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VECTOR MAGNETIC POTENTIALS OF THE TOROIDAL SOLENOID

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$$\mathrm{cth}\,\mu = \frac{r^2 + a^2}{2r\cdot a\cdot \sin\theta_s}\,,\quad \mathrm{ctg}\,\theta = \frac{r^2 - a^2}{2r\cdot a\cos\theta_s}\,,\quad r = a\left(\frac{\mathrm{ch}\,\mu + \cos\theta}{\mathrm{ch}\,\mu - \cos\theta}\right)^{1/2},\quad \mathrm{tg}\,\theta_s = \frac{\mathrm{sh}\,\mu}{\sin\theta}\,.$$

For $r \to \infty$ one has: $\mu \approx \frac{2a}{r}\cdot \sin\theta_s$, $\sin\theta \approx \frac{2a}{r}\cdot\cos\theta_s$. For $r \to 0$: $\mu \approx \frac{2r}{a}\cdot\sin\theta_s$, $\theta = \pi - \frac{2r}{a}\cos\theta_s$. Let $\mu = \mu_0$ correspond to the torus T_0 on which the solenoidal winding is performed. The surface of this solenoid is given by the equation: $(\rho - d)^2 + z^2 = R^2$. Here d and R are the radii of the axial line and of the solenoid cross section. They are expressed in terms of a and μ_0 : $d = a\cdot\mathrm{cth}\,\mu_0$, $R = \frac{a}{\mathrm{sh}\,\mu_0}$. For $\mu > \mu_0$ $(<\mu_0)$ the point x, y, z is inside (outside) of the torus T_0. The cartesian components of the vector potential in the Coulomb gauge satisfy the following system of equations:

$$\Delta A_x = -\delta(\mu - \mu_0)\cdot A\cdot\cos\phi,\quad \Delta A_y = -\delta(\mu - \mu_0)\cdot A\cdot\sin\phi,\quad A = \frac{g}{a^2}(\mathrm{ch}\,\mu - \cos\theta)\sin\theta$$

$$\Delta A_z = -\delta(\mu - \mu_0)\frac{g(\mathrm{ch}\,\mu_0 - \cos\theta)}{a^2\,\mathrm{sh}\,\mu_0}\cdot(1 - \mathrm{ch}\,\mu_0\cos\theta).$$

Due to axial symmetry A_z does not depend on the angle ϕ, whereas A_x and A_y depend trivially: $A_x = A_\rho\cos\phi$, $A_y = A_\rho\sin\phi$. It follows then that $A_\phi = A_y\cos\phi - A_x\sin\phi = 0$. Setting $A_\rho = \tilde{A}_\rho\sqrt{\mathrm{ch}\,\mu - \cos\theta}$, $A_z = \tilde{A}_z\sqrt{\mathrm{ch}\,\mu - \cos\theta}$ one obtains the equations for $\tilde{A}_\rho$, $\tilde{A}_z$:

$$\left(\partial^2_\mu + \mathrm{cth}_\mu\partial_\mu + \partial^2_\theta + \frac{1}{4} - \frac{1}{\mathrm{sh}^2\mu}\right)\tilde{A}_\rho = -\frac{g}{a^2}\,\frac{\sin\theta\cdot\delta(\mu - \mu_0)}{(\mathrm{ch}\,\mu_0 - \cos\theta)^{3/2}}\,,$$

$$\left(\partial^2_\mu + \mathrm{cth}_\mu\partial_\mu + \partial^2_\theta + \frac{1}{4}\right)\tilde{A}_z = -\frac{g}{a^2}\,\frac{\delta(\mu - \mu_0)}{\mathrm{sh}\,\mu_0}\,\frac{1 - \mathrm{ch}\,\mu_0\cdot\cos\theta}{(\mathrm{ch}\,\mu_0 - \cos\theta)^{3/2}}\,. \tag{2.1}$$

Solve (2.1) expanding $\tilde{A}_\rho$ and $\tilde{A}_z$ in $\sin\theta$, $\cos n\theta$: $\tilde{A}_z = \Sigma R^0_n(\mu)\cdot\cos n\theta$, $\tilde{A}_\rho = \Sigma R^1_n(\mu)\cdot\sin n\theta$. In the same way one develops the right-hand sides of (2.1)[6]:

$$\frac{\sin\theta}{(\mathrm{ch}\,\mu - \cos\theta)^{3/2}} = \frac{4\sqrt{2}}{\pi}\sum_{n=1} Q^0_{n-1/2}(\mathrm{ch}\,\mu)\cdot n\cdot\sin n\theta,\quad \frac{1 - \mathrm{ch}\,\mu\cdot\cos\theta}{(\mathrm{ch}\,\mu - \cos\theta)^{3/2}} = \frac{2\sqrt{2}}{\pi}\cdot$$

$$\left\{\frac{1}{2}Q^0_{1/2}(\mathrm{ch}\,\mu) + \sum_{n=1}\cos n\theta\cdot\left[\left(n + \frac{1}{2}\right)Q^0_{n+1/2}(\mathrm{ch}\,\mu) - \left(n - \frac{1}{2}\right)Q^0_{n-3/2}(\mathrm{ch}\,\mu)\right]\right\}.$$

Here and further $Q^\mu_\nu(z)$ and $P^\mu_\nu(z)$ are the Legendre functions of the 2nd and 1st order, correspondingly. Inserting these expansions into (2.1) one obtains equations for $R_n(\mu)$:

$$\frac{d^2R^0_n}{d\mu^2} + \mathrm{cth}\,\mu\cdot\frac{dR^0_n}{d\mu} + \left(\frac{1}{4} - n^2\right)R^0_n =$$

$$= \tilde{g}\,\frac{\delta(\mu - \mu_0)}{\mathrm{sh}\,\mu_0}\cdot\left[\left(n - \frac{1}{2}\right)Q^0_{n-3/2}(\mathrm{ch}\,\mu_0) - \left(n + \frac{1}{2}\right)\cdot Q^0_{n+1/2}(\mathrm{ch}\,\mu_0)\right],$$

$$\frac{d^2R^1_n}{d\mu^2} + \mathrm{cth}\,\mu\,\frac{dR^1_n}{d\mu} + \left(\frac{1}{4} - n^2 - \frac{1}{\mathrm{sh}^2\mu}\right)R^1_n =$$

$$= -2\tilde{g}\cdot\delta(\mu - \mu_0)\cdot n\cdot Q^0_{n-1/2}(\mathrm{ch}\,\mu_0)\,.$$

Here $\tilde{g} = \frac{2\sqrt{2}}{\pi}g$. The solutions of these equations continuous and finite everywhere (including the boundary of torus T_0) are:

$$R^0_n = A^0_n\cdot\begin{cases} P^0_{n-1/2}(\mathrm{ch}\,\mu)\cdot Q^0_{n-1/2}(\mathrm{ch}\,\mu_0) & (\mu < \mu_0)\\ Q^0_{n-1/2}(\mathrm{ch}\,\mu)\cdot P^0_{n-1/2}(\mathrm{ch}\,\mu_0) & (\mu > \mu_0)\end{cases}$$

$$A^0_n = \tilde{g}\cdot\left[\left(n + \frac{1}{2}\right)Q^0_{n+1/2}(\mathrm{ch}\,\mu_0) - \left(n - \frac{1}{2}\right)Q^0_{n-3/2}(\mathrm{ch}\,\mu_0)\right]\quad\text{for } n \neq 0;$$

$$A^0_0 = \frac{1}{2}\tilde{g}\cdot Q^0_{1/2}(\mathrm{ch}\,\mu_0)$$

$$R^1_n = -\tilde{g}\cdot Q^0_{n-1/2}(\mathrm{ch}\,\mu_0)\cdot\begin{cases} P^1_{n-1/2}(\mathrm{ch}\,\mu)\cdot[Q^0_{n+1/2}(\mathrm{ch}\,\mu_0) - Q^0_{n-3/2}(\mathrm{ch}\,\mu_0)] & (\mu < \mu_0)\\ Q^1_{n-1/2}(\mathrm{ch}\,\mu)\cdot[P^0_{n+1/2}(\mathrm{ch}\,\mu_0) - P^0_{n-3/2}(\mathrm{ch}\,\mu_0)] & (\mu > \mu_0).\end{cases}$$

One can easily check that $\mathrm{div}\,A = 0$. Note A_z and A_ρ are even and odd functions of z, correspondingly. At large distances A_ρ and A_z are falling as $1/r^3$: $A_z \approx \frac{4\cdot g}{\pi}\,\frac{a^3}{r^3}\cdot(1 + 3\cdot\cos 2\theta_s)\cdot C_1$, $A_\rho \approx \frac{12g}{\pi}\,\frac{a^3}{r^3}\cdot\sin 2\theta_s\cdot C_1$. Here constant $C_1 = \frac{\pi^2}{32}\,\frac{\mathrm{cth}\,\mu_0}{\mathrm{sh}^2\mu_0}$. However, in special directions A_ρ and A_z decrease more rapidly than $1/r^3$. For example $A_\rho = 0$ in the $z = 0$ plane and on the z axis. Consider now A_z behaviour in the $z = 0$ plane and on the z axis. At the origin $A_z(\rho = 0, z = 0) = \frac{g}{\sqrt{\mathrm{ch}\,\mu_0}}\cdot Q^0_{1/2}\left(\frac{1 + \mathrm{ch}^2\mu_0}{2\,\mathrm{ch}\,\mu_0}\right)$. With changing radius $\rho\ (\equiv\sqrt{x^2 + y^2})$ A_z takes the values

$$\tilde g\cdot\sqrt{1+\mathrm{ch}\mu_0}\cdot\sum_{n=0}(-1)^n Q^0_{n-1/2}(\mathrm{ch}\mu_0)\cdot Q^0_{n+1/2}(\mathrm{ch}\mu_0)\cdot[P^0_{n+1/2}(\mathrm{ch}\mu_0) + P^0_{n+1/2}(\mathrm{ch}\mu_0)]\cdot(n+\frac{1}{2})$$

at the inner boundary ($\rho = d - R$) of the T_0 and

$$\tilde g\sqrt{\mathrm{ch}\mu_0 - 1}\cdot\sum_{n=0}(n+\frac{1}{2})\cdot Q^0_{n+1/2}(\mathrm{ch}\mu_0)\cdot Q^0_{n-1/2}(\mathrm{ch}\mu_0)\cdot[P^0_{n-1/2}(\mathrm{ch}\mu_0) - P^0_{n+1/2}(\mathrm{ch}\mu_0)]$$

at the outer boundary ($\rho = d + R$). Inside the torus (for $\rho = a = \sqrt{d^2 - R^2}$) A_z equals $g\cdot Q^0_{1/2}(\mathrm{ch}\mu_0)\cdot P^0_{-1/2}(\mathrm{ch}\mu_0)$. For a large ρ the potential is negative: $A_z = -g\cdot\frac{8a^3}{\rho^3}\cdot C_1$. On the z axis

$$A_z(0,z) = \frac{R^{1/2}\cdot g\cdot d}{(d^2+z^2)^{3/4}}\cdot Q^0_{1/2}[\frac{d^2+z^2+R^2}{2R(d^2+z^2)^{1/2}}],$$

that equals $g\cdot\frac{16a^3}{|z|^3}\cdot C_1$ for large values of $|z|$.

The above formulas are more transparent if the solenoid is very thin ($\frac{R}{d}\ll 1$ or $\mu_0 \gg 1$). The potentials equal:

$$A_\rho \approx \sqrt{2}\cdot\pi\cdot g\cdot\exp(-2\mu_0)\cdot\sin\theta\cdot\sqrt{\mathrm{ch}\mu - \cos\theta}\cdot P'_{1/2}(\mathrm{ch}\mu),$$

$$A_z = \frac{\pi g}{\sqrt{2}}\cdot\exp(-2\mu_0)\cdot\sqrt{\mathrm{ch}\mu - \cos\theta}\cdot[P^0_{-1/2}(\mathrm{ch}\mu) - P^0_{1/2}(\mathrm{ch}\mu)\cdot\cos\theta]$$

outside T_0 and $A_\rho \approx g\cdot\exp(-\mu)\cdot\sin\theta$, $A_z \approx -g\cdot\exp(-\mu)\cdot\cos\theta$ inside it. In the plane $z=0$ A_z grows from $2\pi g\cdot\exp(-2\mu_0) = \frac{\pi g R^2}{2d^2}$ at the origin up to $g\cdot\exp(-\mu_0) = \frac{gR}{2d}$ at the inner boundary of T_0. Inside the solenoid A_z vanishes at $\rho = d$ and takes the negative value $-\frac{gR}{2d}$ at the outer boundary of T_0. For greater value of ρ A_z remains negative and goes to zero: $A_z \approx -\pi g\frac{a^3}{\rho^3}\cdot\exp(-2\mu_0) = -\frac{\pi g d R^2}{4\rho^3}$.

Now consider $A_z(\rho, z)$ as a function of z for ρ fixed. At the z axis $A_z(0,z) = \frac{2\pi g\exp(-2\mu_0)}{[1+(\frac{z}{d})^2]^{3/2}} = \frac{\pi g}{2}\frac{R^2 d}{(z^2+d^2)^{3/2}}$. For fixed $\rho<d$ the function A_z is positive for all z ; for $\rho>d$ A_z is negative for small z and positive for greater ones. So, the zeroes of A_z in the (ρ, z) plane (for $R \ll d$) lie at the line which originates at the point $(d, 0)$ and has asymptotics $z = \pm\rho/\sqrt{2}$.

Consider now integrals along the closed paths. Let a contour C_2 (fig.1) passes through the hole of T_0. C_2 may be chosen to be a circumference which is an intersection of the meridional plane ϕ =const and the surface of the torus T which encompasses the solenoid T_0 and corresponds to the $\mu<\mu_0$ (and the same a). Along this contour μ=const and θ varies from 0 to 2π: $\oint A_\ell d\ell = a\int_0^{2\pi}\frac{A_\theta}{\mathrm{ch}\mu - \cos\theta}d\theta$. Here A_θ is the component of $\vec{A}$ along the C_2: $A_\theta = -[\mathrm{sh}\mu\cdot\sin\theta\cdot A_\rho + (1-\mathrm{ch}\mu\cos\theta)A_z]\cdot(1-\mathrm{ch}\mu\cos\theta)^{-1}$. The integration gives: $\frac{8ga}{\pi}\sum Q^0_{n-1/2}(\mathrm{ch}\mu_0)\cdot Q^0_{n+1/2}(\mathrm{ch}\mu_0)$. This integral should be equal to the magnetic field flow through the solenoid cross-section: $\iint H_\phi d\rho\cdot dz = g\iint\frac{d\rho\cdot dz}{\rho} = 2\pi g(d - \sqrt{d^2 - R^2}) = 2\pi g a\cdot(\mathrm{cth}\mu_0 - 1)$.

The identity of these expressions may be proved if one takes the integral $\int_0^{2\pi}\frac{\cos\theta}{\mathrm{ch}\mu - \cos\theta}d\theta$ in two different ways. Direct integration gives: $2\pi\cdot(\mathrm{cth}\mu - 1)$. On the other hand, the integrand may be viewed as the product of two factors: $\frac{\cos\theta}{\sqrt{\mathrm{ch}\mu - \cos\theta}}\cdot\frac{1}{\sqrt{\mathrm{ch}\mu - \cos\theta}}$. Expanding them in $\cos n\theta$ and integrating one obtains: $\frac{8}{\pi}\sum Q_{n-1/2}(\mathrm{ch}\mu)\cdot Q_{n+1/2}(\mathrm{ch}\mu)$.

Comparison of two different integrations completes the proof. Consider next the circle C_1 with radius $R_0 > d + R$. Along this contour $\oint A_\ell d\ell = R_0\int A_{\theta_s}d\theta_s = R_0\int_0^{\pi}(A_\rho\cdot\cos\theta_s - A_z\cdot\sin\theta_s)d\theta_s$. From the relation $\sin\theta = \mathrm{ctg}\theta_s\cdot[(\frac{r^2+a^2}{2ra\sin\theta_s}) - 1]^{-1/2}$ it follows that the integrand is an odd function of $(\frac{\pi}{2} - \theta_s)$. So, the above integral vanishes.

3. The functions A_ρ and A_z obtained in the preceeding section are solutions of the Maxwell equations. An alternative way is to solve the equation $H_\phi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} = \frac{g}{\rho}\theta[R - \sqrt{(\rho-d)^2+z^2}]$ with the gauge condition $\mathrm{div}\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{\partial A_z}{\partial z} = 0$. The latter is fulfilled if one puts: $A_\rho = -\frac{1}{\rho}\frac{\partial\psi}{\partial z}$, $A_z = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}$. The function ψ satisfies the equation: $\psi_{\rho\rho} - \frac{1}{\rho}\psi_\rho + \psi_{zz} = g\cdot\theta[R - \sqrt{(\rho-d)^2+z^2}]$. Using toroidal coordinates and substituting $\psi = \tilde\psi\cdot\sqrt{\mathrm{ch}\mu - \cos\theta}$ one ob-

tains the following equation for $\tilde{\psi}$: $\tilde{\psi}_{\mu\mu} - \mathrm{cth}\mu\cdot\tilde{\psi}_{\mu} + \tilde{\psi}_{\theta\theta} + \frac{1}{4}\tilde{\psi} = \frac{\tilde{g}\theta(\mu-\mu_0)}{(\mathrm{ch}\mu - \cos\theta)^{3/2}}$. Now we expand both sides in $\cos n\theta$: $\tilde{\psi} = \Sigma\psi_n(\mu)\cdot\cos n\theta$, $(\mathrm{ch}\mu - \cos\theta)^{-3/2} = -\frac{2\sqrt{2}}{\pi}\frac{1}{\mathrm{sh}\mu}\cdot[Q^1_{-1/2} + 2\sum_{n=1} Q^1_{n-1/2}\cos n\theta]$ and get the equation for $\psi_n(\mu)$: $\ddot{\psi}_n - \mathrm{cth}\mu\dot{\psi}_n + (\frac{1}{4} - n^2)\psi = -2\tilde{g}\frac{\theta(\mu-\mu_0)}{\mathrm{sh}\mu}\cdot Q^1_{n-1/2}(\mathrm{ch}\mu)$

(the dot above ψ_n means derivative with respect to μ). The substitution $\psi_n = \mathrm{sh}\mu\cdot F_n$ reduces this equation to a more familiar one: $\ddot{F}_n + \mathrm{cth}\mu\cdot\dot{F}_n + (\frac{1}{4} - n^2 - \frac{1}{\mathrm{sh}^2\mu})F_n = -\frac{2\tilde{g}}{\mathrm{sh}^2\mu}Q^1_{n-1/2}(\mathrm{ch}\mu)\cdot\theta(\mu-\mu_0)$.
It has solutions: $F_n = -2\tilde{g}\frac{1}{n^2-1/4}\cdot P^1_{n-1/2}(\mathrm{ch}\mu)\cdot\int_{\mu_0}^{\infty}\frac{d\mu}{\mathrm{sh}\mu}\cdot[Q^1_{n-1/2}(\mathrm{ch}\mu)]^2 d\mu$ for $\mu<\mu_0$ and $F_n = -2\tilde{g}\frac{1}{n^2-1/4}\cdot\{Q^1_{n-1/2}(\mathrm{ch}\mu)\cdot\int_{\mu_0}^{\mu}P^1_{n-1/2}(\mathrm{ch}\mu)Q^1_{n-1/2}(\mathrm{ch}\mu)\frac{d\mu}{\mathrm{sh}\mu} + P^1_{n-1/2}(\mathrm{ch}\mu)\int_{\mu}^{\infty}[Q^1_{n-1/2}(\mathrm{ch}\mu)]^2\frac{d\mu}{\mathrm{sh}\mu}\}$ for $\mu>\mu_0$. A_ρ, A_z obtained in such a way should coincide with those found in the preceding section. This gives the following recurrent relations for integrals occurring in F_n :

$$2C_n - C_{n-1} - C_{n+1} = [(n+\tfrac{1}{2})Q^0_{n+1/2} - (n-\tfrac{1}{2})Q^0_{n-3/2}]\cdot Q^0_{n-1/2},$$

$$\frac{2nC_n}{n^2-1/4} - \frac{C_{n+1}}{n+1/2} - \frac{C_{n-1}}{n-1/2} = Q^0_{n-1/2}\cdot(Q^0_{n+1/2} - Q^0_{n-3/2}),$$

$$2D_n - D_{n-1} - D_{n+1} = 1 + [(n+\tfrac{1}{2})Q^0_{n+1/2} - (n-\tfrac{1}{2})Q^0_{n-3/2}]P^0_{n-1/2},$$

$$\frac{2nD_n}{n^2-1/4} - \frac{D_{n+1}}{n+1/2} - \frac{D_{n-1}}{n-1/2} = Q^0_{n-1/2}(P^0_{n+1/2} - P^0_{n-3/2}),$$

where

$$C_n = \int_{\mu}^{\infty}[Q^1_{n-1/2}(\mathrm{ch}\mu)]^2\frac{d\mu}{\mathrm{sh}\mu},$$

$$D_n = \int_{\mu}^{\infty}Q^1_{n-1/2}(\mathrm{ch}\mu)P^1_{n-1/2}(\mathrm{ch}\mu)\frac{d\mu}{\mathrm{sh}\mu}$$

and the argument of the Legendre functions is $\mathrm{ch}\mu$. These relations may be reduced to simpler ones: $C_n - C_{n+1} = (n+\frac{1}{2})Q^0_{n+1/2}Q^0_{n-1/2}$, $D_n - D_{n+1} = (n+\frac{1}{2})Q^0_{n+1/2}P^0_{n-1/2}$. The mentioned above coincidence

of the solutions steams from the fact that they are everywhere continuous and finite solutions of the same equations with the same boundary conditions. This guarantees the uniqueness and identity of solutions /1,5/.

4. Vector potentials of the toroidal solenoid may be also computed with the use of the potentials of separate coils. At the surface of the torus T_0 it is convenient to introduce the coordinates ϕ,ψ: $x = (d + R\cos\psi)\cdot\cos\phi$, $y = (d + R\cos\psi)\cdot\sin\phi$, $z = R\sin\psi$. The unit vector tangential to this surface and belonging to the meridional plane $\phi = \mathrm{const}$ is: $\vec{n}_\psi = \vec{e}_z\cdot\cos\psi - (\vec{e}_x\cdot\cos\phi + \vec{e}_y\cdot\sin\phi)\sin\psi$. Then, the solenoid potential at the point with coordinates x, y, z equals $\vec{A}(x,y,z) = -\frac{R\cdot g}{4\pi}$.

$$\iint\frac{\vec{n}_\psi d\psi\, d\phi}{\{[x-(d+R\cos\psi)\cos\phi]^2 + [y-(d+R\cos\psi)\sin\phi]^2 + (z - R\sin\psi)^2\}^{1/2}}. \quad (4.1)$$

Or after the ψ integration:

$$A_z = \int_0^{2\pi} d\phi\cdot(d - \rho\cos\phi)\cdot F(\rho,z,\phi), \quad A_\rho = z\cdot\int_0^{2\pi}\cos\phi\cdot F(\rho,z,\phi)\cdot d\phi,$$

$$F(\rho,z,\phi) = \frac{\sqrt{R}\cdot g}{2\pi}\,\frac{1}{[(\rho\cos\phi - d)^2 + z^2]^{3/4}}\cdot Q^0_{1/2}\{\frac{\rho^2 - 2d\rho\cos\phi + d^2 + z^2 + R^2}{2R[(\rho\cos\phi - d)^2 + z^2]^{1/2}}\}. \quad (4.2)$$

When the solenoid is thin ($\frac{R}{d} \ll 1$) it is possible to carry out in (4.1) an expansion with respect to $\frac{R}{[(x-d\cos\phi)^2 + (y-d\sin\phi)^2 + z^2]^{1/2}}$, valid outside the solenoid:

$$A_x = \frac{R^2zg}{4}\int_0^{2\pi}\frac{\cos\phi\cdot d\phi}{|\vec{r}-\vec{r}'|^3}, \quad A_y = \frac{R^2zg}{4}\int_0^{2\pi}\frac{\sin\phi}{|\vec{r}-\vec{r}'|^3}d\phi, \quad (4.3)$$

$$A_z = -\frac{R^2g}{4}\int_0^{2\pi}\frac{(x-x')\cos\phi + (y-y')\sin\phi}{|\vec{r}-\vec{r}'|^3}d\phi.$$

Here x', y' run along the filament (in which torus T_0 degenerates when $R\to0$) with the radius d lying in the $z=0$ plane: $x' = d\cdot\cos\phi$, $y' = d\sin\phi$. The last expressions may be further simplified:

$$A_\rho(x,y,z) = \frac{R^2 z\cdot g}{4}\int_0^{2\pi}\frac{\cos\phi\, d\phi}{(r^2 + d^2 - 2d\rho\cos\phi)^{3/2}} = -\frac{R^2zg}{2(d\rho)^{3/2}}\cdot\frac{1}{\mathrm{sh}\mu_1}\cdot Q^1_{1/2}(\mathrm{ch}\mu_1),$$

$$A_z(x,y,z) = \frac{R^2 g}{4}\int_0^{2\pi}\frac{d-\rho\cos\phi}{(r^2+d^2-2d\rho\cos\phi)^{3/2}} = \tag{4.4}$$

$$= \frac{R^2 g}{2(d\rho)^{3/2}}\frac{1}{sh\mu_1}\cdot[\rho\cdot Q^1_{1/2}(ch\mu) - d\cdot Q^1_{-1/2}(ch\mu)],$$

Here $ch\mu_1 = \frac{r^2+d^2}{2d\rho}$. Although these formulas look quite differently from those given in sect.2, they are in fact the same if one takes into account the Whipple relation between Legendre functions [6]:

$$Q^\mu_\nu(z) = \exp(i\mu\pi)\cdot\sqrt{\frac{\pi}{2}}\cdot\Gamma(\nu+\mu+1)\cdot(z^2-1)^{-1/4}\cdot P^{-\nu-1/2}_{-\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right).$$

We stress once more that expressions (4.3), (4.4) are valid only if $(\rho-d)^2+z^2 \geq R^2$, i.e., outside the thin solenoid. For arbitrary ρ, z one must use formulas (4.2) or much comfortable ones given in section 2.

Now calculate $\int A_z\cdot dz$ along a straight line parallel to the axis. Using (4.4) one obtains:

$$\int_{-z_0}^{z_0} A\cdot dz = \frac{R^2 g}{4}\int_0^{2\pi}(d-\rho\cos\phi)\cdot d\phi\int_{-z_0}^{z_0}\frac{dz}{(b^2+z^2)^{3/2}} = \frac{R^2 g}{2}\int_0^{2\pi}\frac{d-\rho\cos\phi}{b^2}\left(1+\frac{b^2}{z_0^2}\right)^{-1/2} d\phi,$$

where $b^2 = d^2+\rho^2-2d\rho\cos\phi$. For $z\to\infty$ it follows:

$$\int_{-\infty}^{\infty} A_z dz = \frac{R^2 g}{2}\int_0^{2\pi}\frac{d-\rho\cos\phi}{\rho^2+d^2-2d\rho\cos\phi}d\phi = \frac{R^2 g\pi}{2d}\left(1-\frac{\rho-d}{|\rho-d|}\right).$$

So $\int_{-\infty}^{\infty} A_z dz$ equals $\frac{R^2 g\pi}{d}$ (i.e., to the magnetic field flux) if the integration axis passes inside the torus hole and zero otherwise.

5. The potentials obtained so far are solutions of equations: $\vec{H} = \mathrm{rot}\vec{A}$, $\mathrm{div}\vec{A} = 0$. Finding of potentials is much simplified if one throws away the gauge condition $\mathrm{div}\vec{A} = 0$ and tries to satisfy the equation $\vec{H} = \mathrm{rot}\vec{A}$ only. We obtain now a particular solution of this equation and compare it with the one used in ref. [4]. As inside the solenoid $H_\phi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} = \frac{g}{\rho}$ it is natural to

seek there the solution in the form:

$$A_z = A_1 + A_2\cdot\ln\rho, \quad A_\rho = A_3/\rho,$$

where A_1, A_2, A_3 are functions of z only. From this one obtains inside the solenoid: $\frac{\partial A_3}{\partial z} - A_2 = g$. At the outer boundary of the solenoid ($\rho = d+\sqrt{R^2-z^2}$, $|z| \leq R$) the potential A_ρ equals $\frac{A_3}{d+\sqrt{R^2-z^2}}$.

The simplest way to ensure continuity of A_ρ, when passing from I to II (fig.2), is to assume A_ρ to be independent of ρ and equal to its value at the boundary (I,II), that is $\frac{A_3}{d+\sqrt{R^2-z^2}}$. If one requires A_ρ to vanish as $\rho\to\infty$, then $A_3 = 0$, $A_2 = -g$. So, A_ρ vanishes everywhere and $A_z = A_1 - g\cdot\ln\rho$ inside the solenoid. Now apply the same procedure to A_z. The continuity of A_z at the outer boundary (I,II) and its vanishing for $r\to\infty$ gives $A_1 = -g\ln(d+\sqrt{R^2-z^2})$, $A_z = g\ln\frac{d+\sqrt{R^2-z^2}}{\rho}$ in I and $A_z = 0$ in II. One may easily get continuity at the inner boundary of the solenoid (i.e., for $\rho = d-\sqrt{R^2-z^2}$, $|z| \leq R$) if one takes A_z in region III to be independent of ρ and equal to its value at the boundary (I,III), that is $g\cdot\ln\frac{d+\sqrt{R^2-z^2}}{d-\sqrt{R^2-z^2}}$. As for $|z| = R$ the potential $A_z = 0$, is natural to take $A_z = 0$ for $|z| > R$. Summing up, we have: $A_\rho = A_\phi = 0$ everywhere; $A_z = g\cdot\ln\frac{d+\sqrt{R^2-z^2}}{\rho}$ inside the solenoid, $A_z = \ln\frac{d+\sqrt{R^2-z^2}}{d-\sqrt{R^2-z^2}}$ in region III and zero otherwise. One may easily see that $\int A_z dz$ equals 0 or $2\pi g(d-\sqrt{R^2-d^2})$ if the integration takes place along the line passing in II or III, respectively. But the divergence of $\vec{A}$ is different from zero: $\frac{\partial A_z}{\partial z} \neq 0$. If the solenoid radius R tends to zero then the potential in region III (which degenerates into the circle of the radius d lying in the $z=0$ plane) is: $A_z = 2g\frac{\sqrt{R^2-z^2}}{d} = \frac{2\Phi}{\pi R^2}\cdot\sqrt{R^2-z^2}$ (where Φ is the magnetic field flux $\frac{\pi R^2 g}{d}$), i.e., it is concentrated in a thin layer with a width 2R. If for a fixed Φ the

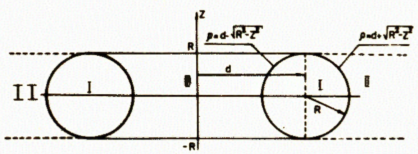

Fig.2. The definition domains of the vector magnetic potentials for the toroidal solenoid when $\mathrm{div}\vec{A} \neq 0$.

radius R goes to zero, then A_z takes the form:

$$A_z = \Phi \cdot \delta(z) \cdot \theta(d-\rho), \tag{5.1}$$

which is identical with that used in[4]. This formula does not work in the vicinity of the solenoid (i.e., for $(\rho-d)^2+z^2<R^2$ or for $\rho=d$, $z=0$ in the exact limit $R=0$). Indeed, the differentiation of (5.1) gives: $H_\rho = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} = \Phi\delta(z)\delta(\rho-d)$. So, the strength of the magnetic field equals infinity at $\rho=d$ rather than $\frac{g}{d}$ as it should be. The potentials (5.1) used in ref.[4] for the calculation of the A.B. effect, were deduced in a quite different way. We want to examine this derivation in more detail. The starting point is the expressions (4.3) which can be presented in the alternative form:

$$A_x = \frac{R^2 zg}{4d} \oint \frac{dy'}{|\vec{r}-\vec{r}'|^3} = -\frac{R^2 g}{4d}\frac{\partial}{\partial z} \oint \frac{dy'}{|\vec{r}-\vec{r}'|},$$

$$A_y = \frac{R^2 g}{4d}\frac{\partial}{\partial z} \oint \frac{dx'}{|\vec{r}-\vec{r}'|}, \tag{5.2}$$

$$A_z = \frac{R^2 g}{4d}\left(\frac{\partial}{\partial x} \oint \frac{dy'}{|r-r'|} - \frac{\partial}{\partial y} \oint \frac{dx'}{|r-r'|}\right),$$

where the integration is performed along the circumference of the radius d lying in the $z=0$ plane; this in turn may be replaced by the integration over the area of the circle with the same radius:

$$\oint \frac{dx'}{|\vec{r}-\vec{r}'|} = \frac{\partial a}{\partial y}, \quad \oint \frac{dy'}{|\vec{r}-\vec{r}'|} = -\frac{\partial a}{\partial x},$$

where

$$a = \iint \frac{dx'dy'}{|\vec{r}-\vec{r}'|}.$$

We substitite this into (5.2):

$$A_x = \frac{R^2 g}{4d}\cdot\frac{\partial^2 a}{\partial x \partial z}, \quad A_y = \frac{R^2 g}{4d}\cdot\frac{\partial^2 a}{\partial y \partial z}, \quad A_z = -\frac{R^2 g}{4d}\cdot\left(\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2}\right). \tag{5.3}$$

Adding and subtracting in A_z the quantity $\frac{R^2 g}{4d}\cdot\frac{\partial^2 a}{\partial z^2}$ and taking into account that $\Delta \cdot \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta^3(\vec{r}-\vec{r}')$ one gets:

$$A_z = \frac{\pi R^2 g}{d} \iint dx'dy'\,\delta^3(\vec{r}-\vec{r}') + \frac{R^2 g}{4d}\cdot\frac{\partial^2 a}{\partial z^2} = \frac{\pi R^2 g}{d}\delta(z)\,\theta(d-\rho) + \frac{R^2 g}{4d}\cdot\frac{\partial^2 a}{\partial z^2} \tag{5.4}$$

or up to the gauge transformation:

$$A_x = A_y = 0, \quad A_z = \frac{\pi R^2 g}{d}\delta(z)\,\theta(d-\rho). \tag{5.5}$$

Expressions (5.5) and (5.1) are identical. Note that A_z - given by (5.5) has a singularity outside the solenoid whereas the starting formulas (4.1)-(4.4), (5.2)-(5.4) do not have. Further, the divergence of $\vec{A}$ is different from zero and has a singularity in the $z=0$ plane for potentials (5.1), (5.5). This means that the transition from (5.3) to (5.5) is performed via a singular gauge transformation. In fact, a simple calculation shows that the singularity of $\frac{\partial^2 a}{\partial z^2}$ exactly compensates the singularity of the first term in (5.4). On the other hand, the infinite value of H_ϕ at $\rho=d$ is due to nonapplicability of (4.3), (4.4), (5.1)-(5.4) in the interior of the solenoid. In this case one should use expressions (4.1), (4.2) or those of section 2.

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Векторные магнитные потенциалы тороидального соленоида

Вычислены векторные магнитные потенциалы тороидального соленоида в кулоновской калибровке $\mathrm{div}\,\vec{A} = 0$. Исследована пространственная зависимость компонент потенциала; показано, что они непрерывны во всем пространстве /включая границу соленоида/ и убывают на больших расстояниях обратно пропорционально третьей степени рассеяния. Полученные в данной работе потенциалы сравниваются с потенциалами работы /4/, которые имеют особенности вне соленоида и соответствуют сингулярной калибровке.

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Vector Magnetic Potentials of the Toroidal Solenoid

Vector magnetic potentials of the toroidal solenoid are calculated in the Coulomb gauge $\mathrm{div}\,\vec{A} = 0$. The components of the potentials are finite and continuous everywhere and behave as $1/r^3$ at large distances. These potentials are compared with those found in ref. /4/, which have discontinuites outside the solenoid and correspond to the singular gauge condition.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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