of quark-antiquark systems into the real and virtual photons /1–3/. In a number of papers /4–6/ these processes are calculated with the use of wave functions of bound states of a quark-antiquark system. As has been noted in /6/, the wave functions of light mesons are not yet well studied for the following two reasons: 1) this region requires the use of the relativistic methods less elaborated then methods of the nonrelativistic quantum mechanics, and 2) the levels of radial and orbital excited states of light mesons are known experimentally much worse than those of excited states of ψ-particles. Therefore, in the case of light mesons it is difficult to study the long-distance behaviour of the wave function determined by solving an equation with a phenomenological confinement potential which describes the mass spectrum of experimentally known meson excitations.

For this reason it would be interesting to study, whithin the same method used for the description of composite mesons, main properties of the behaviour of the decay cross sections of composite electrodynamic systems, as in this case we would be free of the problem of how to take into account the influence of the confinement potentials at long distances.

The aim of this paper is to study the dependence of the cross section of the decay $(\mu^+\mu^-) \to \gamma e^+ e^-$ of a bound state of the muon-antimuon pair $\mu^+\mu^-$ into free electron e^-, positron e^+, and a γ-quantum (Fig. 1) on the invariant mass of a lepton pair squared $q_2^2 = (p_+ + p_-)^2$. To this end we shall use the same method of the covariant simultaneous description of composite systems within a quasipotential approach in quantum field theory proposed by Logunov and Tavkhelidze /7/ which has been employed earlier in /4–6/ to describe an analogous dependence on $q_2^2 = (p_+ + p_-)^2$ of the cross section of the π^0-meson decay as a bound state of a quark q and antiquark $\bar{q}$ into a Dalitz pair $\pi^0 \to \gamma e^+ e^-$. Note that the problem of the description of the $\mu^+\mu^- \to \gamma e^+ e^-$ decay has been studied in /2/ where an essential use has also been made of the single-time Bethe-Salpeter wave function.

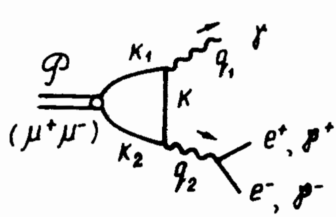

Fig. 1

A formula obtained in [2] for the dependence of the $\mu^+\mu^- \to \gamma e^+e^-$ decay width on $q_2^2 = (p_+ + p_-)^2$ was derived only for $4m^2 \simeq M^2$ (m is the muon mass, M is the mass of a bound state $\mu^+\mu^-$), i.e., in the limit of a vanishing binding energy ($\epsilon_B \to 0$). As is noted by the authors [2], the formula they have found for a fictitious width $\Gamma(q_2^2)$ of the decay of $\mu^+\mu^-$ into a virtual photon with 4-momentum q_2 and a real γ-quantum*

$$\Gamma(q_2^2) = \frac{32\alpha^2}{m^2} \, \frac{|\phi_1'(0)|^2}{4m^2 - q_2^2}, \tag{1}$$

(α - is the fine structure constant, $\phi_1(0)$ is a wave function of the P-state at the origin of coordinates) is valid in the range of maximal values of the invariant mass squared of the e^+e^--pair $q_2^2 \equiv (p_+ + p_-)^2 \to 4m^2 \simeq M^2$ and possesses there a pole singularity producing, in turn, a logarithmic singularity for the decay width $\Gamma((\mu^+\mu^-)_1{}^{++} \to \gamma e^+e^-)$. An analogous result was then obtained by the same method by the authors of [2] for the decay of P-wave 1^{+-}-states of the quarkonium into three gluons and got a wide popularity (see, e.g., [8]).

In the course of description of the decay of a muon-antimuon bound state $(\mu^+\mu^-) \to \gamma e^+e^-$ we shall try to go beyond the scope of the static limit. We shall derive a formula for the decay form factor valid throughout the whole kinematically allowed range of q_2^2: $4m_e^2 \le q_2^2 < M^2$ (m_e is the electron mass) and calculate the probability of decay $\mu^+\mu^- \to 2\gamma$.

2. CALCULATION OF THE $(\mu^+\mu^- \to \gamma e^+e^-)$ DECAY AMPLITUDE WITH THE USE OF THE BETHE-SALPETER AND QUASIPOTENTIAL WAVE FUNCTIONS

According to the formalism of quantum field theory [9], the amplitude of the decay shown in Fig.1 is of the following form

$$T^{\mu\nu}_{\mu^+\mu^- \to \gamma e^+e^-} = [(2\pi)^4 i]^2 \, 4\pi\alpha \int d^4k_1 \, \Gamma^{(\mu^+\mu^-)}_{\alpha\beta}(k_1, q_1 + q_2 - k_1 | \mathcal{P}) \times \tag{2}$$

$$\times S_{\beta\rho}(k_1, m_\mu) \, \gamma^\mu_{\rho\kappa} \, S_{\kappa\phi}(k_1 - q_1; m_\mu) \, \gamma^\nu_{\phi\xi}(q_1 + q_2 - k_1; m_\mu),$$

where $\Gamma^{(\mu^+\mu^-)}_{\alpha\beta}(k_1, q_1 + q_2 - k_1 | \mathcal{P})$ is the vertex function for the transition of a composite particle of a bound state $(\mu^+\mu^-)$ with 4-momentum $\mathcal{P}$ into a muon-antimuon pair, and

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*In the notation of [2] $q_2^2 = a$.

$$S_{\beta\rho}(k_1, m_\mu) = \frac{1}{i(2\pi)^4} \, \frac{(\hat{k}_1 + m_\mu)_{\beta\rho}}{m_\mu^2 - k_1^2 - i\epsilon} \tag{3}$$

is a propagator of a muon with momentum k_1 and mass m_μ.

Since we are interested in the decay of a singlet state $(\mu^+\mu^-)$ which is a pseudoscalar and has zero total spin, we shall represent the vertex function $\Gamma^{(\mu^+\mu^-)}_{\alpha\beta}(k_1, k_2 | \mathcal{P})$ in the form [10,11]:

$$\Gamma^{(\mu^+\mu^-)}_{\alpha\beta}(k_1, k_2 | \mathcal{P}) = \gamma_{5\alpha\beta} \, \Gamma(k_1, k_2 | \mathcal{P}), \tag{4}$$

where the function $\Gamma(k_1, k_2 | \mathcal{P})$ is a scalar.

The wave function $\Psi(k_1, k_2 | \mathcal{P})$ is expressed in terms of the vertex function $\Gamma^{(\mu^+\mu^-)}(k_1, k_2 | \mathcal{P})$ as follows

$$\Psi_{\alpha\beta}(k_1, k_2 | \mathcal{P}) = \frac{\Gamma^{(\mu^+\mu^-)}_{\alpha\beta}(k_1, k_2 | \mathcal{P})}{(m_\mu^2 - k_1^2 - i\epsilon)(m_\mu^2 - k_2^2 - i\epsilon)} \tag{5}$$

and the amplitude (2) in terms of the wave function (5) is given by

$$T^{\mu\nu}_{\mu^+\mu^-} = [(2\pi)^4 i]^{-1} \, 4\pi\alpha \int d^4k_1 \, \mathrm{Sp}\{\Psi_{\alpha\beta}(k_1, k_2 | \mathcal{P}) \, M^{\mu\nu}_{\beta\alpha}(k_1, k_2 | q_1, q_2)\}, \tag{6}$$

where

$$M^{\mu\nu}_{\beta\alpha}(k_1, k_2 | q_1, q_2) = (\hat{k}_1 + m_\mu)_{\beta\rho} \, \gamma^\mu_{\rho\kappa} \, S_{\kappa\phi}(k_1 - q_1; m_\mu) \times \tag{7}$$

$$\times \gamma^\nu_{\phi\xi}(\hat{k}_2 - m_\mu)_{\xi\alpha}.$$

Let us pass now to the wave function given in the space of polarization indices σ_1 and σ_2, spin projections of a fermion and an antifermion with 4-momenta k_1 and k_2, respectively, onto the z-axis:

$$\Psi^{\sigma_1\sigma_2}(k_1, k_2 | \mathcal{P}) = \bar{u}^{\sigma_1}_\alpha(k_1) \, \Psi_{\alpha\beta}(k_1, k_2 | \mathcal{P}) \, v^{\sigma_2}_\beta(k_2), \tag{8}$$

where $u(k_1)$ and $v(k_2)$ are fermion and antifermion bispinors normalized by the invariant conditions: $\bar{u}(k_1)u(k_1) = 2m_\mu$, $\bar{v}(k_2) \cdot v(k_2) = -2m_\mu$. According to (4) and (5), the function (8) may be represented in the form

$$\Psi^{\sigma_1\sigma_2}(k_1, k_2 | \mathcal{P}) = \bar{u}^{\sigma_1}(k_1) \, \gamma_5 \Psi(k_1, k_2 | \mathcal{P}) \, v^{\sigma_2}(k_2), \tag{9}$$

where the scalar part of the wave function $\Psi(k_1, k_2|\mathcal{P})$ is related with the scalar part of the vertex function $\Gamma(k_1, k_2|\mathcal{P})$ by

$$\Psi(k_1, k_2|\mathcal{P}) = (m_\mu^2 - k_1^2 - i\epsilon)^{-1} (m_\mu^2 - k_2^2 - i\epsilon)^{-1} \Gamma(k_1, k_2|\mathcal{P}). \tag{10}$$

Taking into account the formula (7) we rewrite (6) in the form

$$T_{\mu^+\mu^- \to \gamma e^+ e^-} = \sum_{\sigma_1 \sigma_2} \frac{4\pi\alpha}{i} \int \frac{d^4 k_1}{(2\pi)^4} \Psi^{\sigma_1\sigma_2}(k_1, k_2|\mathcal{P})\, T^{\mu\nu}_{\sigma_1 \sigma_2}(k_1, k_2|q_1, q_2), \tag{11}$$

where

$$T^{\mu\nu}_{\sigma_1\sigma_2} = \bar{u}_{\sigma_1}(k_1)\, \gamma^\mu S(k_1 - q_1; m_\mu)\, \gamma^\nu v_{\sigma_2}(k_2) \tag{12}$$

or, taking into account (9),

$$T^{\mu\nu}_{\mu^+\mu^- \to \gamma e^+ e^-} = \frac{4\pi\alpha}{i} \int \frac{d^4 k_1}{(2\pi)^4} \Psi(k_1, k_2|\mathcal{P}) \frac{\mathrm{Sp}\{A^{\mu\nu}(k_1, k_2|q_1, q_2)\}}{(k_1 - q_1)^2 - m_\mu^2}, \tag{13}$$

where

$$\mathrm{Sp}\{A^{\mu\nu}(k_1, k_2|q_1, q_2)\} = \mathrm{Sp}\{\gamma_5(\hat{k}_1 + m_\mu)\, \gamma^\mu (\hat{k}_1 - \hat{q}_1 + m_\mu)\, \gamma^\nu (\hat{k}_2 - m_\mu)\} = \tag{14}$$

$$= 4 m_\mu\, i \epsilon^{\alpha\mu\beta\nu} q_{1\beta} (k_1 + k_2)_\alpha .$$

If we now represent the $(\mu^+\mu^-) \to \gamma e^+ e^-$ decay amplitude in a standard form

$$T^{\mu\nu}_{\mu^+\mu^- \to \gamma e^+ e^-} = F_{\mu^+\mu^- \to \gamma\gamma^*}\, \epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta}, \tag{15}$$

where $F_{\mu^+\mu^- \to \gamma\gamma^*}$ is the form factor of the decay of the bound state $(\mu^+\mu^-)$ into a virtual (γ^*) and a real (γ) photon and in (15) allow for the conservation law

$$\mathcal{P} = k_1 + k_2 = q_1 + q_2 \tag{16}$$

specific for the Feynman-diagram technique, we get for $F_{\mu^+\mu^- \to \gamma e^+ e^-}$ the following expression

$$F_{\mu^+\mu^- \to \gamma e^+ e^-} = 16\pi m_\mu \alpha \int \frac{d^4 k_1}{(2\pi)^4} \Psi(k_1, k_2|\mathcal{P}) \frac{1}{(k_1 - q_1)^2 - m_\mu^2} \tag{17}$$

The formula (17) contains the scalar part of the wave function (5) which is a solution of the Bethe-Salpeter equation with

a kernel constructed out of the Feynman matrix elements of electromagnetic interaction of the system $(\mu^+\mu^-)$.

To study and solve the Bethe-Salpeter equation is a difficult problem. More convenient is a three-dimensional equation for the Bethe-Salpeter wave function depending on one time parameter, which has first been found by Logunov and Tavkhelidze [7].

The single-time wave function (in what follows denoted by ~) was introduced in [13]. In refs. [13,14] it was defined in terms of the Bethe-Salpeter wave function

$$\Psi(x_1, x_2) < 0|\, T\,(\bar{\Psi}(x_1)\, \Psi(x_2))\, |M, \vec{K}> \tag{18}$$

in the following way

$$\tilde{\Psi}_{MK}(p_1, p_2) = \int d^4 x_1 d^4 x_2\, e^{ip_1 x_1 + ip_2 x_2} \delta(\lambda_{\mathcal{P}} x_1 - \tau)\, \delta(\lambda_{\mathcal{P}} x_2 - \tau)\, \Psi_{MK}(x_1, x_2), \tag{19}$$

where $\lambda_{\mathcal{P}} = \mathcal{P}/(\mathcal{P}^2)^{1/2}$, $\mathcal{P} = p_1 + p_2$, $\mathcal{P}^2 > 0$, and the invariant proper time τ defines the space-like plane in the Minkowski space-time of proper times of particles $\tau_1 = \lambda_{\mathcal{P}} x_1$, $\tau_2 = \lambda_{\mathcal{P}} x_2$ and a proper time of the system as a whole $\tau_c = \lambda_{\mathcal{P}} X$. In formula (18) $|M, \vec{K}>$ is a state vector for a bound state as a single particle with mass M and momentum $\vec{K}$. Due to translational invariance $\Psi_{MK}(x_1, x_2) = e^{iKX}\, \Psi(\frac{X}{2}, -\frac{X}{2})$, $(X = x_1 + x_2)$ that allows us to single out of $\tilde{\Psi}_{MK}(p_1, p_2)$ the wave function of relative motion of two particles

$$\tilde{\Psi}_{MK}(p_1, p_2) = (2\pi)^3 \frac{K^\circ}{\sqrt{K^2}}\, \delta^{(3)}\Big[\frac{\sqrt{K^2}}{\sqrt{\mathcal{P}^2}} \vec{\mathcal{P}} - \vec{K}\Big] \exp\big[-i\tau(\sqrt{K^2} - \sqrt{\mathcal{P}^2})\big]\, \tilde{\Psi}_{MK}(q), \tag{20}$$

$$\mathcal{P} = p_1 + p_2, \quad q = \frac{p_1 - p_2}{2}, \quad K^2 = K_0^2 - \vec{K}^2, \quad K^\circ = \sqrt{M^2 + \vec{K}^2},$$

and

$$\tilde{\Psi}_{MK}(q) = \int e^{iqx}\, \delta(\lambda_{\mathcal{P}} x)\, \Psi(x/2; -x/2)\, d^4 x \tag{21}$$

and $x = x_1 - x_2$. By using the transformation law for fermion fields one can show that because of the δ-function in (21) the wave function of relative motion is a function only of a three-dimensional covariant momentum of relative motion in the c.m.s. [15]

$$\overset{\circ}{\vec{p}} \equiv \frac{1}{2}\,(\overrightarrow{\Lambda_K^{-1} q}) = (\overrightarrow{\Lambda_K^{-1} p_1}), \quad \Lambda_K^{-1} K = (M, \vec{0}), \tag{22}$$

i.e.[14]

$$\tilde{\Psi}_{MK}(q) = S_1(\Lambda_K)\, S_2(\Lambda_K)\, \tilde{\Psi}_{M,\vec{K}=0}(\mathring{\vec{p}}), \tag{23}$$

where

$$\tilde{\Psi}_{M,\vec{K}=0}(\mathring{\vec{p}}) = \int d\mathring{\vec{x}}\, e^{-i\mathring{\vec{p}}\mathring{\vec{x}}}\, \Psi_{M,\vec{K}=0}(\mathring{\vec{x}}), \tag{24}$$

$S_{1,2}(\Lambda_K)$ are transformation matrices for bispinors in the c.m.s., $\mathring{\vec{x}} = \mathring{\vec{x}}_1 - \mathring{\vec{x}}_2$, $\mathring{\vec{p}}_1 = -\mathring{\vec{p}}_2 = \mathring{\vec{p}}$ and, respectively, $\mathring{p}_{10} = \mathring{p}_{20} \equiv \mathring{p}_0$.

In the momentum representation the single-time and Bethe-Salpeter wave functions are related by

$$\tilde{\Psi}_{M,\vec{K}=0}(\mathring{\vec{p}}) = \int dq'_0\, \Psi_{M,\vec{K}=0}(\mathring{q}, \mathring{\vec{q}}); \quad \mathring{\vec{p}} \equiv \frac{1}{2}\,\overrightarrow{(\Lambda_K^{-1} q)} = \overrightarrow{(\Lambda_K p)}, \tag{25}$$

where $q'_0 = \frac{K\cdot q}{M}$. Since in the c.m.s. $\vec{\mathcal{P}} = 0$, i.e. $q'_0 = q_0$, the relation (25) turns into the definition of the single-time wave function given in ref.[12].

In refs.[16] for the wave function of two spinor particles (25) a quasipotential equation has been obtained which is projected onto spin states of the bound system

$$2\mathring{p}_0(2\mathring{p}_0 - M)\, \tilde{\Psi}^{\sigma_1\sigma_2}(\mathring{\vec{p}}) =$$

$$= (2\pi)^{-3} \int \frac{d^3\mathring{\vec{k}}}{2\mathring{k}_0}\, V^{\sigma_1\sigma_2}_{\sigma'_1\sigma'_2}(\mathring{\vec{p}}, \mathring{\vec{k}})\, \Psi_M^{\sigma'_1\sigma'_2}(\mathring{\vec{k}}), \tag{26}$$

where the kernel $V(\mathring{\vec{p}}, \mathring{\vec{k}})$ is a quasipotential constructed, according to the developed in[7,17] recipe, from matrix elements of the relativistic scattering amplitude describing the interaction in $(\mu^+\mu^-)$-system.

A formula for the decay amplitude of the bound state into two virtual γ-quanta within the single-time description of two-particle bound states is derived in analogy with that for the amplitude, e.g., of the decay $\pi^0 \to \gamma^*\gamma^*$ given in refs.[4] and [5,6]. As a result, in the single-time description we have, instead of (11)

$$T^{\lambda_1\lambda_2}_{\mu^+\mu^-\to\gamma e^+e^-} = (2\pi)^{-3} \int \frac{d^3\mathring{\vec{k}}}{2\mathring{k}_0}\, \tilde{\Psi}^{\sigma_1\sigma_2}(\mathring{\vec{k}})\, T^{\lambda_1\lambda_2}_{\sigma_1\sigma_2}(k_1, k_2|q_1, q_2), \tag{27}$$

where $T^{\lambda_1\lambda_2}_{\sigma_1\sigma_2}(k_1, k_2|q_1, q_2)$ is the Feynman amplitude of annihilation of the μ^+-meson with polarization σ_1 and μ^--meson with polarization σ_2 into two virtual γ-quanta with polarizations λ_1 and λ_2, respectively.

In what follows we shall make use of the single-time wave function with the spin structure[5]:

$$\tilde{\Psi}^{\sigma_1\sigma_2}(\mathring{\vec{k}}_1) = \bar{u}^{\sigma_1}(k_1)\,\gamma_5\, v^{\sigma_2}(k_2)\, \frac{\tilde{\phi}_{BM}(\mathring{\vec{k}})}{2\mathring{k}_0}, \tag{28}$$

where $\tilde{\phi}_{BM}(\mathring{\vec{k}})$ is a scalar function of the covariant vector of a particle momentum in the c.m.s. (22). The normalization condition for the wave function $\tilde{\Psi}(\vec{p})$ that for the energy-independent quasipotential reads

$$(2\pi)^{-3} \int \frac{d^3\vec{p}}{2p_0}\, \tilde{\Psi}^+_{\sigma_1\sigma_2}(\vec{p})\, 2p_0\, \tilde{\Psi}^{\sigma_1\sigma_2}(\vec{p}) = 2M \tag{29}$$

for the scalar function $\tilde{\phi}_{BM}(\vec{p})$ is given by

$$(2\pi)^{-3} \int d\vec{p}\, |\tilde{\phi}_{BM}(\vec{p})|^2 = M, \tag{30}$$

where M is the bound state mass.

Note that in[2] for the description of the decay width of the $(\mu^+\mu^-)$-system in a state with the total momentum J, its projection M, orbital moment L and total spin S the wave function $\Psi_{JMLS}(k)$ was chosen in the case of a state with $S = 0$ in the form (the notation corresponds to Fig.1):

$$\Psi_{JMLO}(k) = (\hat{k}_2 - m)\,\frac{1}{2}(1+\gamma_4)\,\gamma_5\,(\hat{k}_1 + m)\,\tilde{\phi}_{JMLO}(k) \tag{31}$$

and

$$\tilde{\phi}_{JMLO}(k) = C_1\,\delta(k^\circ)\,\delta_{LJ}\,\tilde{\phi}_{LM}(\vec{k}), \tag{32}$$

where the wave function $\tilde{\phi}_{LM}(k)$ is a solution of the nonrelativistic Schrödinger equation with orbital moment L. And the amplitude $T^{\lambda_1\lambda_2}$ was taken in the form

$$T^{\lambda_1\lambda_2} = \epsilon^{\lambda_1}_{\mu}\,\epsilon^{\lambda_2}_{\nu}\, A^{\mu\nu}, \tag{33}$$

where

$$A_{\mu\nu} = C_2\, e^2 \int d^4k\, \mathrm{Sp}\,[\Psi_{JMLO}(k)\, N_{\mu\nu}(k)] \tag{34}$$

and

$$N_{\mu\nu} = \gamma_\mu S(k_1 - q_1; m)\gamma_\nu + \gamma_\nu S(k_2 - q_2; m)\gamma_\mu . \tag{35}$$

The constants C_1 and C_2 are defined by the normalization of wave functions and amplitudes. The factor $\frac{1}{2}(1+\gamma_4)$ in (31) was introduced in [2] because of the use of the approximation (32), and in this case the relativistic wave functions were expressed in terms of the nonrelativistic ones with large components only (for details see [1]).

It is evident that the expression (6) coincides with the formula (34) (taken from ref. [2]) up to the factor $\frac{1}{2}(1+\gamma_4)$ which is not present in (6) because we are working with the relativistic wave functions. In this way we observe that the substitution of (31) into (34) leads to the same formula (27) as calculations within the single-time formalism*. In what follows, as the wave function, we will use, instead of the Schrödinger nonrelativistic wave function like in [1,2] an approximate solution of the relativistic single-time equation (26) with a quasipotential corresponding to the one-photon exchange[18].

3. CALCULATION OF THE DEPENDENCE OF THE FORM FACTOR $F_{\mu^+\mu^-\to\gamma e^+e^-}$ ON THE SQUARED INVARIANT MASS OF AN e^+e^--PAIR

As is shown in [6], the substitution of (12) and (28) into (27), calculation of the Spur, and integration over angular variables in (27) result in the following expression for the $\mu^+\mu^-\to\gamma e^+e^-$ decay form factor

$$\tilde{F}_{\mu^+\mu^-\to\gamma\gamma^*}(x) = (1-x)^{-1}\,[1 + (4J_N)^{-1} J(x)]. \tag{36}$$

This form factor is normalized to

$$\tilde{F}_{\mu^+\mu^-\to\gamma\gamma^*}(x) = F_{\mu^+\mu^-\to\gamma e^+e^-}(q_1^2, 0)\; f^{-1}_{\mu^+\mu^-\to 2\gamma}, \tag{37}$$

where $f_{\mu^+\mu^-\to 2\gamma}$ is a constant of the decay of the $(\mu^+\mu^-)$-system into two γ-quanta, ($\tilde{F}_{\mu^+\mu^-\to\gamma\gamma^*}(x) \equiv \tilde{F}(x)$)

$$J(x) = \int_0^\infty d\chi_k \,\ln|X(x,\chi_k)|\cdot \phi_{BM}(\chi_k) \tag{38}$$

and

$$X(x,\chi_k) = [1 - xe^{-\chi_k}(M_{\mu^+\mu^-}/m_\mu - e^{-\chi_k})] \times [1 - xe^{\chi_k}(M_{\mu^+\mu^-}/m_\mu - e^{\chi_k})]^{-1},$$

$$4\pi\,\phi_{BM}(\chi_k) = k_1\,\tilde{\phi}_{BM}(k_1), \quad k_1 = |\vec{k}_1|, \quad x = (p_+ + p_-)^2/M^2_{\mu^+\mu^-}, \tag{39}$$

$$k_1^\circ = m_\mu \operatorname{ch}\chi_k, \quad k_1 = m_\mu \operatorname{sh}\chi_k, \quad \chi_k = \ln[(k_1^\circ + k_1)/m_\mu],$$

the rapidity χ_k is conjugate to the muon relative momentum.

The integral J_N in (36) defines the decay constant

$$f_{\mu^+\mu^-\to 2\gamma} = 32\cdot m_\mu \cdot M_{\mu^+\mu^-}\cdot \alpha\cdot J_N \tag{40}$$

and given by

$$J_N = \int_0^\infty d\chi_k \cdot \phi_{BM}(\chi_k)\cdot \chi_k . \tag{41}$$

To get a better understanding of the physical meaning of the decay constant $f_{\mu^+\mu^-\to 2\gamma}$ (40), we represent the wave function in the form[19]

$$\phi_{BM}(\chi_k) = \int_0^\infty dr\cdot r\cdot \phi_{BM}(r)\cdot \sin(m_\mu r \chi_k), \tag{42}$$

where the wave function $\phi_{BM}(r)$ is defined, instead of the usual Fourier transformation, by the expansion over unitary infinite-dimensional representations of the Lorentz group, the group of motions in the Lobachevsky space[19]

$$\phi_{BM}(r) = \frac{2m_\mu}{(2\pi)^3}\int \frac{d\overset{\circ}{\vec{k}}}{2\overset{\circ}{k}_0}\,\xi(\overset{\circ}{\vec{k}}, \vec{r})\cdot \tilde{\phi}_{BM}(\overset{\circ}{\vec{k}}), \tag{43}$$

where the functions

$$\xi(\overset{\circ}{\vec{k}}, \vec{r}) = (\overset{\circ}{k}\cdot n/m_\mu)^{-1 - irm_\mu} \quad (0 \le r < \infty, \;\; n = (1, \vec{n}),\; \vec{n}^2 = 1) \tag{44}$$

realize a unitary representation of the Lorentz group[20]. As is noted in[19], in the nonrelativistic limit $(m\to\infty)\,\xi(\vec{k},\vec{r}) \to e^{i\vec{k}\cdot\vec{r}}$ and the transformation (43) turns into the threedimensional Fourier transformation from the momentum to three-dimensional coordinate space. By substituting (42) into (41) we calculate the integral

$$J_N = \frac{\pi}{2m_\mu^2}\cdot \phi_{BM}(0), \tag{45}$$

where $\phi_{BM}(0)$ is the wave function in the relativistic configurational representation at $r = 0$. Then, for the $\mu^+\mu^-\to 2\gamma$ decay constant (40) we get[5]:

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*The factor $(2\overset{\circ}{k}_0)^{-1}$ in (27) is due to different normalizations of nonrelativistic and relativistic wave functions.

two γ-quanta including the radiation corrections was estimated to be

$$\Gamma(e^+e^- \to 2\gamma) = (0.7984 \pm 0.0001) \times 10^{10} \mathrm{sec}^{-1}. \qquad (56)$$

If $\Gamma(\mu^+\mu^- \to 2\gamma)$ is calculated by (54) with the new wave function

$$\chi_{BM}^{(\mu^+\mu^-)}(\vec{r}) = \frac{1}{\sqrt{M_{\mu^+\mu^-}}} \cdot \phi_{BM}(\vec{r}) \qquad (57)$$

(instead of $\phi_{BM}(0)$), which in the nonrelativistic limit is normalized as follows

$$\int d^3\vec{r}\, |\chi_{BM}^{(\mu^+\mu^-)}(\vec{r})|^2 = 2 \qquad (58)$$

then, taking into account (54), the formula (55) becomes

$$\Gamma(e^+e^- \to 2\gamma) = \frac{4\pi a^2}{m_\mu^2} |\chi_{BM}^{(\mu^+\mu^-)}(0)|^2 \cdot \frac{m_e}{m_\mu}, \qquad (59)$$

where m_e is the electron mass.

Using the function $\chi_{BM}^{(e^+e^-)}(0)$ that describes the width of decay of bound (e^+e^-) state into two γ-quanta we represent (59) by

$$\Gamma(e^+e^- \to 2\gamma) = \frac{4\pi a^2}{m_e^2} \cdot |\chi_{BM}^{(e^+e^-)}(0)|^2. \qquad (60)$$

The wave function $\chi_{BM}^{(e^+e^-)}(\vec{r})$, the Fourier transform of a solution of the nonrelativistic Schrödinger equation with a Coulomb potential in the momentum space, is of the following form for the S-state at $r = 0$ /22/

$$\chi_{BM}^{(e^+e^-)}(0) = \sqrt{\frac{(m_e \cdot a)^3}{8\pi}}. \qquad (61)$$

Inserting (61) into (60) we arrive at the known expression for the width of the parapositronium decay into two photons (see, e.g. /22/)

$$\Gamma(e^+e^- \to 2\gamma) = \frac{1}{2} \cdot m_e \cdot a^5. \qquad (62)$$

5. MODEL WAVE FUNCTION

To calculate the form factor of the decay $(\mu^+\mu^-) \to \gamma e^+e^-$ (36) and the probability of the decay $\mu^+\mu^- \to 2\gamma$ (53), we shall make use of the following relation for the relativistic wave function /18/:

$$\phi_{BM}(\chi_k) = \frac{C_0 \,\mathrm{sh}\, \chi_k}{(\mathrm{ch}\, \chi_k - M_{\mu^+\mu^-}/2m_\mu)^2}, \qquad (63)$$

that is an approximate solution to the quasipotential equation /18/:

$$\begin{aligned} &\mathrm{ch}\,\chi_p (M_{\mu^+\mu^-}/2m_\mu - \mathrm{ch}\,\chi_p)\, \phi_{BM}(\chi_p) = \\ &= \frac{m_\mu^2}{2\cdot(2\pi)^2} \int_0^\infty d\chi_k \{\mathrm{ch}(\chi_p - \chi_k) + \mathrm{ch}(\chi_p + \chi_k) - 1\} \times \\ &\times \int_{|\chi_p - \chi_k|}^{\chi_p + \chi_k} dy\; \mathrm{sh} y\; V_0(2m_\mu \,\mathrm{sh} y/2)\, \phi_{BM}(\chi_k) \end{aligned} \qquad (64)$$

and has a correct nonrelativistic limit and asymptotics as $\chi_k \to \infty$ /18/. In the expression (63) the normalization constant C_0 is defined by the condition (see the formula (30))

$$\int_0^\infty |\phi_{BM}(\chi_k)|^2\, \mathrm{ch}\,\chi_k\, d\chi_k = \frac{1}{8} \frac{M_{\mu^+\mu^-}}{m_\mu}. \qquad (65)$$

In the equation (64) we make use of the following parametrization of the momentum transfer in the quasipotential /18/:

$$q^2 = (p-k)^2 = (2m \cdot \mathrm{sh}(y/2))^2 \qquad (66)$$

and

$$\mathrm{ch} y = \mathrm{ch}\,\chi_p \cdot \mathrm{ch}\,\chi_k - (\vec{n}_p \cdot \vec{n}_k)\, \mathrm{sh}\,\chi_p \cdot \mathrm{sh}\,\chi_k \qquad (67)$$

and the following expression for the quasipotential

$$V_0(2m_\mu \mathrm{sh}(y/2)) = -\frac{4\pi a}{q^2} = -\frac{\pi a}{m_\mu^2 \mathrm{sh}^2(y/2)} \qquad (68)$$

corresponding to the one-photon exchange in electrodynamics /18/.

6. RESULTS OF THE NUMERICAL CALCULATION

An analytic calculation of the form factor of the decay $\mu^+\mu^- \to \gamma e^+e^-$ (36) with the wave function (63) is rather difficult because of the composite integrand of (38), therefore $\tilde{F}(x)$ has been calculated numerically. Before discussing the calculated $\tilde{F}(x)$ we consider the function

$$f(x, \chi_k) = \left(4\int_0^\infty d\chi_k\, \phi_{BM}(\chi_k)\, \chi_k\right)^{-1} \ln|X(x, \chi_k)|\, \phi_{BM}(\chi_k) \qquad (69)$$

entering into the expression (36) as follows

$$\tilde{F}(x) = (1-x)^{-1}\left[1 + \int_0^\infty d\chi_k\, f(x, \chi_k)\right]. \qquad (70)$$

Due to the weak coupling in the $(\mu^+\mu^-)$-system, in the course of calculation of $f(x, \chi_k)$ we made use of the quantity $M_{\mu^+\mu^-} \approx 2m_\mu$ as a mass of the $(\mu^+\mu^-)$ bound state. In Fig.2 we have drawn the dependence of $f(x, \chi_k)$ on the muon rapidity χ_k for some values of the squared invariant mass of a lepton pair x. It is seen that the function $f(x, \chi_k)$ is negative and its absolute values in the range $0 \le \chi_k \le 8$ are smaller than unity. Consequently, the value of the integral $\int_0^\infty d\chi_k\, f(x, \chi_k) \simeq -5 \cdot 10^{-3}$ for $0{,}01 \le x \le 0{,}15$ being a negative additional term to unity in brackets of (70), does not influence the behaviour of the form factor $\tilde{F}(x)$ defined by the pole term $(1-x)^{-1}$.

Calculating $\Gamma(\mu^+\mu^- \to 2\gamma)$ by the formula (53) with the use of the function (63) we get the following value for the probability of the paramuonium decay into two γ-quanta:

$$\Gamma(\mu^+\mu^- \to 2\gamma) = 1.648\times10^{12}\,\mathrm{sec}^{-1}. \qquad (71)$$

Substituting (71) into (55) we obtain the value of the probability of the decay $e^+e^- \to 2\gamma$:

$$\Gamma(e^+e^- \to 2\gamma) = 0.797\times10^{10}\,\mathrm{sec}^{-1} \qquad (72)$$

that is in good agreement with experiment /24/:

$$\Gamma^{exp}(e^+e^- \to 2\gamma) = (0.799 \pm 0.011)\times10^{10}\,\mathrm{sec}^{-1}. \qquad (73)$$

7. CONCLUSION

In this paper we have shown that the form factor of the decay of the bound state of a $(\mu^+\mu^-)$-pair into the e^+e^--Dalitz pair and a γ-quantum smoothly increases with the invariant mass

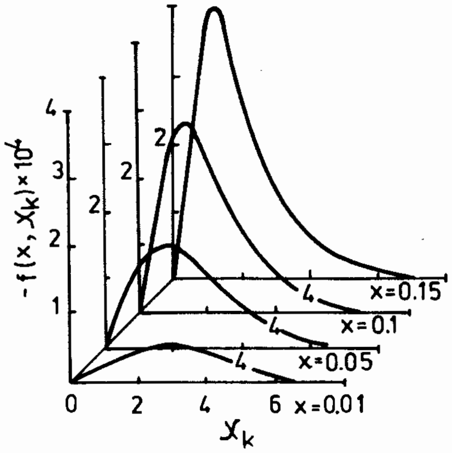

Fig.2

of the e^+e^--pair from a minimal kinematically allowed to a maximal value. It is just the difference from a theoretically possible behaviour of the form factor of the decay $\pi^0 \to \gamma e^+e^-$ /6/. The reason is as follows: the Coulomb electromagnetic interaction in the $\mu^+\mu^-$-system results in a weak coupling in that system, whereas the pion, quark-antiquark system, is characterized by a strong coupling. The total decay width is calculated for the bound state of a $\mu^+\mu^-$-pair into two γ-quanta. It is shown that the method as applied to the parapositronium gives the probability of the decay $(e^+e^-) \to 2\gamma$ in agreement with experiment.

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Козлов Г.А. и др. E2-83-630
О распаде связанного состояния $\mu^+\mu^-$-пары в e^+e^- - пару Далица и γ -квант

Исследуется поведение формфактора распада связанной системы $\mu^+\mu^-$, находящейся в состоянии с полным спином равным нулю, на e^+e^- -пару и γ-квант. Вычислена вероятность распада связанного состояния $\mu^+\mu^-$ на два фотона. Замена в полученном выражении массы μ -мезона на массу электрона приводит к величине вероятности распада парапозитрония на 2 γ-кванта, согласующейся с экспериментальными данными.

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Kozlov G.A. et al. E2-83-630
On the Decay of a Bound State of a $\mu^+\mu^-$ Pair into an e^+e^- Dalitz Pair and a γ -Quantum

The behaviour of the form factor decay is studied for a bound $\mu^+\mu^-$-system in a state with zero total spin into an e^+e^- -pair and a γ-quantum. The probability is calculated of the decay of the bound $\mu^+\mu^-$ -state into two photons. The change of the muon by electron mass in the expression obtained results in a value for the probability of the parapositronium decay into two γ-quanta in agreement with experiment.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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