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MANY-BODY PROBLEM IN QUANTUM MECHANICS AND HOPF MAPS

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1. Embedding into spaces of higher dimensionalities and imposing relevant supplementary conditions (constraints) can lead to some useful forms of dynamics in the classical and quantum theories/1-6/. The Hopf maps (fiber bundles) bear interesting possibilities. Thus, the fiber bundle $S^3 \to S^2$ can be written as a transformation from a complex 2-component spinor to Cartesian coordinates

$$x_m = \bar\xi \sigma_m \xi \quad (m=1,2,3), \qquad \tau = \bar\xi\xi, \tag{1}$$

where σ_m are the Pauli matrices. Eq. (1) realizes the map at each value of τ ($\rho = \sqrt{\tau}$ and τ are radii of spheres S^2 and S^3, respectively). S^2 is the base space, and $S^1(e^{i\lambda}\xi)$ is a fiber. The change of variables (1) may also be of interest: on the one hand, if we tend to come to the spinors throughout (such a tendency is observed in the contemporary field theory) and, on the other hand, because the quantum mechanics takes a form similar to popular CP^1-models.

2. <u>Two-body problem.</u> The change of variables (1) leads to the Lagrangian

$$L = \frac{m}{2}\dot{\vec x}\dot{\vec x} + \frac{e^2}{\tau} = 2m(\bar\xi\xi)(\dot{\bar\xi}\dot\xi) + \frac{m}{2}(\dot{\bar\xi}\xi - \bar\xi\dot\xi)^2 + \frac{e^2}{\bar\xi\xi} \tag{2}$$

(any other potential $V(\tau) = V(\bar\xi\xi)$, $V(\vec x) = V(\bar\xi\vec\sigma\xi)$ can be accepted), which is invariant under gauge transformations $\xi(t) \to e^{i\lambda(t)}\xi(t)$. It is convenient to omit the second term of the last expression of eq. (2), to assume

$$\tilde L = 2m(\bar\xi\xi)(\dot{\bar\xi}\dot\xi) + \frac{e^2}{\bar\xi\xi} \qquad \text{(or with any other } V\text{)} \tag{3}$$

(recall an analogous approach in electrodynamics) and to impose the supplementary conditions (SC) $\dot{\bar\xi}\xi - \bar\xi\dot\xi = 0$ in classical theory and

$$(\xi\bar\pi - \bar\xi\pi)|\,\rangle = 0, \qquad \text{or} \quad \left(m(\bar\xi\xi)(\dot{\bar\xi}\xi - \bar\xi\dot\xi) + i\hbar\right)|\,\rangle = 0,$$

or
$$\left(\xi\frac{\partial}{\partial\xi} - \bar\xi\frac{\partial}{\partial\bar\xi}\right)\langle\xi,\bar\xi|\,\rangle = 0 \tag{4}$$

in quantum theory. The Lagrangian $\tilde L$ is invariant under the phase transformations only with $\lambda = \text{const}(t)$ ($(\bar\xi\xi)(\dot{\bar\xi}\xi - \bar\xi\dot\xi)$ is a relevant integral of motion), but is O_4-symmetric, while the Lagrangian L and SC (4) are only O_3-symmetric. When energy (Hamiltonian) is fixed, equations of motion turn out to be those for a 4-dimensional oscillator /1/ (see ref. /6/ for details in terms adopted here and for a group-theoretical meaning of SC).

A Schrödinger picture consideration can be performed independently of the above Heisenberg picture approach. To this end we solve the relations

$$\frac{\partial}{\partial\xi_\alpha} = \frac{\partial x_m}{\partial\xi_\alpha}\frac{\partial}{\partial x_m}, \qquad \frac{\partial}{\partial\bar\xi_\alpha} = \frac{\partial x_m}{\partial\bar\xi_\alpha}\frac{\partial}{\partial x_m} \tag{5}$$

with respect to $\partial/\partial x_m$. This set is overdetermined and as a compatibility condition we get SC (4) again. Then, applied to the functions satisfying SC (4) the relations

$$2\tau\frac{\partial}{\partial x_m} = \xi\sigma_m^T\frac{\partial}{\partial\xi} + \bar\xi\sigma_m\frac{\partial}{\partial\bar\xi} \tag{6}$$

and the following connection between Laplacians in R_3 and R_4

$$\Delta_{(3)} \equiv \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_m} = \frac{1}{4\tau}\Delta_{(4)} \equiv \frac{1}{\bar\xi\xi}\frac{\partial}{\partial\xi_\alpha}\frac{\partial}{\partial\bar\xi_\alpha} \tag{7}$$

are valid. The Schrödinger equation, say, for Green functions

$$\left[-\frac{\hbar^2}{2m}\Delta_{(3)} - \frac{e^2}{\tau} - E\right]G(\vec x,\vec x_0,E) = -i\hbar\,\delta(\vec x - \vec x_0) \tag{8}$$

can be replaced by the Schrödinger equation in R_4

$$\left[-\frac{\hbar^2}{8m}\Delta_{(4)} - e^2 - E\rho^2\right]\tilde G(\xi,\bar\xi,\xi_0,\bar\xi_0,e^2) = -i\hbar\frac{\pi}{4}\delta(\xi-\xi_0)\delta(\bar\xi-\bar\xi_0), \tag{9}$$

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Полубаринов И.В.
Проблема нескольких тел в квантовой механике и отображения Хопфа

В задаче многих тел в квантовой механике рассматривается переход от декартовых координат к спинорным переменным.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Polubarinov I.V.
Many-Body Problem in Quantum Mechanics and Hopf Maps

Transformation from the Cartesian coordinates to spinor variables is considered in many-body problem in quantum mechanics.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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