объединенный
институт
ядерных
исследований
дубна

E2-83-358

V.V.Nesterenko

NONLINEAR SIGMA MODEL FOR THE DODD-BULLOUGH-EQUATION

Submitted to "ТМФ"

1983

1. INTRODUCTION

An outstanding property of the nonlinear evolution equations integrable by the inverse scattering method is their intimate relation to the nonlinear two-dimensional sigma-models. For example, the SO(3) -sigma-model in a special parametrization reduces to the sine-Gordon equation[1], the SO(4) -sigma-model can be described by the Lund-Regge set of two nonlinear equations[1,2]. In papers[3-5] it has been shown that in general case the SO(n) -invariant nonlinear sigma-model is equivalent to a system of completely integrable nonlinear equations for $(n-2)$ scalar functions. The sigma-model with the pseudo-orthogonal symmetry group SO(1, 2) turned out to be related either to the Ernst equation[6] or to the Liouville equation[7] or to a new set of two nonlinear equations[8] depending on the parametrization used.

This connection was investigated also in the inverse direction, i.e., for a given nonlinear equation admitting Lax's representation one constructed the corresponding sigma-model. In ref.[9] it has been shown that the linear spectral problem for the sine-Gordon equation enables one to reconstruct the field variable of the SO(3) -sigma-model. Later on this method was generalized to an arbitrary nonlinear evolution equation admitting Lax's representation[10]. The crucial point here was the analysis of the group structure of this representation. It turned out specifically that the sigma-model for the nonlinear Schrödinger equation is the Heisenberg continuous spin chain.

Thus, a new interpretation was given of the relation of these two models established before[11].

In addition to the group-theoretical approach one can use another purely geometric method to construct the sigma-models for a given nonlinear evolution equation. It is based on the geometric origin of many nonlinear equations integrable by the inverse scattering method. More precisely, we would like to use the fact that the majority of these equations describe the intrinsic geometry of some surfaces in Euclidean, pseudo-Euclidean, Riemannian, and affine spaces[12-16]. In this approach the unit normal to the surface described by a given nonlinear equation is a natural candidate for the field variable in the corresponding sigma-model.

The purpose of this paper is the geometric construction of the nonlinear two-dimensional sigma-model for the Dodd-Bullough equation[17,18]

$$\phi_{,11} - \phi_{,22} = e^{\phi} - e^{-2\phi}, \tag{1}$$

where $\phi = \phi(u^1, u^2)$, $\phi_{ij} = \partial^2\phi/\partial u^i \partial u^j$. It will be shown that this equation is connected with the SL(3, R) -sigma-model, the field triplet of which takes values on the sphere in the three-dimensional unimodular affine space. This relation is completely similar to the connection of the sine-Gordon equation with the SO(3) -sigma-model describing a three-component field with values on the sphere of the usual three-dimensional Euclidean space.

For Eq. (1) various Lax's representations were proposed[14,19]. We shall use the geometric interpretation of this equation given in[14]. It has been shown in this paper that Eq. (1) describes the intrinsic geometry of the two-dimensional affine sphere in the three-dimensional unimodular affine space like the sine-Gordon equation determines the metric of the sphere in the usual three-dimensional Euclidean space[1,12]. The linear equations that define the moving frame on these surfaces can be used as the Lax representation for both the sine-Gordon equation and the Dodd-Bullough equation.

The affine normal on the sphere in the three-dimensional unimodular affine space will be considered as a field variable of the nonlinear sigma-model corresponding to Eq. (1). The equations determining this normal will play the role of equations of motion in the obtained sigma-model.

The paper is arranged as follows. In section 2 the basic facts from the affine differential geometry are given. In section 3 we construct for the Dodd-Bullough equation the SL(3, R) - sigma-model, the geometry of the affine sphere in the three-dimensional unimodular affine space being used. To illustrate the proposed method, we give in section 4 the geometric derivation of equations of motion in the SO(3) -sigma-model corresponding to the sine-Gordon equation. In conclusion we discuss shortly the problems that should be explored by further investigation of the proposed sigma-model.

2. AFFINE DIFFERENTIAL GEOMETRY

Let us give here the basic formulae of the affine differential geometry[20-23] to be used further.

We consider the three-dimensional affine space A^3 with coordinates x^1, x^2, x^3 and with the transformation group

$$x'^{\alpha} = c^{\alpha}_{\beta} x^{\beta} + d^{\alpha},$$

where the matrix c is from the SL(3, R) group, i.e., $\det(c^\alpha_\beta) = 1$. The Greek indices take values 1,2,3; and the Latin indices, 1,2. Let $\vec{x}(u^1,u^2)$ be a position vector of the surface in A^3 and let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the moving affine frame on this surface subjected to the condition

$$(\vec{e}_1, \vec{e}_2, \vec{e}_3) = \det(e^\beta_\alpha) = 1 . \tag{3}$$

The equations determining the motion of this basis along the surface are

$$d\vec{x} = \omega^i \vec{e}_i , \tag{4}$$

$$d\vec{e}_\alpha = \omega^\beta_\alpha \vec{e}_\beta ,$$

where ω^i and ω^β_α are the Maurer-Cartan forms of the affine unimodular transformations group (3). Differentiating the condition (3) and using (4) we obtain

$$\omega^\alpha_\alpha = 0, \tag{5}$$

so the one-forms ω^β_α take values in the Lie algebra of the SL(3, R) group. The structure equations of A^3, i.e., the integrability conditions of the linear equations (4), have the form

$$\omega^j \wedge \omega^3_j = 0,$$

$$d\omega^i = \omega^j \wedge \omega^i_j , \tag{6}$$

$$d\omega^\beta_\alpha = \omega^\gamma_\alpha \wedge \omega^\beta_\gamma .$$

Making use of the Cartan lemma, we get from the first equation in (6)

$$\omega^3_i = a_{ik}\,\omega^k , \quad a_{ik} = a_{ki} . \tag{7}$$

The linear forms ω^i and ω^β_α obeying eqs. (6) and (5) determine the surface $\vec{x}(u^1, u^2)$ in the space A^3 up to its affine transformations (2) as a whole. It is the Rodon theorem /23/ in the affine differential geometry.

To classify the surfaces in the affine geometry one uses the invariants constructed by tensors of the fundamental differential forms. The first form (affine metric) is quadratic

$$I = |a|^{-1/4} a_{ik}\, du^i du^k = \tilde{g}_{ik}\, du^i du^k , \tag{8}$$

where

$$\omega^i = du^i, \quad \tilde{g}_{ik} = |a|^{-1/4} a_{ik} , \quad a = \det(a_{ik}), \tag{9}$$

$$a_{ik} = (\vec{x}_{,ik} , \vec{x}_{,1} , \vec{x}_{,2}) = (\vec{x}_{,ik} [\vec{x}_{,1} \times \vec{x}_{,2}])$$

and the second form is the cubic one (the so-called Fubini-Pick form)

$$II = T_{ijk}\, du^i du^j du^k = |a|^{-1/4} (\vec{x}_{,1} , \vec{x}_{,2} , d^3\vec{x}) - dI . \tag{10}$$

The tensors $\tilde{g}_{ik}$ and T_{ijk} are symmetric and connected by the apolar relation

$$\tilde{g}^{ij}\, T_{ijk} = 0 , \quad \tilde{g}_{ij}\, \tilde{g}^{jk} = \delta^k_i . \tag{11}$$

According to the usual rules of the Riemannian differential geometry one constructs the invariant κ, Gauss' curvature of the affine metric $\tilde{g}_{ij}$. The mean curvature $\tilde{H}$ and the total affine curvature $\tilde{K}$ are defined by

$$2\tilde{H} = -A^i_i , \quad \tilde{K} = \det(A^j_i) , \tag{12}$$

where

$$A_{ij} = \tilde{\nabla}^k T_{ijk} - \tilde{H}\tilde{g}_{ij} \tag{13}$$

and $\tilde{\nabla}_j$ means the covariant differentiation with respect to the affine metric $\tilde{g}_{ij}$. These invariants are connected by the relation

$$\tilde{H} = \kappa - J, \tag{14}$$

where J is the invariant of the Fubini-Pick cubic form

$$J = \frac{1}{2} T_{ijk}\, T^{ijk} . \tag{15}$$

3. THE DODD-BULLOUGH EQUATION AND SL(3, R)-SIGMA-MODEL

In terms of the invariants $\tilde{H}$ and $\tilde{K}$ the affine sphere is defined by /21/

$$\tilde{K} = \frac{1}{\rho^2} , \quad \tilde{H} = \frac{1}{\rho} , \tag{16}$$

where ρ is a constant having the dimension $\ell^{3/2}$ and determining the affine distance from the sphere surface to its centre (the length of the affine radius).

As examples of the affine spheres with the centre in the coordinate origin we give here the following surfaces of the three-dimensional Euclidean space $xyz = 1$, $(x^2+y^2)z = 1$.

In the asymptotic coordinate set ξ, η on the affine sphere

$$\tilde g_{11} = \tilde g_{22} = 0, \quad \tilde g_{12} = (\vec x_{,\xi\eta}\, \vec x_{,\xi}\, \vec x_{,\eta})^{1/2} = \exp\phi(\xi,\eta) \tag{17}$$

we have [21]

$$J = A \cdot B \exp(-3\phi), \quad \kappa = -\phi_{,\xi\eta} \exp(-\phi), \tag{18}$$

where A and B are the nonvanishing components of the tensor T_{ijk}. For the sphere A and B can be taken without loss of generality as constants [21]. Substituting (17) and (18) into (14) we obtain the Dodd-Bullough equation

$$\phi_{,\xi\eta} = -\tilde H e^{\phi} - A \cdot B \cdot e^{-2\phi}, \tag{19}$$

where $\tilde H$, A and B are constants. By appropriate choosing these constants and introducing the "laboratory" coordinates $u^1 = \xi + \eta$, $u^2 = \xi - \eta$ one can transform eq. (19) to (1).

In paper [14] the Lax representation for eq. (19) has been constructed. For this aim eqs. (4) describing the moving frame on the affine sphere were used.

To construct the sigma-model for eq. (19) we consider the definition of the affine sphere in terms of the position vector of the surface and its affine normal, rather than in terms of the invariants $\tilde K$ and $\tilde H$ (16).

The affine normal is very important in affine differential geometry. It is given by

$$\vec N = \frac{1}{2} \tilde g^{ij} \tilde\nabla_i \tilde\nabla_j x = \frac{1}{2} \tilde\Box \vec x. \tag{20}$$

where $\tilde\Box$ is the covariant Laplace-Beltrami operator for the affine metric tensor $\tilde g_{ij}$

$$\tilde\Box = \frac{1}{\sqrt{|\tilde g|}} \left(\frac{\partial}{\partial u^i} \tilde g^{ij} \sqrt{|\tilde g|} \frac{\partial}{\partial u^j} \right), \quad \tilde g = \det(\tilde g_{ij}). \tag{21}$$

The sphere in the affine unimodular space A^3 is the surface the affine normals of which cross at one point. Let $\vec X(\xi,\eta)$ be the parametric definition of the sphere, and $\vec N(\xi,\eta)$ the field of the affine normal on it. Then the affine sphere with the centre at the coordinate origin is given by the following vector equation [21]

$$\vec x + \rho \vec N = 0, \tag{22}$$

where ρ is the « length » of the affine radius (see eq. (16)). Substituting (20) into (22) we get the differential equation for the position vector of the affine sphere

$$\vec x + \frac{\rho}{2} \tilde\Box \vec x = 0. \tag{23}$$

In the asymptotic coordinate set (ξ, η) defined by eq. (17)

$$(\vec x_{,\xi\xi}\, \vec x_{,\xi}\, \vec x_{,\eta}) = (\vec x_{,\eta\eta}\, \vec x_{,\xi}\, \vec x_{,\eta}) = 0 \tag{24}$$

the equation of motion for the SL(3, R)-sigma-model (23) takes the form

$$\vec x_{,\xi\eta} + \frac{1}{\rho} (\vec x_{,\xi\eta}\, \vec x_{,\xi}\, \vec x_{,\eta}) = 0. \tag{25}$$

The invariance of (24) and (25) under transformations (2) is obvious.

4. THE SINE-GORDON EQUATION AND SO(3)-SIGMA-MODEL

The connection of eq. (1) with the SL(3, R)-sigma-model (24), (25) is analogous to that of the sine-Gordon equation with the SO(3)-sigma-model, the three-component field of which takes values on the sphere in the usual three-dimensional Euclidean space [1,12]. Indeed, let $\vec r(\xi,\eta)$ be a parametric representation of the sphere with the radius R in Euclidean space. Then the position vector $\vec r(\xi,\eta)$ has to obey the equation similar to (22)

$$\vec r(\xi,\eta) + R \cdot \vec n(\xi,\eta) = 0, \tag{26}$$

where $\vec n(\xi,\eta)$ is the unit normal on the surface in Euclidean space

$$\vec n(\xi,\eta) = \frac{[\vec r_{,\xi} \times \vec r_{,\eta}]}{|[\vec r_{,\xi} \times \vec r_{,\eta}]|}. \tag{27}$$

Transferring in (26) $R\cdot\vec{n}$ into the right-hand side and squaring this equation, one obtains the common definition of the sphere in Euclidean space

$$r^2 = R^2. \tag{28}$$

The normal $\vec{n}$ can be expressed by the Gauss derivative formulae[24]

$$\nabla_i \vec{r}_{,j} = b_{ij}\vec{n}. \tag{29}$$

Here b_{ij} is the tensor of the second fundamental quadratic form of the surface, and ∇_j denotes the covariant differentiation with respect to the induced metric on the surface

$$g_{ij} = \vec{r}_{,i}\,\vec{r}_{,j}, \quad i,j = 1,2. \tag{30}$$

Contracting eq. (29) with respect to indices i,j one gets

$$\Box\vec{r} = 2H\cdot\vec{n}, \tag{31}$$

where $H = b^i_i/2$ is mean curvature of the surface and $\Box = \nabla_i\nabla^i$ is the covariant Laplace-Beltrmai operator for g_{ij}.

For the sphere[25]

$$H = \frac{1}{R}, \tag{32}$$

and as a consequence

$$\vec{n} = \frac{R}{2}\Box\vec{r}. \tag{33}$$

Substituting now (33) into (26) we obtain

$$\vec{r} + \frac{R^2}{2}\Box\vec{r} = 0. \tag{34}$$

The sine-Gordon equation arises in the geometry of the sphere by a special parametrization on it (the so-called Tchebyshev coordinate set[12,24])

$$g_{11} = \vec{r}_{,1}\vec{r}_{,1} = g_{22} = \vec{r}_{,2}\vec{r}_{,2} = 1, \tag{35}$$
$$g_{12} = g_{21} = \vec{r}_{,1}\vec{r}_{,2} = \cos a(\xi,\eta).$$

The function $a(\xi,\eta)$ cannot be arbitrary but it has to satisfy the Gauss equation[25], which reduces for the sphere to the sine-Gordon equation

$$a_{,\xi\eta} = \frac{1}{R^2}\sin a. \tag{36}$$

Taking into account (35) we obtain from (34) the equation of motion for the $SO(3)$ -sigma-model

$$\vec{r}_{,\xi\eta} + \frac{1}{R^2}\vec{r}(\vec{r}_{,\xi}\vec{r}_{,\eta}) = 0, \tag{37}$$

$$(\vec{r}_{,\xi})^2 = (\vec{r}_{,\eta})^2 = 1. \tag{38}$$

5. CONCLUSION

The nonlinear sigma-model (25) connected with the Dodd-Bullough equation (1) is invariant under the noncompact $SL(3,R)$ group. As a consequence, it has no instanton solutions. Nevertheless it is worth-while to construct the Backlund transformations for the field variables in this model and to look for the infinite series of the conservation laws in the spirit of papers[3-5,26,27]. For the quantization of this theory it is necessary to ascertain whether eqs. (25) are the Euler equations for some Lagrange density.

The sigma-models with the fields taking values on the spheres S^n are dynamical systems with the squared constraints[1]. The $SL(3,R)$ -sigma-model constructed here is the system with the cubic constraints on the dynamical variables. This follows at least from the fact that the affine sphere is defined by the algebraic equation of the third degree in its coordinates[23].

REFERENCES

1. Pohlmeyer K. Commun.Math.Phys., 1976, 46, No. 3, p. 209-221.
2. Lund F. Phys.Rev., 1977, D15, No. 6, p. 1540.
3. Eichenherr H., Forger M. Commun.Math.Phys., 1981, 82, No. 2, p. 227-255.
4. Rehren K.-H., Pohlmeyer K. J.Math.Phys., 1979, 20, No. 12, p. 2628-2632.
5. Eichenherr H., Honerkamp J. J.Math.Phys., 1981, 22, No. 2, p. 374-376.
6. Chinea F.J. Phys.Rev., 1981, D24, No. 4, p. 1053-1055.
7. Nesterenko V.V. JINR, E2-82-745, Dubna, 1982.
8. Nesterenko V.V. JINR, E2-82-761, Dubna, 1982.
9. Neveu A., Papanicolaou N. Commun.Math.Phys., 1978, 58, No. 1, p. 31-64.
10. Orfanidis S.J. Phys.Rev., 1980, D21, No. 6, p. 1513-1522.
11. Zakharov V.E., Takhtajan L.A. Teor.Mat.Fiz., 1979, 38, No. 1, p. 26-35.

12. Barbashov B.M., Nesterenko V.V. Fortschritte der Physik, 1980, B28, H.8/9, p. 427-464.
13. Sasaki R. Nucl.Phys., 1979, B154, No. 2, p. 343-357.
14. Nesterenko V.V. Lett. Math.Phys., 1980, 4, p.451-456.
15. Barbashov B.M., Nesterenko V.V. Commun.Math.Phys., 1981, 78, p. 499-506.
16. Barbashov B.M., Nesterenko V.V., Chervyakov A.M. Teor.Mat. Fiz., 1982, 52, No. 1, p. 3-13.
17. Dodd R.K., Bullough R.K. Proc.Roy. Soc.Lon., 1977, A352, p. 481-503.
18. Zhiber A.V., Shabat A.B. Dokl.Akad.Nauk SSSR 1979, 247, p. 1103 (in Ruaaian).
19. Mikhailov A.V. JETP Letters, 1979, 30, p. 433 (in Russian).
20. Blaschke W. Vorlesungen über differential Geometrie, Springer, Berlin, 1923.
21. Schirokow P.A., Schirokow A.P. Affine differential Geometrie, Leipzig, 1962.
22. Chern S.-S., Terng C.-L. Rocky Mountin Journal of Math., 1980, 10, No. 1, p. 105-124.
23. Favard J. Cours de Geometrie Differentialle Locale, Paris, Gauthier-Villars, 1957.
24. Dubrovin B.A., Novikov S.P., Fomenko A.T. Modern Geometry., M., "Nauka", 1979 (in Russian).
25. Eisenhart L.P. An Introduction to Differential Geometry with Use of the Tensor Calculus. Princeton University Press, Princeton, 1940.
26. Eichenherr H., Forger M. Nucl.Phys., 1979, B155, p. 381; 1980, B164, p. 528.
27. Eichenherr H., Pohlmeyer K. Phys. Lett., 1979, B89, No. 1, p. 76-78.

Нестеренко В.В. E2-83-358

Нелинейная сигма-модель для уравнения Додда-Буллоу

Показано, что уравнение Додда-Буллоу связано с нелинейной двумерной $SL(3, R)$ -сигма-моделью, в которой триплет безмассовых полей принимает значения на сфере 3-мерного унимодуляторного аффинного пространства, аналогично тому, как уравнение синус-Гордона связано с $SO(3)$ -сигма-моделью, описывающей трехкомпонентное поле со значениями на сфере обычного 3-мерного евклидова пространства. Получены уравнения движения для $SL(3, R)$ -сигма-модели.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1983

Nesterenko V.V. E2-83-358

Nonlinear Sigma Model for the Dodd-Bullough-Equation

It is shown that the Dodd-Bullough equation is intimately connected with the nonlinear two-dimensional $SL(3, R)$ -sigma-model, the triplet of massless fields of which takes values on the sphere in the three-dimensional unimodular affine space. This relation is completely similar to the connection of the sine-Gordon equation with the $SO(3)$ -sigma-model describing the three-component field with values on the sphere of the usual three-dimensional Euclidean space. The equations of motion for the $SL(3, R)$ -sigma-model are written explicitly.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1983