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ядерных
исследований
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A.V.Koudinov, M.A.Smondyrev

# STRONG COUPLING EXPANSION FOR THE ANHARMONIC OSCILLATOR

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1. In this paper we investigate a strong coupling expansion for the quantum-mechanical O(N)-symmetric oscillator with an arbitrary power anharmonicity. The Hamiltonian of the system is as follows:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{m^2}{2}\sum_{i=1}^{N} x_i^2 + \frac{g}{N^{n-1}}\left(\sum_{i=1}^{N} x_i^2\right)^n . \tag{1.1}$$

Simple dimensional arguments allow us to guess quite easily the general form of the expansion, say, for the ground-state energy:

$$E_0 = g^{1/(n+1)}\left[d_0 + \sum_{k=1}^{\infty} d_k \left(m^2/g^{2/(n+1)}\right)^k\right]. \tag{1.2}$$

The strong coupling expansion has evident advantage as compared to the conventional perturbation theory in powers of g. Simon has shown [1] that the expansion (1.2) converges for large g in contrast with the asymptotic perturbation series. The strong coupling expansion works equally well for the potentials with positive and negative $m^2$. Several first terms of the expansion (1.2) can be used to evaluate approximately the energy levels in a wide range of coupling constant.

Unfortunately, a consistent construction of the strong coupling expansion is not yet achieved even in quantum mechanics, to say nothing of quantum field theory. To calculate approximately the coefficients $d_k$, different approaches have been used, from the traditional variational methods to fashionable lattice approximations of path integrals.

The most straightforward way is to compare the expansion (1.2) with the exact values of energy levels computed numerically. Thus, Hioe, MacMillen and Montroll [2,3] have considered one-dimensional oscillators (N = 1) with quartic, sextic, and octic anharmonicities (i.e. n = 2,3,4). They found the coefficients $d_0$, $d_1$, and $d_2$ of the expansion (1.2) for the ground-state and excited energy levels.

In the well-known series of papers (see, e.g., ref. [4]) Bender and coauthors made an attempt to construct a strong coupling expansion starting with the lattice approximation of path integrals, which is equally applicable in quantum field theory. However, additional dimensional parameter (the lattice spacing a) distorts the general form (1.2): the energy is expanded now in wrong powers of g. Moreover, the series thus obtained has no appropriate limit when a tends to zero. So, to keep the energy finite in the continuum limit, one has to use rather sophisticated procedures, for example, the renormalization of the coupling constant even in quantum mechanics. Things look unsuitable to us.

In our opinion, it is more consistent to construct the strong coupling expansion with the help of $1/N$-expansion. In a previous paper /5/ we have obtained analytically six coefficients of the $1/N$-expansion for the ground energy level of the oscillator with the Hamiltonian (1.1). When this series is reexpanded in the limit of large g, the expansion (1.2) with correct powers of the coupling constant is generated automatically. Each strong coupling coefficient is represented then as an asymptotic power series in $1/N$ and below $d_0 \div d_3$ are found up to the order $N^{-4}$. To sum these asymptotic power series, we use the Pade-method taking into account the behaviour of the sum when $N \to 0$. This enables us to calculate the coefficients of the strong coupling expansion with high accuracy.

Applications to multidimensional quartic, sextic, and octic oscillators are easy and provide a number of strong coupling formulae.

A simple relation between the ground and first excited energy levels of different oscillators allows us to obtain the strong-coupling expansion for the first excited energy level also. The comparison with numerical results demonstrates that these formulae can be successfully applied to calculate energy levels in a wide range of coupling constant. The strong coupling expansion fails only for rather small values of g. But even here we can get proper results using Pade-approximations, that points to the self-consistency of considerations.

2. The $1/N$ -expansion for the ground-state energy of the oscillator with the Hamiltonian (1.1) is of the form:

$$E_0/\omega = N\epsilon_0(\lambda) + \sum_{\ell=0}^{\infty} \epsilon_{\ell+1}(\lambda)/N^{\ell} \; ; \qquad (2.1)$$

where $\lambda$ is a dimensionless coupling constant and $\omega$ is a characteristic energy scale of the system defined by the equations:

$$\frac{m^2}{\omega^2} = 1 - \frac{4n}{2^n}\lambda; \quad \lambda = \frac{g}{\omega^{n+1}}; \qquad (2.2)$$

The coefficients $\epsilon_0(\lambda) \div \epsilon_5(\lambda)$ have been found analytically in ref. /5/. In the limit of large g the solutions of eqs. (2.2) can be obtained as a series in powers of a small parameter $\Delta = m^2/g^{2/(n+1)}$:



$$\lambda = \frac{2^n}{4n}\Big[1 - \frac{4}{(8n)^{2/(n+1)}}\Delta + \frac{32}{(n+1)(8n)^{4/(n+1)}}\Delta^2 + \frac{64}{(n+1)^2(8n)^{6/(n+1)}}\Delta^3 + \ldots\Big]; \qquad (2.3)$$

$$\omega = \frac{1}{2}(8ng)^{1/(n+1)}\Big[1 + \frac{4}{(n+1)(8n)^{2/(n+1)}}\Delta + \frac{8(n-2)}{(n+1)^2(8n)^{4/(n+1)}}\Delta^2 + \frac{32(2n^3 - 11n + 12)}{3(n+1)^3(8n)^{6/(n+1)}}\Delta^3 + \ldots\Big].$$

Then we get the expansion (1.2) for the ground-state energy, where the coefficients $d_k$ are represented as asymptotic power series in $1/N$:

$$d_k = d_{k,0} \cdot N + \sum_{\ell=0}^{\infty} d_{k,\ell+1}/N^{\ell}. \qquad (2.4)$$

The coefficients $d_{k,\ell}$ are found by substituting the expansions (2.3) into the analytical expressions for $\epsilon_k(\lambda)$ from ref. /5/. We give here six coefficients of the expansion (2.4) for $d_k$ with $k = 0,1,2,3$ computed by means of SCHOONSHIP. General expressions for $d_{k,4}$ and $d_{k,5}$ are rather cumbrous to be written out here. So, we give only their values for $n = 2,3,4$.

$$d_0 = d_{0,0} \cdot N + \sum_{\ell=1}^{4} d_{0,\ell+1}/N^{\ell}; \qquad (2.5)$$

$$d_{0,0} = (8n)^{1/(n+1)}\frac{n+1}{8n};$$

$$d_{0,1} = (8n)^{1/(n+1)}\Big[\frac{1}{2}\sqrt{\frac{n+1}{2}} - \frac{1}{2}\Big];$$

$$d_{0,2} = (8n)^{1/(n+1)}\frac{n-1}{n+1}\Big[(-2n^2 + 15n + 53)/72 - \sqrt{\frac{n+1}{2}}\Big];$$

$$d_{0,3} = (8n)^{1/(n+1)}\frac{n-1}{(n+1)^2}\Big[(-2n^2 + 15n + 29)/18 + \sqrt{\frac{n+1}{2}}(4n^4 + 4n^3 + 45n^2 - 76n - 985)/432\Big];$$

$n = 2$ $\quad d_{0,4} = -0.028863732; \quad d_{0,5} = -0.169597632;$

$n = 3 \quad d_{0,4} = -0.873982226\,; \quad d_{0,5} = 0.634055252;$

$n = 4 \quad d_{0,4} = -4.207310062; \quad d_{0,5} = 10.64877473;$

$$d_1 = d_{1,0} \cdot N + \sum_{\ell=0}^{4} d_{1,\ell+1} / N^{\ell}\,; \tag{2.6}$$

$$d_{1,0} = \frac{1}{2(8n)^{1/(n+1)}}\,;$$

$$d_{1,1} = \frac{1}{(n+1)(8n)^{1/(n+1)}}\left[-2 + \sqrt{\frac{n+1}{2}}(3-n)\right];$$

$$d_{1,2} = \frac{n-1}{(n+1)^2(8n)^{1/(n+1)}}\left[(8n^2 - 5n - 25)/6 + 2\sqrt{\frac{n+1}{2}}(3-n)\right];$$

$$d_{1,3} = \frac{n-1}{(n+1)^3(8n)^{1/(n+1)}}\left[(16n^3 - 10n^2 - 34n)/3 + \sqrt{\frac{n+1}{2}}(4n^5 - 8n^4 - 711n^3 - 703n^2 + 3443n - 9)/216\right];$$

$n = 2 \quad d_{1,4} = -0.109684358; \quad d_{1,5} = 0.397479995\,;$

$n = 3 \quad d_{1,4} = -0.872149956; \quad d_{1,5} = 6.963762885\,;$

$n = 4 \quad d_{1,4} = -4.147755390; \quad d_{1,5} = 48.31375163\,;$

$$d_2 = d_{2,0} \cdot N + \sum_{\ell=0}^{4} d_{2,\ell+1} / N^{\ell}\,; \tag{2.7}$$

$$d_{2,0} = -\frac{1}{(n+1)(8n)^{3/(n+1)}}\,;$$

$$d_{2,1} = \frac{1}{(n+1)(8n)^{3/(n+1)}}\left[4(2-n) - \sqrt{\frac{n+1}{2}}(n^2 - 10n + 13)\right];$$

$$d_{2,2} = \frac{n-1}{(n+1)^3(8n)^{3/(n+1)}}\left[(50n^3 - 233n^2 - 113n + 494)/9 - \sqrt{\frac{n+1}{2}}(6n^2 - 60n + 78)\right];$$



$$d_{2,3} = \frac{n-1}{(n+1)^4(8n)^{3/(n+1)}}\left[(400n^4 - 2064n^3 + 604n^2 + 2964n - 1400)/9 + \sqrt{\frac{n+1}{2}}(4n^6 - 36n^5 - 2743n^4 + 8254n^3 + 16000n^2 - 41338n + 15827)/72\right];$$

$n = 2 \quad d_{2,4} = 0.139486868\,; \quad d_{2,5} = -0.189125736\,;$

$n = 3 \quad d_{2,4} = 1.784398735\,; \quad d_{2,5} = -2.528634057\,;$

$n = 4 \quad d_{2,4} = 8.569686490\,; \quad d_{2,5} = -2.093925700\,;$

$$d_3 = d_{3,0} \cdot N + \sum_{\ell=0}^{4} d_{3,\ell+1} / N^{\ell}\,;$$

$$d_{3,0} = -\frac{4(n-4)}{3(n+1)^2(8n)^{5/(n+1)}}\,; \tag{2.8}$$

$$d_{3,1} = \frac{1}{(n+1)^3(8n)^{5/(n+1)}}\left[(-32n^2 + 176n - 192)/3 + \sqrt{\frac{n+1}{2}}(-6n^3 + 110n^2 - 386n + 330)/3\right];$$

$$d_{3,2} = \frac{n-1}{(n+1)^4(8n)^{5/(n+1)}}\left[(608n^4 - 6056n^3 + 9580n^2 + 13076n - 21168)/27 + \sqrt{\frac{n+1}{2}}(-60n^3 + 1100n^2 - 3860n + 3300)/3\right];$$

$$d_{3,3} = \frac{n-1}{(n+1)^5(8n)^{5/(n+1)}}\left[(7296n^5 - 77536n^4 + 174928n^3 + 9232n^2 - 247264n + 123264)/27 + \sqrt{\frac{n+1}{2}}(60n^7 - 1040n^6 - 89085n^5 + 683225n^4 - 582910n^3 - 3024870n^2 + 5225215n - 2089635)/324\right];$$

$n = 2 \quad d_{3,4} = -0.047456455\,; \quad d_{3,5} = -0.148399675\,;$

$n = 3 \quad d'_{3,4} = 0.585407615; \quad d_{3,5} = -9.752809560;$

$n = 4 \quad d_{3,4} = 9.933767530; \quad d_{3,5} = -106.1521824.$

We stress once more that $1/N$ -expansion presented above is asymptotic. Thus, $d_{0,\ell}$ are proportional to $\epsilon_\ell(\lambda)$ when $m^2 = 0$ (or $\lambda = \frac{2^n}{4n}$ ), and the asymptotics of $\epsilon_\ell(\lambda)$ for $\ell \to \infty$ is described in ref.[5].

3. Before turning to the summation of the $1/N$ -series for the strong-coupling coefficients $d_k$, let us determine the asymptotics of the ground state energy in the limit $N \to 0$. Consider first an anharmonic oscillator with a slightly different choice of the coupling constant: $g' = g/N^{n-1}$. The ground state energy is then defined by the Schrödinger equation for the radial part of the wave function, which can be brought into the form:

$$\frac{d^2\chi}{d\rho^2} - \frac{1}{4}[m^2 + 2g'\rho^{n-1} + \frac{N(N-4)}{4\rho^2} - \frac{2E_0}{\rho}]\chi = 0; \tag{3.1}$$

$$\rho = r^2; \quad \chi(\rho) = \rho^{-N/4}R(\rho).$$

Dolgov, Eletsky and Popov[6] have shown that for $N \to 0$ and $g'$ fixed the ground state energy tends to zero, namely

$$E_0 = N\epsilon(g'); \quad N \to 0.$$

Thus in eq. (3.1) both the centrifugal term and the term with energy are negligible and in some cases exact solutions can be found. Henceforth the function $\chi$ will be interpreted as a solution of eq. (3.1) in the limit $N \to 0$:

$$\frac{d^2\chi}{d\rho^2} - \frac{1}{4}[m^2 + 2g'\rho^{n-1}]\chi = 0; \qquad \chi \underset{\rho \to \infty}{\to} 0. \tag{3.2}$$

In ref.[6] it is shown that

$$\epsilon(g') = -\frac{1}{\chi}\frac{d\chi}{d\rho}\Big|_{\rho=0}; \tag{3.3}$$

and there the exact solutions for $\chi(\rho)$ and $\epsilon(g')$ are given also for $n = 2,3$. We need only the strong coupling expansions for these quantities. So, after the scale transformation $\rho \to \rho/(g')^{1/(n+1)}$ we get from eqs. (3.2), (3.3):



$$\frac{d^2\chi}{d\rho^2} - \frac{1}{4}[\frac{m^2}{(g')^{2/(n+1)}} + 2\rho^{n-1}]\chi = 0; \tag{3.4}$$

$$\epsilon(g') = (g')^{1/(n+1)}[-\frac{1}{\chi}\frac{d\chi}{d\rho}]\Big|_{\rho=0}.$$

It is evident that the expansion for $\epsilon(g')$ is of the form:

$$\epsilon(g') = (g')^{1/(n+1)}[c_0 + c_1\Delta' + c_2(\Delta')^2 + c_3(\Delta')^3 + \dots];$$

$$\Delta' = m^2/(g')^{2/(n+1)}.$$

It makes no difficulty to obtain the coefficient $c_0$: when $\Delta' = 0$, the solution of the eq. (3.4) is as follows:

$$\chi_0 \sim \sqrt{\rho}\, K_{\frac{1}{n+1}}\left(\frac{\sqrt{2}\rho^{\frac{n+1}{2}}}{n+1}\right) \tag{3.5}$$

and we have immediately:

$$c_0 = \frac{(n+1)^{\frac{n-1}{n+1}}}{2^{1/(n+1)}} \frac{\Gamma(\frac{n}{n+1})}{\Gamma(\frac{1}{n+1})}. \tag{3.6}$$

One can also calculate the coefficient $c_1$ (see Appendix A):

$$c_1 = \frac{1}{2^{\frac{2n+1}{n+1}}(n+1)^{\frac{n-1}{n+1}}} \frac{\Gamma^2(\frac{2}{n+1})\Gamma(\frac{3}{n+1})}{\Gamma(\frac{1}{n+1})\Gamma(\frac{4}{n+1})}. \tag{3.7}$$

As to the coefficients $c_2$ and $c_3$, it is rather difficult to find them in a general form, but in particular cases of $n = 2,3$ we find them from the exact solutions of ref.[6].

Now we return to our former notation:

$$g' \to g/N^{n-1}; \quad \Delta' \to \Delta \cdot N^{\frac{2(n-1)}{n+1}}; \quad E = N\epsilon(g').$$

The strong-coupling expansion for N tending to zero is then of the form:

$$E_0 \simeq g^{1/(n+1)} \sum_{k=0} c_k \Delta^k N^{\left[\frac{n-1}{n+1}(2k-1)+1\right]};$$

and the asymptotics of the coefficients $d_k$ is:

$$\frac{d_k}{N} \simeq c_k N^{\frac{n-1}{n+1}(2k-1)} . \tag{3.8}$$

The main goal of this section is to ascertain the asymptotics (2.8) and to find the coefficients $c_k$ (3.6), (3.7). In the next section this information will be used to choose an adequate method for summing up asymptotic 1/N -expansion (2.4).

4. The strong coupling coefficients were written down above as series in powers of 1/N:

$$\frac{d_k}{N} = d_{k,0} + \sum_{\ell=1}^{5} d_{k,\ell}/N^\ell . \tag{4.1}$$

To sum this series we use the Padé-approximation. Bearing in mind that the coefficients $d_{k,\ell}$ are obtained for six different values of $\ell$ ($\ell$ = 0,1,...,5) and the asymptotic behaviour (3.8), we take the Padé-approximation to be of the form:

$$\frac{d_k}{N} = \left[\frac{a_0 + a_1/N + a_2/N^2}{1 + \beta_1/N + \beta_2/N^2 + \beta_3/N^3}\right]^{\frac{n-1}{n+1}(2k-1)} . \tag{4.2}$$

It is natural that the accuracy of the approximation is increased with increasing N, and the coefficients $c_k$ are approximated by the expression $(a_2/\beta_3)^{\frac{n-1}{n+1}(2k-1)}$ with the least accuracy. The discrepancy between the exact and approximate values of $c_k$ is an intrinsic criterium of the applicability of our method.

Let us note also that one can easily obtain the strong coupling expansion for the first excited energy level, when taking into account its connection with the ground energy level of the oscillator with another number of components and scaled coupling constant:



$$E_1(N, g) = E_0(N+2, g(1+2/N)^{n-1}) . \tag{4.3}$$

Numerical results for different oscillators obtained by the Pade-approximation (4.2) with the formulae (2.5)-(2.8) are collected in Tables 1.1-3.3. There one can also find the exact values of the coefficients $c_k$ and the values of the strong coupling coefficients computed by Hioe, Mac Millen and Montroll [2,3]. These are placed for the purpose of comparison. Note that we have not found any references with the data concerning the strong coupling expansions for the multidimensional oscillators with N > 1.

A. $V(r) = \frac{m^2}{2} r^2 + \frac{g}{N} r^4$

Table 1.1.

| k | $C_k$ | $C_k$ (exact value) |
|---|---|---|
| 0 | 0,578596 | 0,578617 |
| 1 | 0,167432 | 0,167399 |
| 2 | -0,014124 | -0,014070 |
| 3 | 0,002028 | 0,001957 |

Table 1.2. N = 1

| | k | $d_k$ | $d_k$ (Hioe et al.) |
|---|---|---|---|
| Eo | 0 | 0,667982 | 0,667986259 |
| | 1 | 0,143674 | 0,14367 |
| | 2 | -0,008634 | -0,0088 |
| | 3 | 0,000824 | - |
| E1 | 0 | 2,393643 | 2,39364402 |
| | 1 | 0,357804 | 0,35780 |
| | 2 | -0,014372 | -0,0140 |
| | 3 | 0,000866 | - |

Table 1.3

| | k | $d_k$ (N=2) | $d_k$ (N=3) |
|---|---|---|---|
| Eo | 0 | 1,172413 | 1,659659 |
| | 1 | 0,325472 | 0,516043 |
| | 2 | -0,024695 | -0,043116 |
| | 3 | 0.002872 | 0.005402 |
| E1 | 0 | 2.697113 | 3.104899 |
| | 1 | 0.563234 | 0.763187 |
| | 2 | -0.031267 | -0.049472 |
| | 3 | 0.002561 | 0.004674 |

B. $V(r) = \frac{m^2}{2} r^2 + \frac{g}{N^2} r^6$

Table 2.1.

| k | $C_k$ | $C_k$ (exact value) |
|---|---|---|
| 0 | 0,568557 | 0,568428 |
| 1 | 0,157698 | 0,157841 |
| 2 | -0,010669 | -0,010627 |
| 3 | 0,001225 | 0,001106 |

Table 2.2. N=1.

| | k | $d_k$ | $d_k$ (Hioe et al.) |
|---|---|---|---|
| Eo | 0 | 0.680739 | 0.680707 |
| | 1 | 0.129433 | 0.12939 |
| | 2 | -0.005519 | -0.0052 |
| | 3 | 0.000340 | - |
| E1 | 0 | 2.579763 | 2.57975 |
| | 1 | 0.301964 | 0.30193 |
| | 2 | -0.007254 | -0.0071 |
| | 3 | 0.000212 | - |



Table 2.3

| | k | $d_k$ (N=2) | $d_k$ (N=3) |
|---|---|---|---|
| $E_o$ | 0 | 1.097131 | 1.489427 |
| | 1 | 0.317843 | 0.523017 |
| | 2 | -0.019598 | -0.037695 |
| | 3 | 0.001578 | 0.003298 |
| $E_1$ | 0 | 2.648199 | 2.906163 |
| | 1 | 0.520246 | 0.737824 |
| | 2 | -0.020422 | -0.036666 |
| | 3 | 0.000919 | 0.001993 |

C. $V(r) = \frac{m^2}{2} r^2 + \frac{g}{N^3} r^8$

Table 3.1.

| k | $C_k$ | $C_k$ (exact value) |
|---|---|---|
| 0 | 0.581006 | 0.579859 |
| 1 | 0.149223 | 0.149889 |
| 2 | -0.008662 | - |
| 3 | 0.000602 | - |

Table 3.2. N=1.

| | k | $d_k$ | $d_k$ (Hioe et al.) |
|---|---|---|---|
| $E_o$ | 0 | 0.704438 | 0.70405 |
| | 1 | 0.120458 | 0.12005 |
| | 2 | -0.004167 | -0.0039 |
| | 3 | 0.000168 | - |
| $E_1$ | 0 | 2.731769 | 2.7315 |
| | 1 | 0.272984 | 0,2730 |
| | 2 | -0.004866 | -0.0047 |
| | 3 | 0.000085 | - |

Table 3.3

| | k | $d_k$ (N=2) | $d_k$ (N=3) |
|---|---|---|---|
| $E_0$ | 0 | 1.072125 | 1.413095 |
| | 1 | 0.312195 | 0.527727 |
| | 2 | -0.017095 | -0.035155 |
| | 3 | 0.001008 | 0.002305 |
| $E_1$ | 0 | 2.642749 | 2.810516 |
| | 1 | 0.497992 | 0.727527 |
| | 2 | -0.016078 | -0.031255 |
| | 3 | 0.000467 | 0.001106 |

5. The strong coupling expansion fails to work at small values of the coupling constant. When $g > 0.1$ ($m^2 = 1$), our strong coupling formulae approximate the energy levels with an accuracy from $10^{-4}\%$ to $10^{-2}\%$. In the case of still smaller g the conventional perturbation theory works and the asymptotics of energy levels for $g \to 0$ is a common knowledge: $E_0 \to 0.5$ m; $E_1 \to 1.5$ m.

We can use one more Padé-approximation to continue the strong coupling expansion to the point $g = 0$. The appropriate form of Padé-approximation is as follows:

$$E = \frac{m}{\sqrt{\Lambda}}(d_0 + d_1\Lambda + d_2\Lambda^2 + d_3\Lambda^3) \simeq$$

$$\sim E_{as}\left[\frac{1 + \alpha_1\Lambda + \alpha_2\Lambda^2}{1 + \beta_1\Lambda}\right]^{1/2} .$$

The approximate values of asymptotic energies are presented in Table 4.1 for one-dimensional oscillators.

These values demonstrate the self-consistency of the method proposed for evaluating the coefficients of the strong coupling expansion.

To conclude, we would like to stress once more that the strong coupling coefficients are found with an extremely high accuracy especially when taking into consideration that only six terms of the 1/N -power series were used. Thus, in the case of quantum mechanics the problem of constructing the strong



Table 4.1 $g = 0$

| | n = 2 | n = 3 | n = 4 |
|---|---|---|---|
| $E_0/m$ | 0.502 | 0.514 | 0.517 |
| $E_1/m$ | 0.51 | 1.56 | 1.59 |

coupling expansion is solved practically, since our method is self-consistent and needs neither any numerical fits nor sophisticated summation procedure. We reckon that the only but rather essential defect of the method is that it cannot be transferred into the quantum field theory where the problem of strong coupling remains still unsolved.

APPENDIX A

Let us pass from eq. (3.4) to the Ricatti equation:

$$f(\rho) = -\frac{1}{\chi(\rho)}\frac{d\chi}{d\rho}\ ; \quad \epsilon(g') = (g')^{1/(n+1)} f(\rho = 0); \qquad (A.1)$$

$$f'(\rho) = f^2(\rho) - \frac{1}{4}[\Delta' + 2\rho^{n-1}]\ ;$$

when $\Delta' = 0$ the function $f(\rho)$ thus introduced turns into:

$$f_0(\rho) = -\frac{d}{d\rho}\ln\chi_0(\rho) .$$

Define now the function:

$$F(\rho) = \frac{df(\rho)}{d\Delta'} \ . \qquad (A.2)$$

Differentiating (A.1) with respect to $\Delta'$, we get the equation:

$$\frac{dF}{d\rho} = 2f(\rho)F(\rho) - 1/4;$$

the solution of which has the form

$$F(\rho) = \frac{1}{4}\int_\rho^\infty ds \exp[2\int_s^\rho f(t)dt].$$

From the definition of the function $F(\rho)$ (see eq. (A.2)) it is evident that when $\Delta'$ tends to zero $c_1 = F(\rho=0)$. Taking into account that in this limit $f(\rho) \to f_0(\rho)$, we obtain:

$$c_1 = \frac{1}{4\chi_0^2(0)} \int_0^\infty ds\, \chi_0^2(s); \qquad (A.3)$$

Using the expression (3.5), one can find:

$$\chi_0(0) \sim \frac{2^{\nu/2-1}}{\nu^\nu} \Gamma(\nu); \quad \nu = \frac{1}{n+1};$$

and

$$c_1 = \frac{\nu^{2\nu}}{2^\nu \Gamma^2(\nu)} \int_0^\infty ds\, s\, K_\nu^2(\sqrt{2}\, \nu \cdot s^{1/2\nu}). \qquad (A.4)$$

While evaluating the integral in eq. (A.4) we used the formula:

$$\int_0^\infty dx\, x^\lambda K_\nu^2(x) = \frac{2^{-2-\lambda}}{\Gamma(1-\lambda)} \Gamma\left(\frac{1-\lambda}{2}+\nu\right) \Gamma^2\left(\frac{1-\lambda}{2}\right) \Gamma\left(\frac{1-\lambda}{2}-\nu\right).$$

As a result, we have

$$c_1 = \frac{2^{\nu-2}}{\nu^{2\nu-1}} \frac{\Gamma^2(2\nu)\Gamma(3\nu)}{\Gamma(\nu)\Gamma(4\nu)} \qquad (A.5)$$

from which one can get eq. (3.7).

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Кудинов А.В., Смондырев М.А.

E2-83-412

Разложение сильной связи для ангармонического осциллятора


Разложение сильной связи для ангармонического осциллятора получено с помощью 1/N-разложения.



Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1983



Koudinov A.V., Smondyrev M.A.

E2-83-412

Strong Coupling Expansion for the Anharmonic Oscillator


Strong coupling expansion for anharmonic oscillator has been obtained using the 1/N-expansion.



The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1983