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QUANTUM FIELD THEORETIC TREATMENT OF THE CASIMIR EFFECT.

Finiteness of the Casimir Force up to Second Order of Perturbation Theory

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1. INTRODUCTION

In this second part we continue our treatment of the Casimir effect [1] with more physical questions. Whereas in ref. [2] we have considered the general formalism, the perturbation theory and explicit expressions for the Casimir force up to second order of perturbation theory (all these results are more or less formal), we will investigate here these results more concretely. At first we will compare our closed expression for the photon Green function with corresponding Green functions of massless scalar field theory, repeat the calculation of the Casimir force in zeroth order, and then give an absolutely convergent expression in second order of perturbation theory. It turns out that this force as a directly measurable quantity (contrary to the energy) is finite without any renormalization. This is a very surprising result also if one has in mind that for loop diagrams the Z-factors of multiplicative renormalization cancel out and, maybe, the Z-factor of the energy operator is essential one. It is, however, worthwile to note that the surface energy density is not a finite quantity, but its divergent part is independent of the distance a between two plates, so that it does not influence the Casimir force.

Technically, we proceed in the following way. The investigation of the photon Green function and the calculation of the Casimir force in zeroth order are done in x-space. A more complicated expression in second order of perturbation theory is transformed into momentum space, after a Wick rotation we obtain an absolutely convergent expression which is suited for further detailed investigations. We restrict ourselves here to the calculation of the Casimir force in the limit $a \to \infty$. At first sight this seems to be a strange and unphysical limit. But if we take into account that the essential physical length in electrodynamics is the Compton wave length of the electron and compare it with the macroscopic values of distances available in Casimir experiments, then it is reasonable to study such a limit.

2. GREEN FUNCTIONS WITH BOUNDARY CONDITIONS

In ref. [2] we have shown that in QED the standard perturbation theory remains also in the presence of boundary conditions, we have only to replace the usual free photon propagator $D^c_{\mu\nu}$ by a more complicated one which satisfies the necessary boundary conditions

$$D^c_{\mu\nu}(x-y) \to {}^sD^c_{\mu\nu}(x,y) = D^c_{\mu\nu}(x-y) + \tilde{D}^c_{\mu\nu}(x,y) \qquad (2.1)$$

with

$$\tilde{D}^c_{\mu\nu}(x,y) = \int \frac{d^3p}{(2\pi)^3} \frac{-i}{2\Gamma} P_{\mu\nu}(p)\, e^{ip_\alpha(x-y)^\alpha} e^{i\Gamma|x_3-a_i|} h^{-1}_{ij} e^{i\Gamma|y_3-a_j|},$$

$$\Gamma = \sqrt{p_0^2 - p_1^2 - p_2^2 + i\epsilon}, \quad \alpha = 0,1,2, \quad h_{ij} = e^{i\Gamma|a_i - a_j|}. \qquad (2.2)$$

Here $P_{\mu\nu}(p)$ is a projection operator defined in momentum space by

$$P_{\mu\nu}(p) = \begin{cases} g_{\mu\nu} - \dfrac{p_\mu p_\nu}{\Gamma^2} & \text{for } \mu,\nu \neq 3 \\ 0 & \text{for } \mu = 0 \text{ or } \nu = 3\,. \end{cases} \qquad (2.3)$$

It is important and simplifies all calculations that in all our expression the contracted part

$$\tilde{D}^c(x,y) = \frac{1}{2} g^{\mu\nu} \tilde{D}^c_{\mu\nu}(x,y) =$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{-i}{2\Gamma} e^{ip_\alpha(x-y)^\alpha} e^{i\Gamma|x_3-a_i|} h^{-1}_{ij} e^{i\Gamma|y_3-a_j|} \qquad (2.4)$$

of the additional term to the photon propagator appears. We will show here that

$${}^sD^c(x,y) = D^c(x-y) + \tilde{D}^c(x,y),$$

$$D^c(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{-p^2 - i\epsilon} = \int \frac{d^3p}{(2\pi)^3} \frac{i}{2\Gamma} e^{ip_\alpha(x-y)^\alpha + i\Gamma|x_3-y_3|}. \qquad (2.5)$$

is the propagator of a scalar massless field theory. (Note that this is not the case for the directly contracted modified photon propagator (2.1))

$$g^{\mu\nu}\, {}^sD^c_{\mu\nu}(x,y) = (4 - (1-\alpha))D^c(x-y) + 2\tilde{D}^c(x,y). \qquad (2.6)$$

We prove the following properties of the propagator which determine it uniquely.

$$\Box\, {}^sD^c(x,y) = \delta(x-y) \quad (x,y \,\bar{\in}\, S), \qquad (2.7a)$$

$${}^sD^c(x,y) = 0 \quad \text{for } x \in S \text{ or } y \in S. \qquad (2.7b)$$

Let us first consider the boundary condition

$${}^sD^c(x,y)\Big|_{x_3=a_k} = \int \frac{d^3p}{(2\pi)^3} \frac{i}{2\Gamma} e^{ip_\alpha(x-y)^\alpha + i\Gamma|a_k-y_3|} + $$

$$+ \int \frac{d^3p}{(2\pi)^3} \frac{-i}{2\Gamma} e^{ip_\alpha(x-y)^\alpha} e^{i\Gamma|a_k-a_i|} h^{-1}_{ij} e^{i\Gamma|y_3-a_j|}. \qquad (2.8)$$

It is easy to see that the first term cancels the second if we take into account the definition $h_{ij} = \exp i\Gamma|a_i - a_j|$. Second, we have to show that the differential equation (2.7a) is fulfilled. Because of $\Box D^c(x-y) = \delta(x-y)$ we have to show

$$\Box \tilde{D}^c(x,y) = 0 \quad (x,y \,\bar{\in}\, S).$$

This can be done is each region of the variables separately. Consider for instance the region $x_3 \geq a_1$, $y_3 \geq a_1$. Then we have

$$\Box \tilde{D}^c(x,y) = \int \frac{d^3p}{(2\pi)^3} e^{ip_\alpha(x-y)^\alpha} \frac{-i}{2\Gamma}(-p_\alpha p^\alpha + \Gamma^2) \times$$

$$\times e^{i\Gamma|x_3-a_i|} h^{-1}_{ij} e^{i\Gamma|y_3-a_j|} = 0$$

because of

$$-q_\alpha q^\alpha + \Gamma^2 = -q_0^2 + q_1^2 + q_2^2 + q_0^2 - q_1^2 - q_2^2 = 0.$$

It is interesting to remark that our Green function represents three in principle separately defined Green functions: the first defined to the left of the left plate, the second one defined in between the two plates, and the third one defined to the right of the right plate. From this point of view it is very surprising that such a compact expression as eq. (2.2) does exist .

Now it is possible to write down other representations of the scalar propagator. The most useful, besides our momentum space representation, is the x-space representation obtained by the reflection principle. In this manner the following representation can be received

$$
{}^{s}D^{c}(x,y) = \begin{cases} D^{c}(x_a - y_a ; x_3 - y_3) - D^{c}(x_a - y_a ; x_3 + y_3 - 2a_0) & \text{for } \begin{matrix} x_3 \le a_0 \\ y_3 \le a_0 \end{matrix} \\ \sum\limits_{n=-\infty}^{\infty} [D^{c}(x_a - y_a ; x_3 - y_3 + 2an) - D^{c}(x_a - y_a ; x_3 + y_3 + 2an)] & \text{for } \begin{matrix} a_0 \le x_3 \le a_1 \\ a_0 \le y_3 \le a_1 \end{matrix} \\ D^{c}(x_a - y_a ; x_3 - y_3) - D^{c}(x_a - y_a ; x_3 + y_3 - 2a_1) & \text{for } \begin{matrix} a_1 \le x_3 \\ a_1 \le y_3 \end{matrix} \\ 0 \text{ for all other regions} & \end{cases} \tag{2.9}
$$

Here is $a = |a_0 - a_1|$, $a_0 < a_1$. This representation is very suited for discussion of the x-space behaviour and the regularity structure in this space. On the other hand, our momentum space representation as a closed representation is very suited for Feynman diagram calculations. Such calculations would be very complicated if we would apply such infinite sums.

Let us add some remarks concerning the connection of the vector and scalar theory. In all calculations (see ref. /2/ eqs. (3.19)) the vector propagator turns finally to the scalar propagator. Thereby that part of the propagator which corresponds to the full space without boundary conditions drops out. In this sense the determination of the Casimir force for the electromagnetic field reduces to the corresponding scalar problem, and there appears the factor 2 corresponding to the two degrees of freedom of the electromagnetic field.

3. FINITENESS OF THE CASIMIR FORCE

Here we will show that the Casimir force is finite without any subtractions up to second order of perturbation theory. In zeroth order this will be shown by explicit calculations. In second order we first show that the usual divergent renorma-

lization constants compensate each other, later on we transform the full expression in momentum space where it appears finally as an absolutely convergent integral expression.

a) Casimir Force in Zeroth Order of Perturbation Theory

Our starting point is the(derived in ref. /2/) explicit expression for the energy per unit surface element

$$
E(a) = \lim_{\delta \to 0} \int_{-\infty}^{\infty} dx_3\, i \tilde{D}^{c}_{\rho\rho}(0; x_3, x_3 + \delta), \quad \tilde{D}^{c}_{\rho\rho}(x,y) = -\partial_{xy} \tilde{D}^{c}(x,y) . \tag{3.1}
$$

Using the representation of the Green function (2.8) we have

$$
\begin{aligned} E^{\delta}(a) &= -i \int_{-\infty}^{a_0} dx_3\, D^{c}_{\rho\rho}(0; 2x_3 + \delta - 2a_0) - i \int_{a_1}^{\infty} dx_3\, D^{c}_{\rho\rho}(0; 2x_3 + \delta - 2a_1) + \\ &+ i \int_{a_0}^{a_1} dx_3 \Big[\sum_{n \neq 0} D^{c}_{\rho\rho}(0; -\delta + 2an) - \sum_{n} D^{c}_{\rho\rho}(0; 2x_3 + 2an + \delta) \Big] = \\ &= i(a_1 - a_0) \sum_{n \neq 0} D^{c}_{\rho\rho}(0; -\delta + 2an) - 2i \int_{-\infty}^{\infty} dx_3\, D^{c}_{\rho\rho}(0; 2x_3) . \end{aligned} \tag{3.2}
$$

Here we have used

$$
\int_{a_0}^{a_1} dx_3 \sum_{n} D^{c}_{\rho\rho}(0; 2x_3 + 2an + \delta) = \int_{-\infty}^{\infty} dx_3\, D^{c}_{\rho\rho}(0; 2x_3) ,
$$

a-independent terms are dropped everywhere. The limit $\delta \to 0$ can be performed, so that we are able to determine the force to

$$
F(a) = -\frac{d}{da} E(a) = -\frac{d}{da}\, a \sum_{n \neq 0} i D^{c}_{\rho\rho}(0; 2an) . \tag{3.3}
$$

This is the well-known result and shows that the Casimir force is finite without any subtractions. Note that this is obviously not the case for other quantities.

b) Compensation of Divergences in Second Order of Perturbation Theory

It is a more difficult problem to show that also the expression for the Casimir force up to second order of perturbation theory

$$F = \lim_{\delta\to 0}\left(-\frac{d}{da}\right)\Big\{ \int d^4\xi \Big[(i\delta(\xi) - \Pi(\xi^2))\,\partial_{\xi\xi} + \Big[\frac{\partial}{\partial\xi_\rho}\frac{\partial}{\partial\xi^\rho}\xi^2\,\Pi(\xi^2)\Big]\frac{\partial^2}{\partial\xi_0^2}\Big] \times$$

$$\times \int_{-\infty}^{\infty} dx_3\,\tilde{D}^c(\xi_a;\xi_3 + x_3, x_3 + \delta) - \int d^4\xi\,\Pi(\xi^2)\,\partial_{\eta\eta}\int d^4z' \int_{-\infty}^{\infty} dx_3 \times \tag{3.4}$$

$$\times\, \tilde{D}^c(z'_a;\, z'_3, x_3 + \delta)\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\xi^\mu}\tilde{D}^c(\xi_a + z'_a + \eta_a;\, \xi_3 + z'_3, x_3 + \eta_3 + \delta)\Big|_{\eta=0}\Big\}$$

is a finite quantity. For completeness we have included the expression of zeroth order which was already discussed. At first we remark: the only region where divergences may appear (after the limit $\delta\to 0$) is the neighbourhood of the point $\xi^2 = 0$ because just there the polarization operator as well as the photon Green function are singular. More correctly we have singularities at the point $\xi = 0$ (we use time-ordered Green functions which have some arbitrariness at this point) and on the light-cone. We will show that the singularities at $\xi = 0$ cancel out for the Casimir force. From standard perturbation theory it is well known that the polarization operator can be represented by

$$\Pi(\xi^2) = A\delta(\xi) + \Pi^{ren}(\xi^2), \tag{3.5}$$

whereby A is a divergent constant and Π^{ren} a well defined distribution. For the renormalized polarization operator there exists a similar expression $\Pi^{ren} = d\delta(\xi^2) + \hat{\Pi}$. Here d is a finite constant which depends on the renormalization conditions. For our purpose it is sufficient to take the ansatz

$$\Pi(\xi^2) = A\delta(\xi)$$

and insert it into the expression (3.4) for the Casimir force. Then we see:

1. The second Π-containing term vanishes

$$\Big[-\frac{\partial}{\partial\xi_\rho}\frac{\partial}{\partial\xi^\rho} + 2\frac{\partial^2}{\partial\xi^2}\Big]\xi^2 A\delta(\xi) = 0\,.$$

2. Because of

$$\int d^4z\,\tilde{D}^c(z_a;\, z_3, x_3)\frac{\partial}{\partial z_\mu}\frac{\partial}{\partial z^\mu}\tilde{D}^c(z_a;\, z_3, x_3 + \eta) = \tag{3.6}$$

$$= -\tilde{D}^c(0, x_3, x_3 + \eta)$$

the third Π-depending term takes the form

$$\int d^4\xi\, A\delta(\xi)\,\partial_{\xi\xi}\int_{-\infty}^{\infty} dx_3\,\tilde{D}^c(\xi_a;\, \xi_3 + x_3, x_3 + \delta)\,.$$

Therefore the remaining contributions compensate each other, so that the typical perturbation theoretical divergences drop out. Of course, also the similar finite terms $d\delta(\xi)$ do not contribute. That means: Up to second order of perturbation theory the Casimir force is not influenced by the renormalization procedure, renormalization conditions play no role.

c) Finiteness of the Casimir Force in Second Order of Perturbation Theory

The finiteness of eq. (3.4) can be shown in x-space as well as in momentum space. In x-space an essential point is the discussion of the test function-like properties of

$\frac{d}{da}\int_{-\infty}^{\infty} dx_3\,\tilde{D}^c(\xi_a;\, x_3 + \xi_3, x_3 + \delta)$. We will, however, solve this problem in momentum space. Here we can see that all expressions are finite, we arrive at absolutely convergent integrals which are very suited for further investigations. The standard Fourier transform leads us to the following expression

$$F = -\frac{d}{da}E\,, \tag{3.7}$$

$$E = \int\frac{d^4k}{(2\pi)^4}\,\tilde{\Pi}(k^2)\Big[\frac{2k_0^2\, a}{k_3^2 - \Gamma^2}\frac{1}{(\sin\Gamma a)^2}(1 - \cos\Gamma a\cdot\cos k_3 a) -$$

$$-\left(\frac{2k_0^2}{k_3^2\Gamma^2}+\frac{4k_0^2\Gamma^2}{(k_3^2-\Gamma^2)^2}\right)\frac{1}{\Gamma\sin\Gamma a}(e^{-i\Gamma a}-\cos k_3 a)\Big] +$$

$$+\int\frac{d^4k}{(2\pi)^4}\,\tilde{\Pi}'(k^2)\,\frac{-2k_0^2\Gamma^2}{k_3^2-\Gamma^2}\,\frac{1}{\Gamma\sin\Gamma a}(e^{-i\Gamma a}-\cos k_3 a), \qquad (3.8)$$

$$\tilde{\Pi}(k^2) = \int d^4x\, e^{-ikx}\,\Pi(x^2), \qquad \tilde{\Pi}'(k^2) = \frac{\partial}{\partial k^2}\tilde{\Pi}(k^2). \qquad (3.9)$$

The calculation is lengthy but without any difficulties. In the Appendix we list some useful integral formulae which simplify the calculations. From this expression we subtract the a-independent divergent constant

$$E_{div} = -\int\frac{d^4k}{(2\pi)^4}\tilde{\Pi}(k^2)\Big[\frac{4ik_0^2}{k_3^2-\Gamma^2}\,\frac{1}{\Gamma}+\frac{8ik_0^2\Gamma^2}{(k_3^2-\Gamma^2)^2}\,\frac{1}{\Gamma}\Big] -$$

$$-\int\frac{d^4k}{(2\pi)^4}\,\tilde{\Pi}'(k^2)\,\frac{4ik_0^2\Gamma^2}{k_3^2-\Gamma^2}\,\frac{1}{\Gamma} \qquad (3.10)$$

without any change of the Casimir force. The result is

$$F = -\frac{d}{da}\Big\{\int\frac{d^4k}{(2\pi)^4}\tilde{\Pi}(k^2)\Big[\frac{2k_0^2}{k_3^2-\Gamma^2}\,\frac{a}{(\sin\Gamma a)^2}(1-\cos\Gamma a\cdot\cos k_3 a) -$$

$$-\left(\frac{2k^2}{k_3^2-\Gamma^2}+\frac{4k_0^2\Gamma^2}{(k_3^2-\Gamma^2)^2}\right)\frac{1}{\Gamma\sin\Gamma a}(e^{i\Gamma a}-\cos k_3 a)\Big] +$$

$$+\int\frac{d^4k}{(2\pi)^4}\,\tilde{\Pi}'(k^2)\,\frac{-2k_0^2\Gamma^2}{k_3^2-\Gamma^2}\,\frac{1}{\Gamma\sin\Gamma a}(e^{i\Gamma a}-\cos k_3 a)\Big\}. \qquad (3.11)$$

In comparison with eq. (3.9) the sign of some exponentials has been changed: $e^{-i\Gamma a}\to e^{i\Gamma a}$. This however, is an essential point because according to our prescription $\Gamma=\sqrt{k_0^2-k_1^2-k_2^2+i\epsilon}$ should have a positive imaginary part. The new expression allows now an analytical continuation into the upper half-plane

$$\Gamma\to i\gamma, \quad \gamma=\sqrt{k_4^2+k_1^2+k_2^2}\,,$$

$$k_0 = ik_4\,, \quad k_E^2 = k_1^2+k_2^2+k_3^2+k_4^2, \quad k_E = (k_4, k_1, k_2, k_3)$$

so that (3.11) takes the form

$$F = -\frac{d}{da}\Big\{i\int\frac{d^4k_E}{(2\pi)^4}\tilde{\Pi}(-k_E^2)\Big[\frac{2ak_4^2}{k_E^2}\,\frac{1}{(\mathrm{sh}\gamma a)^2}(1-\mathrm{ch}\gamma a\cos k_3 a) -$$

$$-\left(\frac{2k_4^2}{k_E^2}-\frac{4k_4^2\gamma^2}{(k_E^2)^2}\right)\frac{1}{\gamma\,\mathrm{sh}\gamma a}(e^{-\gamma a}-\cos k_3 a)\Big] + \qquad (3.12)$$

$$+\,i\int\frac{d^4k_E}{(2\pi)^4}\,\tilde{\Pi}'(-k_E^2)\,\frac{2k_4^2\gamma^2}{k_E^2}\,\frac{1}{\gamma\,\mathrm{sh}\gamma a}(e^{-\gamma a}-\cos k_3 a)\Big\}.$$

If we take into account

$$\tilde{\Pi}(0)=0, \qquad |\tilde{\Pi}(-\infty)| < (k_E^2)^{\delta}, \quad \delta \text{ finite},$$

then it is obvious that (3.12) is an absolutely convergent expression. This proves finally the finiteness of the Casimir force in second order of perturbation theory.

4. SECOND-ORDER CORRECTIONS TO THE CASIMIR FORCE IN THE LIMIT $a\to\infty$

The starting point for further considerations is the expression (3.12) for the Casimir force. Its detailed investigation needs explicit calculations. Here we will determine the leading correction terms in the limit $a\to\infty$ only. At first sight it seems to be very strange to calculate such a useless quantity because one measures the Casimir force for small distances, otherwise it is impossible to measure anything. If we, however, compare the experimentally realized distance of order $1\mu m$ with a characteristic quantity of electrodynamics like the Compton wave length of the electron then the experimentally realized distances are very large as compared to this quantity.

For practical calculations of the $a\to\infty$ asymptotics there are several possibilities. We choose the most simple method which consists in successive approximation of eq. (3.12).

At first we remark, that each expression contains the factor $e^{-\gamma a}$. This gives for $\gamma \neq 0$ an exponentially decreasing part. Therefore the region $\gamma \approx 0$ only could lead to an asymptotic power behaviour.

Second we see that in our expression (3.12) it is possible to perform the k_3-integration in the complex k_3-plane, we close the original integration path $-\infty \leq k_3 \leq \infty$ in the upper or lower half-plane in dependence on the convergence of the corresponding infinite half-circles (the $\cos k_3 a$ -term must be decomposed as $\cos k_3 a = \frac{1}{2}(e^{ik_3 a} + e^{-ik_3 a})$. Thereby we have to respect the singularities in the complex plane. There are

poles at $k_3 = \pm i\gamma$: $\dfrac{1}{k_3^2 + \gamma^2}$,

double poles at $k_3 = \pm i\gamma$: $\dfrac{1}{(k_3^2 + \gamma^2)^2}$,

cuts $-\gamma^2 - k_3^2 \geq 4m_e^2$.

So the original integral is represented by integrals over paths around the singularities in the complex plane. By direct calculation it can be checked that we have no contributions from the poles. The contributions of the simple poles compensate each other that directly corresponds to the cancellation of terms shown in section 3b. The explicit considerations of the double poles will be omitted here. So, there remain the cut-contributions near $\gamma \approx 0$. At $\gamma = 0$ the cut starts at $k_3 = \pm 2im_e$. For the terms containing the exponential function $\exp \pm ik_3 a$ coming from the term $\cos k_3 a$ this leads again to an exponentially decreasing term. So it is clear: Power-type contributions should be expected from terms not containing the $\cos k_3 a$ factor with the restricted integration region $\int dk_3 \int_{k_\delta} d^3k_E$,

k_δ: $k_4^2 + k_1^2 + k_2^2 = \gamma^2 \leq \delta$. Here the following approximations are allowed

$$k_E^2 = k_3^2 + \gamma^2 \approx k_3^2 \, , \quad F(\gamma) = a_1 \gamma^{\alpha_1} + a_2 \gamma^{\alpha_2} + \ldots \approx a_1 \gamma^{\alpha_1} \quad (\alpha_1 \leq \alpha_2 \leq \ldots).$$

Because of $\int_0^\infty dx\, x^\lambda e^{-ax} \sim a^{-\lambda-1}$ the leading term for $a \to \infty$ is determined by the smallest power of x. Taking these leading terms only we get

$$F(a) \underset{a\to\infty}{\sim} -\frac{d}{da}\, i^2 \int \frac{d^3 k_E}{(2\pi)^4} \int_{2m_e}^{\infty} dq \left(\tilde{\Pi}((q - i\epsilon)^2) - \tilde{\Pi}((q + i\epsilon)^2)\right) \times \tag{4.1}$$

$$\times \left(-\frac{2ak_4^2}{q^2} \frac{1}{(\mathrm{sh}\,\gamma a)^2} + \frac{2k_4^2}{q^2} \frac{1}{\gamma\, \mathrm{sh}\,\gamma a} e^{-\gamma a} \right), \quad q = ik_3 . \tag{4.1}$$

For the explicit evaluation of this expression we introduce polar coordinates $k_4 = \gamma \cos\theta$, $k_1 = \gamma \sin\theta \sin\phi$, $k_2 = \gamma \sin\theta\cos\phi$, $d^3k_E = \gamma^2 \sin\theta d\theta\, d\phi\, d\gamma$, integrate over the angles, use the substitution $\gamma a = x$, and thus, the result is

$$F(a) \underset{a\to\infty}{\sim} -\frac{d}{da} \frac{1}{a^4} \int_{2m_e}^{\infty} dq \frac{\tilde{\Pi}(q^2 - i\epsilon) - \tilde{\Pi}(q^2 + i\epsilon)}{2} \frac{4}{3} \frac{1}{(2\pi)^4} \int_0^\infty dx \left[\frac{x^3 e^{-x}}{\mathrm{sh}\,x} - \frac{x^4}{(\mathrm{sh}\,x)^2}\right] -$$

$$\sim -\frac{d}{da} \frac{1}{a^4} \frac{\pi}{8} |B_4| \int_{2m_e}^{\infty} dq \frac{\tilde{\Pi}(q^2 - i\epsilon) - \tilde{\Pi}(q^2 + i\epsilon)}{2} . \tag{4.2}$$

Because of $\tilde{\Pi}(0) = 0$ it is possible to transform the integral over the discontinuity to an integral over the polarization operator itself

$$F(a) \underset{a\to\infty}{\sim} -\frac{d}{da} \frac{1}{a^4} \frac{\pi}{4}\, i\, |B_4| \int_0^\infty dq \frac{\tilde{\Pi}(-q^2)}{q^2} . \tag{4.3}$$

At this place we have to take into account the explicit expression for the polarization operator of standard perturbation theory [3]

$$\Pi_{\mu\nu}(z - z') = -ie^2\, \mathrm{Tr}[\gamma_\mu \hat{S}(z - z')\gamma_\nu \hat{S}(z' - z)],$$

$$\Pi_{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\, \tilde{\Pi}_{\mu\nu}(p) ,$$

$$\tilde{\Pi}_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu} k^2)\, \tilde{\Pi}(k^2) , \tag{4.4}$$

$$\tilde{\Pi}(k^2) = \frac{e^2}{(2\pi)^2} \int_0^1 dx\, x(1 - x) \ln(1 - x(1 - x) \frac{k^2}{m_e^2}) ,$$

$$\tilde{\Pi}(0) = 0 .$$

For the integrated polarization operator we obtain

$$\int^{\infty} dq \frac{\tilde{\Pi}(-q^2)}{q^2} = \frac{ie^2}{(2\pi)^2} \int_0^1 dx\, x(1-x) \int_0^{\infty} \frac{dq}{q^2} \ln(1 + x(1-x) \frac{q^2}{m_e^2}) =$$

$$= \frac{ie^2}{(2\pi)^2} \frac{1}{m_e} \frac{3\pi^2}{8 \cdot 16} . \tag{4.5}$$

So as a final result we receive

$$F(a) \underset{a\to\infty}{=} - \frac{\hbar c \pi^2}{240 a^4} (1 - \frac{3}{4 \cdot 16\pi} \frac{e^2}{a m_e}). \tag{4.6}$$

Here we have included also the zeroth order. The correction term seems to be too small for an experimental verification at the present time and in near future. Remark that this results can be interpreted as an example for an improved convergence of perturbation theory.

APPENDIX

Here we will list some formulae that facilitate the Fourier transform of eq. (3.4). Taking into account the definitions (2.4), (2.5), (2.2), the explicit calculation yields

$$\frac{\partial}{\partial \xi_\rho} \frac{\partial}{\partial \xi^\rho} \tilde{D}^c(\xi_\alpha; \xi_\xi, x_3) = - \int \frac{d^3 q}{(2\pi)^3} \delta(\xi_3 - a_i) e^{i\Gamma|\xi_3 - a_i|} h^{-1}_{ij} e^{i\Gamma|x_3 - a_j|} e^{iq_\alpha \xi^\alpha},$$

$$\partial_{\xi\xi} \tilde{D}^c(\xi_\alpha; \xi_3 + x_3, x_3) = \int \frac{d^3 q}{(2\pi)^3} (\delta(\xi_3 + x_3 - a_i) + i\Gamma(1 + \frac{q_1^2 + q_2^2}{\Gamma^2})) \times$$

$$\times e^{i\Gamma|x_3 + \xi_3 - a_i|} h^{-1}_{ij} e^{i\Gamma|x_3 - a_j|} e^{iq_\alpha \xi^\alpha} ,$$

$$\frac{\partial}{\partial \xi_0} \frac{\partial}{\partial \xi_0} \tilde{D}^c(\xi_\alpha; \xi_3 + x_3, x_3) = \int \frac{d^3 q}{(2\pi)^3} e^{iq_\alpha \xi^\alpha} \frac{iq_0^2}{2\Gamma} e^{i\Gamma|\xi_3 + x_3 - a_i|} \times$$

$$\times h^{-1}_{ij} e^{i\Gamma|x_3 - a_j|},$$

$$\int_{-\infty}^{\infty} dx_3 \partial_{\xi\xi} \tilde{D}^c(\xi^\alpha; \xi_3 + x_3, x_3) =$$

$$= \int \frac{d^3 q}{(2\pi)^3} e^{iq_\alpha \xi^\alpha} (\frac{-q_1^2 - q_2^2}{\Gamma^2} + i(1 + \frac{q_1^2 + q_2^2}{\Gamma^2})\Gamma|a_i - a_j - \xi_3|) e^{i\Gamma|a_i - a_j - \xi_3|} h^{-1}_{ij},$$

$$\int_{-\infty}^{\infty} dx_3 \frac{\partial}{\partial \xi_0} \frac{\partial}{\partial \xi_0} \tilde{D}^c(\xi^\alpha; \xi_3 + x_3, x_3) =$$

$$= \int \frac{d^3 q}{(2\pi)^3} e^{iq_\alpha \xi^\alpha} \frac{iq_0^2}{2\Gamma} (\frac{i}{\Gamma} + |a_i - a_j - \xi_3|) e^{i\Gamma|a_i - a_j - \xi_3|} h^{-1}_{ij} .$$

The matrix multiplication gives

$$h^{-1}_{ij} e^{-iq_3(a_i - a_j)} = \frac{1}{i \sin\Gamma a} (\cos k_3 a - e^{-i\Gamma a}),$$

$$H^{-1}_{\ell i} = h^{-1}_{ij} |a_j - a_k| e^{i\Gamma|a_j - a_k|} h^{-1}_{\ell k} = \frac{a}{2(\sin\Gamma a)^2} \begin{pmatrix} 1, & -\cos\Gamma a \\ -\cos\Gamma a, & 1 \end{pmatrix}$$

$$H^{-1}_{ij} e^{-iq_3(a_i - a_j)} = \frac{a}{(\sin\Gamma a)^2} (\cos q_3 a - e^{-i\Gamma a}).$$

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Бордаг М., Робашик Д. E2-83-489

Эффект Казимира с точки зрения квантовой теории поля. Конечность силы Казимира во втором порядке по теории возмущений

Исследуется формальное выражение для силы Казимира во втором порядке по теории возмущений, выведенное в предыдущей работе /2/. Показано, что обычные в теории возмущений ультрафиолетовые расходимости и соответствующие Z-факторы сокращаются так, что в этом приближении не требуется перенормировки. Зависящие от расстояния /a/ члены в соответствующем выражении для энергии вакуума конечны, так что сила Казимира как непосредственно измеряемая величина также конечна. Определяется сила Казимира в пределе $a \to \infty$, который только и имеет физический смысл.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1983

Bordag M., Robaschik D. E2-83-489

Quantum Field Theoretic Treatment of the Casimir Effect. Finiteness of the Casimir Force up to Second Order of Perturbation Theory

Here we investigate the formal expression for the Casimir force up to order e^2 derived in a foregoing paper /2/. At first we show that the usual perturbation theoretical UV-divergences and the corresponding Z-factors cancel so that there is no renormalization in this approximation. Moreover it turns out that the distance (a) dependent parts of suitable vacuum energy expressions are finite, so that the Casimir force as directly measurable quantity is also finite. Finally we determine the Casimir force in the physical limit $a \to \infty$.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1983