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D.I.Kazakov

CALCULATION OF FEYNMAN INTEGRALS BY THE METHOD OF "UNIQUENESS"

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1. Statement of the Problem

In the recent years a remarkable progress has been achieved in calculating higher-order corrections to various quantities. Different and very involved methods of multiloop calculations have been developed [1,2,3]. They enable us to advance in a number of loops in Feynman diagrams integrated by these methods. Nevertheless new approaches arise which not only simplify the calculations considerably but allow us to evaluate more camplicated diagrams. The present paper presents the description and development of one of these approaches based on the so-called "uniqueness" relation.

The method is aimed at calculating massless Feynman integrals of the propagator type dependent on one external momentum or one external coordinate. All the calculations are performed within the dimensional regularization and MS scheme. The dependence on a single dimensional parameter is determined by pure dimensional considerations and is power-like. The aim of the calculation is the coefficient function depending on a D =dimension of space-time. For $D=4-2\varepsilon$ it is the Laurent series in ε , and of interest are the coefficients of negative and of few first positive powers of ε .

2. Formulation of the Method

For the completeness we give here all the main formulae including already present in non-numerous references on the topic [4,5]. Hereafter we use the notation of ref.[4].

All the calculations will be performed in the coordinate space. The lines of graphs are associated with simple powers like $1/(x^2)^{\alpha}$, α being called the index of the line. For the ordinary line it is $(D-2)/2$ in a D -dimensional space due to the well-known Fourier transform

$$\int \frac{d^D p\, e^{ipx}}{p^2} = \frac{\pi^{D/2}\, 2^{D-2}}{[x^2]^{\frac{D-2}{2}}} \frac{\Gamma(D/2-1)}{\Gamma(1)}\,, \qquad (1)$$

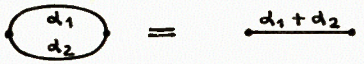
where Γ is the Euler Γ-function. For an arbitrary index ν it reads

$$\int \frac{d^D p\, e^{ipx}}{(p^2)^\nu} = \frac{\pi^{D/2}\, 2^{D-2\nu}}{[x^2]^{D/2-\nu}}\, a(\nu)\,,\quad a(\nu) \equiv \frac{\Gamma(D/2-\nu)}{\Gamma(\nu)}\,. \qquad (2)$$

We will also need the concept of the index of a vertex, of a triangle and of a diagram - the sum of the indices of constituent lines. The line, vertex, and triangle will be called "unique" if their indices are equal to 0 , D , and $D/2$, respectively.

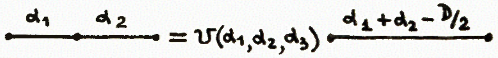
Calculations are carried out according to the following rules:

1. The contribution of a simple loop is an ordinary product



(3)

2. Chains are integrated using the graphical identity



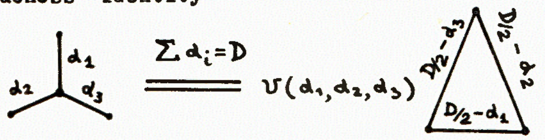
(4)

where

$$\upsilon(\alpha_1,\alpha_2,\alpha_3) = \pi^{D/2} \prod_{i=1}^{3} a(\alpha_i) \qquad (5)$$

and index $\alpha_3 = D-\alpha_1-\alpha_2$ is determined by the uniqueness condition.

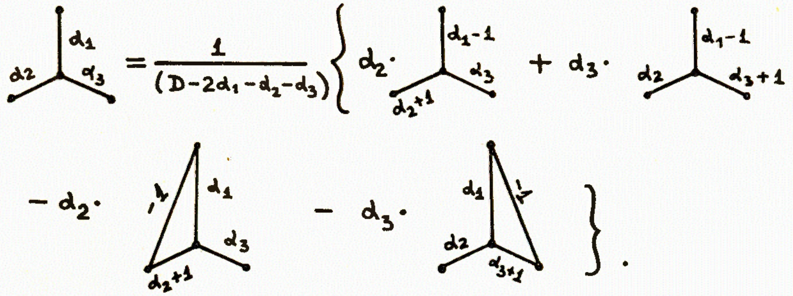
3. The main element in integration of the vertex is the well known "uniqueness" identity /6/



(6)

connecting the "unique vertex" with "unique triangle".

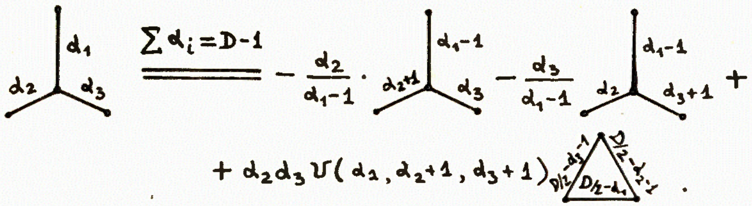
4. For the vertex with arbitrary index the following equality is valid



(7)

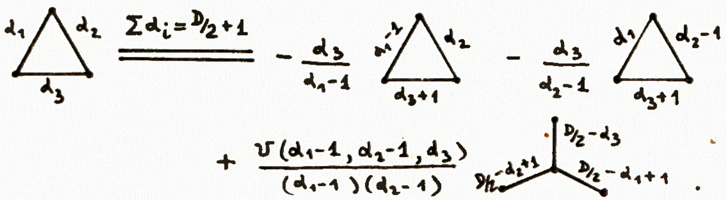
It can be obtained by integration by parts.

5. In case of the one-step-deviation from uniqueness, i.e. when the index of the vertex is $D-1$, eq. (7) due to (6) takes the form given in ref. /5/



(8)

6. There is an extra useful relation which is the inversion of (8) for the triangle one-step-deviating from the uniqueness, i.e. with the index equal to $D/2+1$ /5/:



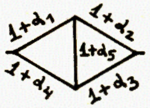
(9)

7. In case when the indices do not obey the desired properties of uniqueness there exists a point group of transformations allowing the change of their values. The summary of these transformations is given in ref. /4/. They include the insertion of a point into a line, transition to a dual (in a sense of the Fourier transform) diagram, conformal mapping of inversion. We discuss the application of these techniques below.

3. Illustration

A. Two-loop integrals

We illustrate the simplicity and efficiency of the "uniqueness" technique calculating the self-energy diagram

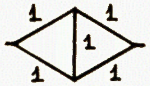
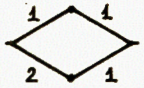
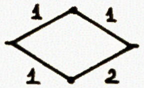
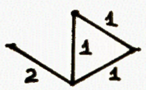
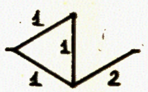


(10)

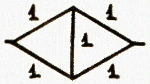
The majority of our graphs are reduced to it during the calculation. Hereafter the value of a diagram means the value of a coefficient function. The dimension of space-time is $D = 4-2\varepsilon$, $S_1 = 3+d_1+d_2+d_5$ and $S_2 = 3+d_3+d_4+d_5$ are the indices of two vertices of the integration, $t_1 = 3+d_1+d_4+d_5$ and $t_2 = 3+d_2+d_3+d_5$ are the indices of the left and right triangles, $d = 5+d_1+d_2+d_3+d_4+d_5$ is the total index of the diagram.

Let for simplicity all $d_i = 0$. Then using eq. (7) for the lower vertex we get

(11)

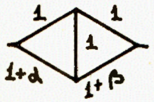
 $= \frac{1}{-2\varepsilon}\Big\{$  $+$  $-$  $-$  $\Big\}$.

A further calculation is straightforward due to eq. (3) and (4). The result is



$$= -\frac{1}{\varepsilon} v(1,1,2-2\varepsilon)\left[v(2,1,1-2\varepsilon) - v(2,1+\varepsilon,1-3\varepsilon)\right] \quad (12)$$

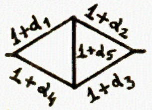
In the same manner we calculate a more complicated integral

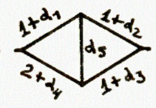
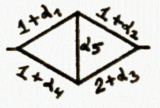


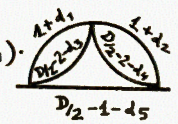
$$= -\frac{1}{(2\varepsilon+\alpha+\beta)} v(1,1,2-2\varepsilon)\Big[(1+\alpha)v(2+\alpha,1+\beta,1-2\varepsilon-\alpha-\beta) \quad (13)$$

$$-(1+\alpha)v(2+\alpha,1+\varepsilon+\beta,1-3\varepsilon-\beta) + (\alpha \leftrightarrow \beta)\Big].$$

Another characteristic example is the diagram with two triangles and one vertex one-step-deviating from the uniqueness, i.e. with $t_1 = D/2+1$, $t_2 = D/2+1$, $S_2 = D-1$. Using eq. (8) for the lower vertex we get



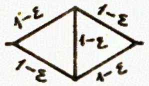
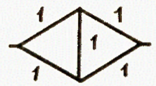
$= -\frac{1+d_4}{d_5}$  $- \frac{1+d_3}{d_5}$  (14)

$+ (1+d_3)(1+d_4)\, v(1+d_5, 2+d_3, 2+d_4) \cdot$ 

.

The first diagram in eq. (14) is integrated now due to the "uniqueness" of the left triangle, the second one – due to the "uniqueness" of the right triangle. The third diagram is trivial.

Consider now the graph with ordinary lines, i.e. with indices equal to $1-\varepsilon$. The recurrent relation (7) does not produce any integrable diagrams in this case. One should make the transformation of indices. It is useful to go to the dual diagram. For this purpose we carry out the Fourier transform of the initial graph and treat the obtained momentum diagram as a coordinate one with transformed indices. In this particular case the dual diagram is topologically equivalent to the initial one. As a result, we have

 $= \frac{\prod_{i=1}^{5} a(1-\varepsilon)}{a(1-3\varepsilon)}$  .

The resulting diagram has already been considered above.

We see that the basis of integrability is always some kind of the uniqueness. The diagrams with uniqueness are straightforwardly integrated due to eq. (6). The next class of integrable diagrams consists of those one-step-deviating from the uniqueness. A single application of the recurrent eq. (7)-(9) reduces them to the sum of diagrams with the uniqueness. Note, however, that to be integrable the diagram should be one-step-deviating from the uniqueness in three parameters simultaneously. For instance, for diagrams (11), (13) one-step-deviating from the uniqueness there are three lines (d_1, d_2 and d_5), for diagram (14) these are t_1, t_2 and S_2.

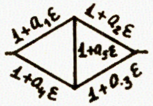
In general for the diagram under consideration there are four variants of its complete integrability (up to the obvious permutations):

I.	II.	III.	IV.
$d_1 = 0$	$t_1 = 3-\varepsilon$	$t_1 = 3-\varepsilon$	$d_1 = 0$
$d_2 = 0$	$t_2 = 3-\varepsilon$	$S_1 = 3-2\varepsilon$	$d_4 = 0$
$d_5 = 0$	$S_2 = 3-2\varepsilon$	$S_2 = 3-2\varepsilon$	$d = 5-3\varepsilon$
$d_3+d_4 \neq -2\varepsilon$	$d_5 \neq 0$	$d_1 \neq 0$	$d_2+d_5 \neq -2\varepsilon$

Three one-step deviations from the uniqueness (three lines, two triangles, and one vertex, two vertices and one triangle, two lines and total index) ensure the integrability of a diagram after applying the recurrent eq. (7). (In the last case one should also use the insertion of a point into the central line). The fulfilment of inequalities guarantees that the denominators in eq.(7) are not zero. Otherwhise one should introduce the regularization keeping all the uniqueness conditions unchanged. It leads to the derivatives of Γ -functions in the final result.

To be sure that the diagram (10) is calculable, one should establish whether there is some kind of the uniqueness condition (I-IV); If not, whether it can be obtained with the help of point transformations. The latter is easily established in two-three steps. Otherwhise a diagram is not integrated by the proposed methods.

Of a practical interest is the diagram with $\alpha_i = a_i \varepsilon$, where a_i are some numbers. In this case we are not interested in the exact expression for the diagram, rather in first few terms of the expansion in ε . Four variants of the total integrability (I-IV) enable us to get four terms of the expansion for an arbitrary diagram. The result is:



$$= \frac{\exp\left[-2\left(\gamma\varepsilon + \frac{\zeta(2)}{2}\varepsilon^2\right)\right]}{(1-2\varepsilon)} \Big\{ 6\zeta(3) + 9\zeta(4)\varepsilon +$$
$$+\big[42 + 30(a_1+a_2+a_3+a_4) + 45a_5 + 10(a_1^2+a_2^2+a_3^2+a_4^2) + 15a_5^2 + 15a_5(a_1+a_2+$$
$$+a_3+a_4) + 10(a_1a_2+a_3a_4) + 5(a_1a_3+a_2a_4) + 10(a_1a_4+a_2a_3)\big]\zeta(5)\varepsilon^2 +$$
$$+\big[90 + 75(a_1+a_2+a_3+a_4) + \frac{225}{2}a_5 + 25(a_1^2+a_2^2+a_3^2+a_4^2) + \frac{75}{2}a_5^2 + \frac{75}{2}a_5(a_1+a_2+$$
$$+a_3+a_4) + 25(a_1a_2+a_3a_4) + \frac{25}{2}(a_1a_3+a_2a_4) + 25(a_1a_4+a_2a_3)\big]\zeta(6)\varepsilon^3 -$$
$$-\big[46 + 42(a_1+a_2+a_3+a_4) + 45a_5 + 14(a_1^2+a_2^2+a_3^2+a_4^2) + 15a_5^2 + 33a_5(a_1+$$
$$+a_2+a_3+a_4) + 50(a_1a_2+a_3a_4) + 31(a_1a_3+a_2a_4) + 14(a_1a_4+a_2a_3) + 6a_5(a_1^2+$$
$$+a_2^2+a_3^2+a_4^2) + 6a_5^2(a_1+a_2+a_3+a_4) + 24a_5(a_1a_2+a_3a_4) + 12a_5(a_1a_3+a_2a_4) +$$
$$+12(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) + 12(a_1^2a_2 + a_2^2a_1 + a_3^2a_4 + a_4^2a_3) +$$
$$+6(a_1^2a_3 + a_3^2a_1 + a_2^2a_4 + a_4^2a_2)\big]\zeta^2(3)\varepsilon^3 \Big\}. \qquad (15)$$

We use here the well-known expression for the Γ -function

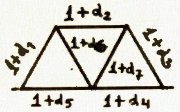
$$\Gamma(1+x) = \exp\Big\{-\gamma x + \sum_{n=2}^{\infty} \frac{(-)^n}{n}\zeta(n)x^n\Big\}, \qquad (16)$$

where γ is the Euler constant and $\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}$ is the Riemann ζ -function.

Practically this formula is sufficient for four- and almost for all five-loop calculations. To be precise, we keep the exponential in front of the braces. It can be omitted being absorbed into the redefinition of the angular integration and gives no contribution to the final result.

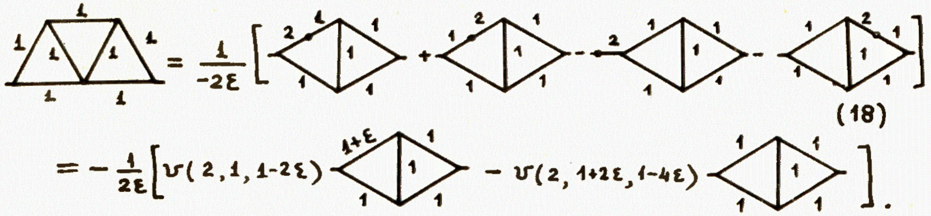
B. <u>Three-loop integrals</u>

Consider now examples of three-loop calculations. The characteristic graph now is



(17)

Let for simplicity all $\alpha_i = 0$. Applying eq. (7) to the left upper vertex we get

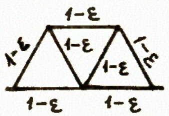


(18)

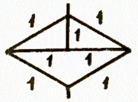
The remaining two-loop diagrams are easily integrated applying eq. (7) to the upper vertex.

We have succeeded in reducing the three-loop diagram to the two-loop ones due to the presence of three lines (α_5 , α_6 , α_7) one-step-deviating from the uniqueness. In general it is always necessary to have three such conditions. They may be: one vertex and two adjoined triangles, three triangles, three lines. If there is no such conditions, one should use the point transformations of indices.

Consider, for instance, the diagram with ordinary lines. It has no kind of the uniqueness. However, the transition to the dual diagram gives

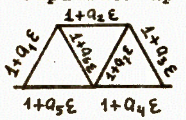


$$= \frac{\prod_{i=1}^{7} a(1-\varepsilon)}{a(1-4\varepsilon)}$$



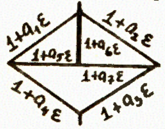
Applying now eq. (7) to the central vertex we immediately reduce this diagram to a sum of two-loop ones.

In general, the diagram (17) can be reduced if there are three parameters one-step-deviating from the uniqueness or if they can be obtained by transforming indices. The latter is easily established in each particular case. Otherwise the diagram cannot be calculated by proposed methods. However, like in the previous case we can get few terms of its expansion in ε. For $\alpha_i = a_i\varepsilon$ we can get two terms of the expansion for arbitrary a_i that corresponds to the expansion up to ε^3 in eq. (15). The result is:



$$= \frac{\exp[-3(\gamma\varepsilon + \frac{\zeta(2)}{2}\varepsilon^2)]}{(1-2\varepsilon)} \Big\{ 20\zeta(5) + \varepsilon\Big[50\zeta(6) + \Big(20+6(a_4+a_5+a_6+a_7)\Big)\zeta^2(3)\Big]\Big\}. \qquad (19)$$

Notice that the coefficient $\zeta(6)$ like coefficient $\zeta(4)$ in eq. (15) is independent of a_i. An analogous expansion can be obtained for the dual diagram



$$= \frac{\exp[-3(\gamma\varepsilon + \frac{\zeta(2)}{2}\varepsilon^2)]}{(1-2\varepsilon)} \Big\{ 20\zeta(5) + \varepsilon\Big[50\zeta(6) - \Big(4+6(a_3+a_4+a_5+a_7)\Big)\zeta^2(3)\Big]\Big\}. \qquad (20)$$

Practically these formulas are sufficient for four and even for the most of five-loop calculations. The calculation of any one diagram nonintegrable by the rules (1)-(7) allows us to extent the expansions (15), (19) and (20) by one order for arbitrary indices.

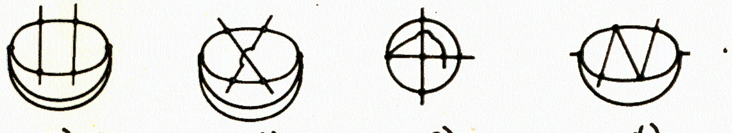
Hence the general conclusion of our consideration is the following. The integrability of a diagram is associated with some kind of the uniqueness. The presence of one uniqueness or simultaneously three one-step deviations from the uniqueness lead to the one-loop reduction of a diagram. The procedure is repeated until all the loops are reduced. If it is not possible at some step, the expan-

sions like (15), (19) are very useful. Being obtained once they can be used further as a table.

4. Five-Loop Calculations in the φ^4 Model

We present here the application of the described technique to the 5-loop renormalization group calculations in the φ^4 model. Recall that the four-loop anomalous dimensions and β-function in this model in the MS scheme were calculated analytically in ref. [7] using the methods of ref. [1,2]. The anomalous dimension γ_2 in a 5-loop approximation was calculated later[8] with the help of integration by parts [3]. Now the 5-loop calculation of the β-function is performed[9]. This has required to calculate nearly 120 diagrams. All of them but four were calculated analytically by the integration by parts. For the remaining four diagrams the Gegenbauer polynomial expansion in x-space enables us to present the answer in the form of three-fold infinite convergent series. The computer summation of these series leads to acceptable accuracy. However, quite understandable is the aspiration for an analytical answer.

To make clear the possibilities of the "uniqueness" technique, we apply it to the calculation of these four diagrams. Graphically they are



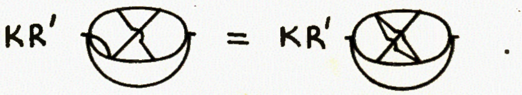
a) b) c) d)

Notice that the very fact of the calculability, i.e. representation of the answer through the ζ-functions for these diagrams is an open question. The authors of ref.[9] succeeded in the analytical summation of the series for the diagram a) and represented the answer in terms of (5) and (6). Whether it is possible for the other diagrams, is an open question.

The method of "uniqueness" gives the following results:

a) For the diagram a) the problem is equivalent to the calculation of the diagram  up to $O(\varepsilon)$. This diagram can be reduced to the V-like diagram (17) which should be also known up to $O(\varepsilon)$. The last task has already been solved in section 3B. In this way the answer was found to coincide with that of the summation obtained in ref.[9].

b) For the diagram b) the problem is equivalent to the calculation of the nonplanar crossed diagram up to $O(\varepsilon)$. This task is not so simple due to the diagram being nonplanar. The point transformations do not give the needed "uniqueness" here. However, we may use the identity following from the independence of KR' for the diagram on external lines:



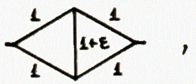
Changing the index of the external enveloping line in one of the diagrams from $1-\varepsilon$ to $1+\varepsilon$, we create the "uniqueness" in the other diagram where this line is internal. In this way the initial problem is reduced to three V-like diagrams which should be known up to $O(\varepsilon)$. The latter is of no problem and gives:

$$KR' = -\frac{\zeta(5)}{\varepsilon^2} - \frac{\frac{5}{2}\zeta(6) - 5\zeta(5) + \frac{17}{5}\zeta^2(3)}{\varepsilon} \quad (21)$$

c) The diagram c) does not contain divergent subgraphs. This means that to evaluate KR', one should know the diagram up to a constant. This enables us to choose the indices of all lines so as to create the needed one-step deviations from the uniqueness. The available arbitrariness is sufficient to reduce all the loops. The result is

$$KR' = \frac{36\zeta^2(3)}{5\varepsilon} \quad (22)$$

d) The main problem is the last diagram. It does not contain divergent subgraphs as well, so we need to know the N-like diagram up to a constant. Choosing the indices in the form $1+\alpha_i\varepsilon$ we reduce the diagram to three V-like ones which should be known up to $O(\varepsilon^2)$ or to nine two-loop diagrams (10) which should be known up to $O(\varepsilon^4)$. Hence the expansions (15), (19) are not sufficient here. The available arbitrariness is sufficient to evaluate eight out of nine obtained diagrams. The last diagram is not calculable by this method. It can be represented as



and we are interested in the coefficient $\sim \varepsilon^4$. In case when the diagram is integrable, this coefficient contains two structures - $\zeta(7)$ and $\zeta(3)\cdot\zeta(4)$. Assuming the same dependence for the diagram of interest, we can try to find the coefficients. Omitting further details we give the final answer:

$$KR' = \frac{integer}{40\varepsilon}\zeta(7),$$

where the integer number can be found from the comparison with the numerical calculation. This comparison gives

$$KR' = \frac{441}{40\varepsilon}\zeta(7) \simeq \frac{11{,}117050783135\ldots}{\varepsilon} \quad (23)$$

The result of numerical calculation is - 11,11705(1). Notice that the deviation in the integer number by unity leads to the discrepancy in the second decimal digit.

This result is equivalent to the knowledge of the diagram (10) up to ε^4 and enables us to extent the expansions (15), (19), (20) by one order for arbitrary indices.

Applying the obtained results (21), (22) and (23) to the β-function together with the answers for other diagrams given in ref. /9/ we may write the final analytical answer in the five-loop approximation: $\left(\mathcal{L}_{int} = -\frac{16\pi^2}{4!} h\varphi^4\right)$

$$\begin{aligned}\beta_{MS}(h) = {} & \frac{3}{2}h^2 - \frac{17}{6}h^3 + \left(\frac{145}{16} + 6\zeta(3)\right)h^4 \\ & - \left(\frac{3499}{96} + 39\zeta(3) - 9\zeta(4) + 60\zeta(5)\right)h^5 \\ & + \left(\frac{767261}{4608} + \frac{7965}{32}\zeta(3) - \frac{1177}{16}\zeta(4) + \frac{2049}{4}\zeta(5)\right. \\ & \left. - \frac{771}{4}\zeta(6) + \frac{45}{2}\zeta^2(3) + \frac{1323}{2}\zeta(7)\right)h^6 + O(h^7). \end{aligned} \quad (24)$$

Note that $\zeta(7)$ appears only in one diagram (as well as $\zeta(3)$ and $\zeta(5)$ in previous orders) and needs a further confirmation.

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Казаков Д.И. E2-83-323

Вычисление фейнмановских интегралов методом "уникальностей"

Дано развитие и обсуждаются приложения метода расчета безмассовых фейнмановских интегралов на основе соотношения "уникальности". Процедура получения ответа состоит из нескольких алгебраических шагов и не содержит ни взятия интегралов от элементарных или специальных функций, ни разложения в бесконечные ряды и их суммирования. Простота и эффективность метода иллюстрируется на примере вычисления двух- и трехпетлевых размернорегуляризованных интегралов. В качестве приложения предложенной техники дано аналитическое вычисление последних диаграмм, завершающих пятипетлевые ренорм-групповые вычисления в теории ϕ^4.

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Kazakov D.I. E2-83-323

Calculation of Feynman Integrals by the Method of "Uniqueness"

We develop and discuss the application of the method to calculate massless Feynman integrals based on the "uniqueness" relation. The procedure consists of several algebraic steps and involves neither integration of elementary, special or any other functions, nor expansions in and summation of infinite series of any kind. The simplicity and efficiency of the method are illustrated by the calculation of two- and three-loop dimensionally regularized integrals. As an application of the proposed technique we give the analytical calculation of the remaining diagrams completing the five-loop renormalization group calculations in the ϕ^4 model.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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