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ON CONFORMAL INVARIANCE IN THE GAUGE THEORIES: YANG-MILLS THEORY

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1. INTRODUCTION

Starting with the assumption that the electromagnetic potential together with one scalar field with zero scale dimension is transformed by some nonbasic representation of the conformal group, in the first part of the present paper [1] we constructed one nontrivial model of the conformal quantum electrodynamics. A similar model was found early in paper [2] where the basic representation was used but the invariance with respect to the special conformal transformations was required up to some restricted class of gauge transformations (see refs. [3,4]). In ref. [5] the same problem is attacked using the nondecomposability of the representation of the conformal group under which the electromagnetic potential is transformed. However, there exist some difficulties to extend results of paper [2] to the nonabelian case.

In the present paper it is shown that results of paper [1], without essential difficulties, can be extended to the nonabelian case, i.e., to the Yang-Mills theory and to the massless chromodynamics. There are found the nontrivial (with nonzero transversal part) conformal invariant two-point functions, the invariant action from which the equations of motion as well as the nonlinear additional conditions are derived. A quantization scheme is outlined which generalizes the Gupta-Bleuler formalizm [6]. Because of the fact that here the nonlinear conformal invariant additional conditions are used which replace the Lorentz condition, the corresponding physical subspace is compatible with the equations of motion. We point out that the Lorentz condition is inconsistent not only in the case of the Yang-Mills theory [3,4] but also in the quantum electrodynamics [7,8]. The canonical quantization procedure for the model under consideration will be considered in the third part of the present paper. This procedure essentially differs from the corresponding procedure considered in ref. [6].

2. INVARIANT TWO-POINT FUNCTIONS

Consider the Yang-Mills field $A_\mu(x)$ and the scalar field R(x) with zero scale dimension which are transformed by the adjoint representation of the local gauge group SU(N) according to the

law

$$A'_\mu(x) = \omega^{-1}(x) A_\mu(x) \omega(x) + \omega^{-1}(x) \partial_\mu \omega(x),$$
$$R'(x) = \omega^{-1}(x) R(x) \omega(x), \tag{2.1}$$

where $\omega(x) \in SU(N)$ and $A_\mu(x) = A^a_\mu(x) t_a$, $S(x) = S^a(x) t_a$ are fields with values on the Lie algebra of SU(N) with generators t_a.

Then consider the following five-component potential

$$\mathcal{A}(x) = \begin{pmatrix} R(x) \\ A_\mu(x) \end{pmatrix}, \tag{2.2}$$

for which, as in paper [1], the following transformation laws with respect to the special conformal transformations

$$\left[\begin{pmatrix} R(x) \\ A_\tau(x) \end{pmatrix}, K_\mu \right] = i \begin{pmatrix} 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu & 0 \\ 2\lambda \cdot g_{\mu\sigma} & [2x_\mu(1 + x^\nu \partial_\nu) - x^2 \partial_\mu] \delta^\rho_\tau + 2ix^\nu (\Sigma_{\mu\nu})^\rho_\tau \end{pmatrix} \begin{pmatrix} R(x) \\ A_\rho(x) \end{pmatrix} \tag{2.3}$$

are supposed. Here $\lambda \neq 0$ is an arbitrary parameter and $(\Sigma_{\mu\nu})^\rho_\tau = i(\delta^\rho_\mu g_{\nu\tau} - \delta^\rho_\nu g_{\mu\tau})$ are generators of the Lorentz group in the four-vector representation. The remaining generators of the conformal group act on the field $\mathcal{A}(x)$ in a standard way. The five-component current

$$J(x) = \begin{pmatrix} D(x) \\ j_\mu(x) \end{pmatrix} \tag{2.4}$$

obeys the following transformation law

$$[J(x), K_\mu] = i \begin{pmatrix} 2x_\mu(4 + x^\nu \partial_\nu) - x^2 \partial_\mu & -2\lambda\, g_{\mu\tau} \\ 0 & [2x_\mu(3 + x^\nu \partial_\nu) - x^2 \partial_\mu] \delta^\rho_\tau + 2ix^\nu (\Sigma_{\mu\nu})^\rho_\tau \end{pmatrix} \begin{pmatrix} D(x) \\ j_\rho(x) \end{pmatrix} \tag{2.5}$$

where D(x) is the scalar field with the scale dimension four.

The two-point function for the five-component potential has the following form

$$\langle \mathfrak{A}(x)\mathfrak{A}^T(y)\rangle_0 = \begin{pmatrix} \langle R(x)R(y)\rangle_0 & \langle R(x)A_\nu(x)\rangle_0 \\ \langle A_\mu(x)R(y)\rangle_0 & \langle A_\mu(x)A_\nu(y)\rangle_0 \end{pmatrix} , \qquad (2.6)$$

where the notation

$$\mathfrak{A}^T(x) = (R(x), A_\mu(x))$$

is used.

The Fourier kernel of the conformal covariant Euclidean two-point Green function is given by (see [1]).

$$(\tilde{G}(p))^{ac}_{bd} = \begin{pmatrix} -\frac{1}{2\lambda^2}\delta^{(4)}(p) & \frac{2i}{\lambda}\frac{p_\nu}{p^2} \\ -\frac{2i}{\lambda}\frac{p_\mu}{p^2} & -\frac{\delta_{\mu\nu}}{p^2} + c\frac{p_\mu p_\nu}{(p^2)^2} \end{pmatrix} (\delta^a_d\delta^c_b - \frac{1}{N}\delta^a_b\delta^c_d), \qquad (2.7)$$

$$(a,b,c,d = 1, \ldots, N).$$

We point out that the corresponding conformal-covariant time-ordered Green function in the Minkowski space M_4 can be found from (2.7) by substituting

$$p_4 \to ip_0, \quad p^2 \to p_0^2 - \underline{p}^2, \quad \delta_{\mu\nu} \to g_{\mu\nu}$$

and the corresponding Whightmann function by the following limit

$$W(x_0, \underline{x}) = \lim_{\epsilon \to 0} G(\underline{x}, -\epsilon - ix_0).$$

Note also that as in the abelian case both the Gree function (2.7) and any of other two-point functions have nonzero transversal parts and that the gauge fixing parameter c is conformal invariant.

The inverse Green function is given by [1]

$$(\tilde{G}^{-1}(p))^{ac}_{bd} = \begin{pmatrix} \frac{\lambda^2(1-c)}{4}(p^2)^2 & \frac{i\lambda}{2}p_\nu p^2 \\ -\frac{i\lambda}{2}p_\mu p^2 & -g_{\mu\nu}p^2 + p_\mu p_\nu \end{pmatrix} (\delta^a_d\delta^c_b - \frac{1}{N}\delta^a_b\delta^c_d). \qquad (2.8)$$

It can be checked that $\tilde{G}^{-1}(p)$ is the Fourier kernel of the conformal covariant two-point Green function of the five-component current (2.4) transforming according to the representation (2.5). As in the abelian case (2.8) is the interweaving operator between the potential (2.3) and current (2.5) representations of the conformal group [1].

As will be seen below it is necessary to introduce the second scalar field $S(x)$ with zero scale dimension which is SU(N) invariant. For this field the conformal symmetry is taken in a weak sense (see [2]), i.e.,

$$[S(x), D] = ix^\nu\partial_\nu S(x) + 1,$$
$$[S(x), K_\mu] = i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)S(x) + 2i r x_\mu, \qquad (2.9)$$

where D is the generator of dilatations and $r \neq 0$ is an arbitrary parameter. It can be checked that the invariant two-point function for the field $S(x)$ is given by [2]

$$\langle S(x)\,S(y)\rangle_0 \sim \ln[\mu^2(x-y)^2],$$

where μ is a parameter with the dimension of mass. Notice that the field $S(x)$ with transformation laws (2.9) was considered in ref. [2] as a scalar partner of the electromagnetic potential when constructing the nontrivial quantum electrodynamics.

We also point out that as in the abelian case [1] the field $R(x)$ has extraordinary properties, i.e., the field $\tilde{R}(x) = R(x) - \langle R\rangle_0$ has the vanishing two-point function, but $\langle\tilde{R}(x)A_\mu(y)\rangle_0 \neq 0$, what is possible if for the field $R(x)$ the nilpotent properties are assumed (see ref. [1]).

3. INVARIANT ACTION

The kinetic part of the Yang-Mills action can be made conformal invariant (with respect to the considered representation), as in the case of electrodynamics [1], by adding to the Lagrangian the following term

$$\mathrm{tr}\{\frac{\lambda}{4}(R\Box\partial^\mu A_\mu - A^\mu\Box\partial_\mu R) + \frac{\lambda^2}{8}(1-c)R\Box^2 R\}. \qquad (3.1)$$

However, the self-interaction part cannot be made invariant with respect to the transformations (2.3) without introducing the second scalar field with zero scale dimension transforming by the law (2.5) and which is SU(N) invariant. Then the conformal invariant action has the following form

$$I = \frac{1}{g^2}\int d^4x \{ tr[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\lambda}{4}(R\Box\partial^\mu A_\mu - A^\mu\Box\partial_\mu R)$$

$$+ \frac{(1-c)\lambda^2}{8} R\Box^2 R] - \frac{\lambda}{2}\partial^\mu Str([D^\nu, R][A_\nu, A_\mu] + (\partial_\mu A_\nu - \partial_\nu A_\mu)[R, A^\nu])$$

$$+ (\frac{\lambda}{r})^2 \partial^\mu S \partial_\mu Str(\partial^\nu R[A_\nu, R] + \frac{1}{2}[R, A^\nu][R, A_\nu]) \qquad (3.2)$$

$$+ (\frac{\lambda}{r})^2 \partial^\mu S \partial^\nu Str(\partial_\nu R[R, A_\mu] + \frac{1}{2}[R, A_\mu][A_\nu, R])\},$$

where $D_\mu = \partial_\mu + A_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. To check the invariance of (3.2) with respect to the special conformal transformation, we point out that the field $S(x)$ takes part in (3.2) only with its derivative $\partial_\mu S$. From (2.9) for the derivative $\partial_\sigma S(x)$ we get

$$[\partial_\sigma S(x), K_\mu] = i\{(2x_\mu + 2x_\mu x^\nu \partial_\nu - x^2\partial_\mu)\partial_\sigma S(x)$$

$$+ 2ix^\nu(\Sigma_{\mu\nu})^\rho_\sigma \partial_\rho S\} + 2ir\, g_{\mu\sigma}. \qquad (3.3)$$

Note that the Yang-Mills Lagrangian can be written down in an equivalent (up to the full divergence) form

$$\mathcal{L}_{YM} = \frac{1}{2} tr\{A^\mu(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu + 2\partial^\mu A^\nu[A_\mu, A_\nu]$$

$$+ A^\mu[A_\nu, A_\mu]A^\nu\} = \mathcal{L}^k_{YM} + \mathcal{L}^1_{YM} + \mathcal{L}^2_{YM}. \qquad (3.4)$$

As has been pointed above, the kinetic term $\mathcal{L}^k_{YM}$ is made conformal-invariant with respect to the special conformal transformations (2.3) by adding the term (3.1), as in the case of electrodynamics /1/. Consider the first of the self-interacting terms in (3.4). The variation of this term by the special conformal transformation is given by

$$\delta\mathcal{L}^1_{YM} = 8(x\delta a)\mathcal{L}^1_{YM} + 2\lambda\delta a^\rho\, tr\{\partial^\mu R[A_\mu, A_\rho]$$

$$+ (\partial_\rho A_\mu - \partial_\mu A_\rho)[R, A^\mu]\}, \qquad (3.5)$$

where a_μ is a parameter of the special conformal transformations. As is shown in ref. /1/, the first term in (3.5) is compensated by the Jacobian of the special conformal transformations. Note

that as follows from (2.3), the field R and its derivative $\partial_\mu R$ are transformed as the basic fields. It can be checked that the second derivative $\partial_\mu\partial_\nu R$ is transformed as a nonbasic field, however, the term $\partial^\mu\partial^\nu[A_\mu, A_\nu] = 0$. Consequently, it is impossible to construct the conformal invariant Lagrangian without introducing one additional field. As such a field, one can introduce the scalar zero-scale-dimensional field transforming under the special conformal transformation by the law (2.9) and which is invariant with respect to the local gauge transformations (2.1), i.e.,

$$S'(x) = S(x).$$

Then to $\mathcal{L}^1_{YM}$ the term $\mathcal{L}^3_{YM}$ is added, which is chosen so that its variation by the special conformal transformation compensates the second term in (3.5):

$$\mathcal{L}^3_{YM} = -2\frac{\lambda}{r}\partial^\mu Str\{\partial^\nu R[A_\nu, A_\mu] + (\partial_\mu A_\nu - \partial_\nu A_\mu)[R, A^\nu]\}. \qquad (3.6)$$

Then from (2.3) and (3.3) we have

$$\delta\mathcal{L}^3_{YM} = -4\lambda\delta a^\rho tr\{\partial^\nu R[A_\nu, A_\rho] + (\partial_\rho A_\nu - \partial_\nu A_\rho)[R, A^\nu]\}$$

$$-4\frac{\lambda^2}{r}\delta a^\rho(\partial_\rho Str\{2\partial^\nu R[A_\nu, R]\} \qquad (3.7)$$

$$+ \partial^\mu Str\{\partial_\rho R[R, A_\mu] + \partial_\mu R[R, A_\rho]\}) + 8(x\delta a)\mathcal{L}^3_{YM}.$$

The first term in (3.7) cancels with the first term of (3.5). To compensate the second term of (3.7), it is necessary to add also the term quadratic in field S, i.e.,

$$\mathcal{L}^5_{YM} = 2(\frac{\lambda}{r})^2\{\partial^\mu S\partial_\mu Str(\partial^\nu R[A_\nu R])$$

$$+ \partial^\mu S\partial^\nu Str(\partial_\nu R[R, A_\mu])\} \qquad (3.8)$$

the variation of which is equal (up to the sign) to the second term of (3.7).

Analogously, we introduce the terms compensating the variation of $\mathcal{L}^2_{YM} = \frac{1}{2}tr\{A^\mu[A_\nu, A_\mu]A^\nu\}$. Thus we have proved the invariance of the action (3.2) with respect to special conformal transformations under consideration. The invariance of this action under the Lorentz and dilatational transformations is evident.

At the end of this section, we notice that the action (3.2), on the whole, is not gauge-invariant. As in the abelian case [1], the term (3.1) making the kinetic part of the action conformal invariant is the gauge term. Other interaction terms with the fields R and S differ from the "ghost" Faddeev-Popov fields terms [6]. In more detail, this question will be considered in the next paper.

4. EQUATION OF MOTION

Varying the action (3.2) with respect to the Yang-Mills field A_μ we get the following equations

$$
\begin{aligned}
\mathcal{A}_\mu(A, R, S) &= (g_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu(x) - \partial^\nu[A_\nu, A_\mu] \\
&+ [A^\nu, \partial_\mu A_\nu - \partial_\nu A_\mu] + [[A_\nu, A_\mu], A^\nu] - \frac{\lambda}{2}\Box\partial_\mu R \\
&- \frac{\lambda}{r}\{\partial_\mu S[D^\nu, R]A_\nu - \partial^\nu S A_\nu[D_\mu, R] + \partial^\nu S[R, [A_\mu, A_\nu]] \\
&+ \partial^\nu S[\partial_\nu A_\mu - \partial_\mu A_\nu, R] + \partial^\nu(\partial_\mu S[R, A_\nu] - \partial_\nu S[R, A_\mu])\} \\
&+ (\frac{\lambda}{r})^2\{\partial^\nu S\partial_\nu S([R, \partial_\mu R] + [[R, A_\mu], R]) \\
&+ \partial_\mu S\partial^\nu S([\partial_\nu R, R] + [[A_\nu, R], R])\} = 0.
\end{aligned} \tag{4.1}
$$

Varying with respect to the field R we have

$$
\begin{aligned}
\tilde{\mathcal{R}}(A, R, S) &= \frac{\lambda}{2}\Box\partial^\mu A_\mu + \frac{1-c}{4}\lambda^2\Box^2 R - \frac{\lambda}{2}\partial^\mu S\{[D^\nu, [A_\mu, A_\nu]] \\
&+ [A^\nu, \partial_\mu A_\nu - \partial_\nu A_\mu]\} - (\frac{\lambda}{r})^2\partial^\nu\{\partial^\mu S\partial_\mu S[A_\nu, R] \\
&+ \partial^\mu S\partial^\nu S[R, A_\nu]\} + (\frac{\lambda}{r})^2\{\partial^\mu S\,\partial_\mu S[A^\nu, [R, A_\nu]] \\
&+ \partial^\mu S\partial^\nu S[A_\nu, [R, A_\mu]]\} = 0.
\end{aligned} \tag{4.2}
$$

And finally, varying with respect to the scalar field S(x) we get

$$
\begin{aligned}
\mathcal{S}(A, R, S) &= \partial^\mu \mathrm{tr}\{[D^\nu, R][A_\nu, A_\mu] + \\
&+ (\partial_\mu A_\nu - \partial_\nu A_\mu)[R, A^\nu]\} - 2\frac{\lambda}{r}\partial^\mu\{\partial_\mu S\,\mathrm{tr}(\partial^\nu R[A_\nu, R] \\
&+ \frac{1}{2}[R, A^\nu][R, A_\nu]\} - \frac{\lambda}{r}\partial^\mu(\partial^\nu S\,\mathrm{tr}\{\partial_\nu R[R, A_\mu]\}) \\
&- \frac{\lambda}{r}\partial^\nu(\partial^\mu S\,\mathrm{tr}[\partial_\nu R, [R, A_\mu]]) \\
&- \frac{\lambda}{r}\partial^\mu\{\partial_\mu S\,\mathrm{tr}([R, A^\nu][R, A_\nu]) + \partial^\nu S\,\mathrm{tr}([R, A_\mu][A_\nu, R])\} = 0.
\end{aligned} \tag{4.3}
$$

Note that in the Lagrangian (3.2) there is no kinetic term of the field S, and consequently, this field is considered as the Lagrange multiplier. Taking the divergence of eq. (4.1), we get

$$
\begin{aligned}
&-\frac{\lambda}{2}\Box^2 R + \partial^\mu[A^\nu, \partial_\mu A_\nu - \partial_\nu A_\mu] + \partial^\mu[[A_\nu, A_\mu], A^\nu] \\
&- \frac{\lambda}{r}\partial^\mu\{\partial_\mu S[D^\nu, R]A_\nu - \partial^\nu S A_\nu[D_\mu, R] + \partial^\nu S[R, [A_\mu, A_\nu]] \\
&+ \partial^\nu S[\partial_\nu A_\mu - \partial_\mu A_\nu, R]\} + (\frac{\lambda}{r})^2\partial^\mu\{\partial^\nu S\partial_\nu S([R, \partial_\mu R] \\
&+ [[R, A_\mu], R])\} + (\frac{\lambda}{r})^2\partial^\mu\{\partial_\mu S\partial^\nu S([\partial_\nu R, R] \\
&+ [[A_\nu, R], R])\} = 0.
\end{aligned} \tag{4.4}
$$

Unlike the abelian case where the field R satisfies the free-field equation $\Box^2 R = 0$ [1], here this field satisfies eq. (4.4). Determining $\Box^2 R$ from (4.4) and substituting it into eq. (4.2) we find

$$
\begin{aligned}
\mathcal{R}(A, R, S) &= \frac{\lambda}{2}\Box\partial^\mu A_\mu - \frac{\lambda}{r}\partial^\mu S\{[D^\nu, [A_\mu, A_\nu]] \\
&+ [A^\nu, \partial_\mu A_\nu - \partial_\nu A_\mu]\} - (\frac{\lambda}{r})^2\partial^\nu\{\partial^\mu S\partial_\mu S[A_\nu, R] \\
&+ \partial^\mu S\partial^\nu S[R, A_\nu]\} + (\frac{\lambda}{r})^2\{\partial^\mu S\partial_\mu S[A^\nu, [R, A_\nu]] + \\
&+ \partial^\mu S\partial^\nu S[A_\nu, [R, A_\mu]]\} + \frac{\lambda^2(1-c)}{2}(\partial^\mu[A^\nu, \partial_\mu A_\nu - \partial_\nu A_\mu] \\
&+ \partial^\mu[[A_\nu, A_\mu], A^\nu] - \frac{\lambda}{r}\partial^\mu\{\partial_\mu S[D^\nu, R]A_\nu - \partial^\nu S A_\nu[D_\mu, R] \\
&+ \partial^\nu S[R, [A_\mu, A_\nu]] + \partial^\nu S[\partial_\nu A_\mu - \partial_\mu A_\nu, R]
\end{aligned} \tag{4.5}
$$

$$+(\frac{\lambda}{r})^2 \partial^\mu\{\partial^\nu S\partial_\nu S([R,\partial_\mu R]+[[R,A_\mu],R])\}$$

$$+(\frac{\lambda}{r})^2 \partial^\mu\{\partial_\mu S\partial^\nu S([\partial_\nu R,R]+[[A_\nu,R],R])\})=0.$$

Consequently, for the nontrivial conformal-invariant Yang-Mills model considered here the field equations are given by (4.1) and eqs. (4.3) and (4.4) are subsidiary conditions fixing the gauge. As a consequence of the conformal invariance of the action (3.2) eqs. (4.1-5) are conformal-covariant also.

A canonical-quantization procedure for the nontrivial conformal-invariant model differing from the correspoinding procedure considered in paper [6] will be considered in the third part of the present paper. Notice that the subsidiary conditions (4.4) and (4.5), by which the physical subspace in the state space is separated, are compatible with the equations of motion (4.1).

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Зайков Р.П. E2-83-44
О конформной инвариантности в калибровочных теориях. Поля Янга-Миллса

В этой работе дается обобщение результатов предыдущей работы [1] для неабелевого случая. Из предположения, что поля Янга-Миллса вместе со скалярным полем нулевой масштабной размерности преобразуются по неосновному представлению конформной группы, получаем, что инвариантный пропагатор имеет ненулевую поперечную часть. Вышеуказанные поля преобразуются также по присоединенному представлению локальной калибровочной группы SU(N). Для построения инвариантного действия необходимо ввести второе скалярное поле с нулевой размерностью, являющееся SU(N) инвариантом. Получены уравнения движения и соответствующие дополнительные условия, которые совместимы с уравнениями движения.

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Zaikov R.P. E2-83-44
On Conformal Invariance in the Gauge Theories. Yang-Mills Theory

In the present paper the results of the first part [1] are generalized to the nonabelian case. From the assumption that the Yang-Mills fields together with the one scalar zero-dimensional fields are transformed by the nonbasic representation of the conformal group, it follows that the covariant propagator has a nonzero transversal part. These fields are transformed also by an adjoint representation of a local gauge group SU(N). For the construction of an invariant action it is necessary to introduce a second zero-dimensional scalar field which is SU(N) invariant. There are derived the corresponding equations of motion as well as subsidiary conditions which are compatible with the equations of motion.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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