объединенный
институт
ядерных
исследований
дубна

E2-83-286

E.A.Ivanov, S.O.Krivonos

NONLINEAR REALIZATION OF THE CONFORMAL GROUP IN TWO DIMENSIONS AND THE LIOUVILLE EQUATION

Submitted to "ТМФ"

1983

1. The starting point of the inverse scattering method (ISM) description of nonlinear differential equations is the definition of a proper L-M pair [1]. Unfortunately, the form of this pair is as a rule simply guessed. Only after that, ISM comes into play. The treatment of ISM in the AKNS-approach [2] allowed one to analyze from the common standpoint most of the integrable two-dimensional equations known to date (KdV, sine-Gordon, Liouville, etc.). Nevertheless, this approach provides as before no answer to the question what is the primary principle defining, in one or another specific case, the concrete structure of the basic differential 1-form Ω_0 for which the zero-curvature representation is written down

$$d^{ext}\Omega_0 = i\,\Omega_0 \wedge \Omega_0 \qquad (1)$$

(the symbols d^{ext} and $\wedge$ mean exterior differentiation and multiplication). The only requirement to be satisfied by Ω_0 is that it should belong to the algebra $sl(2,R)$ (in generalizations of the AKNS-method, Ω_0 takes values in various algebras [3] and superalgebras [4]).

In the present paper we argue that a more general approach is possible, within which the necessary parametrization of Ω_0 emerges in a natural way. It relies upon the idea of embedding of the (super) algebra $\mathcal{G}_0$ to which Ω_0 belongs into a more extensive infinite dimensional (super) algebra $\mathcal{G}$. With this procedure, the fields entering into Ω_0 and satisfying a given integrable equation acquire the meaning of coordinates of a certain coset space of the infinite parameter group G constructed by $\mathcal{G}$. Thus, they support a nonlinear realization of G. The 1-form Ω_0 is identified with the $\mathcal{G}_0$-component of the whole Cartan form Ω on the algebra $\mathcal{G}$. The zero-curvature condition for Ω_0 and the specific parametrization of Ω_0 follow from the Maurer-Cartan equation for Ω and additional dynamical constraints which covariantly reduce the original coset space to its certain connected fully geodesic subspace.

The main advantage of the suggested scheme should be seen in the possibility of reducing the problem of searching for the new equations possessing the zero-curvature representation to the classification task of listing proper (super) algebras $\mathcal{G}$ and $\mathcal{G}_0$. For the time being, we have managed to understand in this language the Liouville equation and its supersymmetric extensions /5/ but it is plausible that other integrable systems can be interpreted in a similar way (the principal question here is which algebras $\mathcal{G}$ are connected with these systems). The constructivity of the method has been already demonstrated by us in ref. /6/ where the N=2 supersymmetric extension of the Liouville equation has been set up with its help.

The present paper contains a more detailed description of our approach by the simplest example of ordinary Liouville equation

$$u_{+-} = m^2 e^{-2u}, \tag{2}$$

where $u_{+-} \equiv \frac{\partial^2 u}{\partial x^+ \partial x^-}$, and $x^{\pm} = x^0 \pm x^1$ are the light cone coordinates of (1+1) - Minkowski space, and $[m^2] = L^{-2}$. In this case, $\mathcal{G}_0 = sl(2,R)$ and $\mathcal{G}$ is the direct sum of two contact algebras $\mathbb{K}^{\pm}(1)$ *) (Sect. 2). The related group G is isomorphic to the conformal group in two dimensions. We choose the basic coset space to be G/SO(1,1), SO(1,1) being the (1+1)-Lorentz group. The field $u(x)$ is identified with the coset parameter associated with the dilatation generator. Equation (2) appears as one of the conditions of the covariant reduction of G/SO(1,1) to the pseudosphere SL(2,R)/SO(1,1). Another reduction, which yields the free equation for $u(x)$, is to the pseudoplane $\mathcal{P}(1,1)$/SO(1,1) where $\mathcal{P}(1,1)$ is the (1+1) Poincaré group. We explain how the relevant zero-curvature representations emerge in this picture (Sect. 2) and how to construct the general solution of eq. (2) (Sect. 3). The Bäcklund transformations relating different solutions of eq. (2) to each other and to those of the free equation are interpreted as the constrained right gauge shifts on the coset space G/SO(1,1) preserving the reduction conditions (Sect. 4).

2. The contact algebras $\mathbb{K}^{\pm}(1)$ are formed by the infinite set of generators $L^n_{\pm}$ which fulfill the commutation relations

$$i[L^n_{\pm}, L^m_{\pm}] = (n-m)L^{n+m}_{\pm}, \quad i[L^n_{+}, L^m_{-}] = 0\,; \; (n,m = -1,0,1,2,\dots). \tag{3}$$

The algebra $\mathcal{G} = \mathbb{K}^{+}(1) \oplus \mathbb{K}^{-}(1)$ coincides with the conformal algebra of (1+1)-dimensional Minkowski space. The standard Virasoro algebra /8/ is a central extension of $\mathcal{G}$ continued to all negative indices $(n,m = -1,-2,\dots)$.

*) We basically follow the terminology of refs. /7/.

The algebra (3) contains several finite dimensional subalgebras. We will be interested in the subalgebra $sl(2,R)$ generated by the following combinations of $L^n_{\pm}$:

$$R_+ = L^{-1}_+ + m^2 L^1_-, \; R_- = L^{-1}_- + m^2 L^1_+, \; U = L^0_+ - L^0_-, \tag{4}$$

$$i[R_+, R_-] = -2m^2 U, \quad i[R_{\pm}, U] = \mp R_{\pm}. \tag{5}$$

In the contraction limit $m=0$ (4) and (5) go over to the algebra of the Poincaré group $\mathcal{P}(1,1)$. We identify U with the generator of the corresponding Lorentz group SO(1,1) and $L^{-1}_{\pm}$ with the translation generators.

Let us consider the nonlinear realization of group G with the algebra $\mathcal{G}$ in the coset space G/H, where H=SO(1,1) is the above Lorentz group. An element of the left coset G/SO(1,1) can be parametrized as follows:

$$g \equiv G/H = e^{ix^{\pm} L^{-1}_{\pm}} e^{i z^{\pm}_1(x) L^1_{\pm}} e^{i z^{\pm}_2(x) L^2_{\pm}} \dots e^{iu(x)(L^0_+ + L^0_-)}. \tag{6}$$

Here, $x^{\pm}$ are (1+1)-Minkowski space coordinates, and $u(x), z^{\pm}_1(x), z^{\pm}_2(x) \dots$ constitute an infinite array of coordinates-fields. The group G acts on the coset (6) from the left:

$$g_0(\lambda)\, g(x, u, z_1, \dots) = g(x', u', z'_1, \dots) \cdot h(\lambda, x), \tag{7}$$

where $g_0(\lambda)$ is an arbitrary element of G:

$$g_0(\lambda) \equiv \exp\Big(i \sum_{n=-1}^{+\infty} \lambda^{\pm}_n L^n_{\pm}\Big) \tag{8}$$

and h belongs to the subgroup H. The dependence of h on the group parameters $\lambda^{\pm}_n$ and the coset space coordinates is uniquely fixed by the commutation relations (3). The arrangement of the group-factors as in (6) is convenient in that the transformation law of coordinates coincide with the ordinary (1+1)-conformal transformation:

$$\delta x^{\pm} = \lambda^{\pm}(x) = \sum_{n=-1}^{+\infty} (x^{\pm})^{n+1} \lambda^{\pm}_n \tag{9}$$

while the group variation of $u(x)$ and the element h depend only on the Minkowski space coordinates $x^{\pm}$, but not on the coordinates-fields:

$$\begin{cases} \delta u(x) = u'(x') - u(x) = \frac{1}{2}(\partial_+ \lambda^+ + \partial_- \lambda^-) \\ h(\lambda, x) = \exp\{i[\frac{1}{2}(\partial_+ \lambda^+ - \partial_- \lambda^-) + \mathcal{O}(\lambda^2)]U\}. \end{cases} \tag{10}$$

The geometry of the coset space G/H is described by the Cartan forms which are introduced by the familiar relation /9/

$$g^{-1}dg = i\sum_{n=-1}^{+\infty} \omega_\pm^n L_\pm^n = i\Omega \equiv i(\Omega_0 + \Omega_1). \tag{11}$$

The forms Ω_0 and Ω_1 are defined so that they lie in the algebra Sl(2,R) (4) and its orthogonal complement, respectively (the latter is spanned by the infinite set of generators $L_+^{-1} - m^2 L_-^1$, $L_-^{-1} - m^2 L_+^1$, $L_+^0 + L_-^0$, $L_\pm^2$, $L_\pm^3, \ldots$).

The whole 1-form Ω transforms under the shifts (7) according to the standard law of nonlinear realizations /9/

$$\Omega' = -i\, h^{-1}(d + i\Omega)h. \tag{12}$$

All the components of Ω, except for that of the generator U, transform homogeneously. Let us quote several first components explicitly

$$\begin{cases} \omega_\pm^{-1} = e^{-u} dx^\pm \\ \omega_\pm^0 = du - 2z_1^\pm dx^\pm \\ \omega_\pm^1 = e^u (dz_1^\pm + (z_1^\pm)^2 dx^\pm - 3 z_2^\pm dx^\pm) \\ \omega_\pm^2 = e^{2u}(dz_2^\pm + 4 z_1^\pm z_2^\pm dx^\pm - 4 z_3^\pm dx^\pm). \end{cases} \tag{13}$$

We will also need to know the structure of components of the form $\Omega_0 = \omega_0^{R+} R_+ + \omega_0^{R-} R_- + \omega_0^U U$:

$$\begin{cases} \omega_0^{R+} = \dfrac{1}{2m^2}(m^2 \omega_+^{-1} + \omega_-^1) \\ \omega_0^{R-} = \dfrac{1}{2m^2}(m^2 \omega_-^{-1} + \omega_+^1) \\ \omega_0^U = z_1^- dx^- - z_1^+ dx^+ . \end{cases} \tag{14}$$

Note that the components of the form Ω_1 associated with the generators $L_+^{-1} - m^2 L_-^1$, $L_-^{-1} - m^2 L_+^1$ result from ω^{R+}, ω^{R-} through the change $m^2 \to -m^2$.

By construction, the 1-form Ω (11) satisfies the Maurer-Cartan equation on the full algebra $\mathcal{G}$:

$$d^{ext}\Omega = i\,\Omega \wedge \Omega . \tag{15}$$

Let us stress that at this stage eq. (15) is satisfied identically and has no any dynamical content. The dynamics arises as a result of covariant reduction of the coset space G/SO(1,1) to its subspace SL(2,R)/SO(1,1) . This reduction is effected by setting the G/SL(2,R)-component of Ω equal to zero

$$\Omega_1 = 0. \tag{16}$$

The constraint (16) is manifestly covariant under the action of the group G . Expanding Ω_1 in generators $L_+^{-1} - m^2 L_-^1$, $L_-^{-1} - m^2 L_+^1$, $L_+^0 + L_-^0$, $L_\pm^2$, $L_\pm^3$, $L_\pm^4, \ldots$ we obtain the infinite sequence of the relations

$$\omega_1^\pm = m^2 \omega_{-1}^\mp , \tag{17$\pm$}$$

$$\omega_0^+ + \omega_0^- = 0, \tag{18a}$$

$$\omega_n^\pm = 0 \quad (n \geq 2). \tag{18b}$$

Each of them yields two equations, for the coefficients of dx^+ and dx^- in the corresponding ω . The equations for the coefficients of dx^+ in ω_n^+ and for those of dx^- in ω_n^- express the higher parameters-fields $z_1^\pm(x)$, $z_2^\pm(x), \ldots$ in terms of the single object, the dilaton $u(x)$:*)

$$z_1^\pm(x) = \partial_\pm u(x), \tag{19a}$$

$$z_2^\pm(x) = \frac{1}{3}\left\{\partial_\pm^2 u(x) + [\partial_\pm u(x)]^2\right\}, \quad etc. \tag{19b}$$

The dynamics is concentrated in the relations (17$\pm$); upon inserting the expressions (19) in the form $\omega_1^\pm$, each of the equations obtained by projecting (17$\pm$) respectively on dx^- and dx^+ reduces to the Liouville equation (2). We prove in Appendix that the rest of eqs. (17) and (18) do not impose additional restrictions on $u(x)$ and are fulfilled identically.

It can be easily seen that the zero-curvature representation for eq. (2) automatically arises in this picture. Indeed, upon imposing the constraint (16) the 1-form Ω (11) reduces to the sl(2,R)-valued 1-form Ω_0^{Red} depending on the single field $u(x)$

---

*) The conditions (17) and (18) are the particular case of constraints of the inverse Higgs phenomenon /10/ having a wide area of application in nonlinear realizations. It immediately follows from the general theorem of ref./10/ that all parameters-fields $z_1^\pm$, $z_2^\pm$, $z_3^\pm, \ldots$ are expressible in terms of $u(x)$. To see this, it is of no need to know the detailed structure of Cartan forms, it is sufficient to analyze the commutation relations (3).

$$\Omega^{Red}=\Omega_0^{Red}=\omega_0^{R_+}R_+ +\omega_0^{R_-}R_- +\omega_0^{U}U=e^{-u}(dx^+R_+ +dx^-R_-)+(u_-dx^- -u_+dx^+)U. \quad (20)$$

Inserting eq. (16) into the initial Maurer-Cartan equation (15) immediately yields the zero-curvature condition for Ω_0^{Red}:

$$d^{ext}\Omega_0^{Red}=i\,\Omega_0^{Red}\wedge\Omega_0^{Red}. \quad (21)$$

It is a simple exercise to verify that eq. (21) is equivalent to the Liouville equation.

Let us discuss shortly the geometric meaning of constraint (16). According to Cartan/11/ equations of this kind (the Pfaff equations) always correspond to extracting some connected fully geodesic submanifold in a given group manifold. In the case we are considering such a submanifold is the two-dimensional pseudosphere SL(2,R)/SO(1,1). **The field u(x) specifies the embedding of this pseudo**sphere into G/SO(1,1). In order to be convinced that the components of the 1-form (20) actually describe a pseudosphere ($\omega_0^{R_+}$, $\omega_0^{R_-}$ - are covariant differentials, ω_0^U is the SO(1,1)-connection), one should construct the relevant invariant interval

$$ds^2=\omega^{R_+}\omega^{R_-}=e^{-2u}dx^+dx^- \quad (22)$$

and evaluate the curvature of the metric. When u(x) is subject to the Liouville equation (2) this curvature is equal to $-\frac{1}{4}m^2$.

Another connected two-dimensional subspace of the coset space G/SO(1,1) is the pseudo-Euclidean plane $\mathcal{P}(1,1)$/SO(1,1). One may perform the covariant reduction to this subspace too. It is achieved by singling out of the whole form Ω its part associated with the Poincaré group generators $L^{-1}_{\pm}$, $L^0_+-L^0_-$ and by nullifying its remaining piece spanned by the generators $L^1_{\pm}, L^0_++L^0_-, L^2_{\pm},\dots, L^n_{\pm},\dots$ The relations (17), (18) are replaced by the following ones:

$$\omega_0^+ +\omega_0^- =0\,,\quad \omega_n^{\pm}=0 \quad (n\geq 1) \quad (23)$$

(they are simply the contraction limit of eqs. (17), and (18)). The higher parameters-fields are expressed in terms of u(x) by the same formulas as before while the field u(x) satisfies now the free equation:

$$u_{+-}=0 \quad (24)$$

for which the zero-curvature condition on the group $\mathcal{P}(1,1)$ emerges (the corresponding Ω_0^{Red} is again given by the expression (20) but with R_+, R_- replaced by the ordinary translation generators L^{-1}_+, L^{-1}_-). As is expected, the curvature of the metric in eq. (22) vanishes in this case. Thus, in the present approach the Liouville

equation (2) and the free equation (24) are described in the uniform manner as the conditions of extracting different connected subspaces in the same coset space G/SO(1,1).

The described mechanism of implementing the zero-curvature representation is advantageous in that the necessary structure of the basic 1-form Ω_0 is completely fixed within its framework by the choice of extended algebra $\mathcal{G}$, the stability group algebra $\mathcal{H}$ and the algebra $\mathcal{G}_0$ to which Ω_0 belongs. The choice of the two-dimensional Lorentz group SO(1,1) as H in the present case is dictated by the minimality requirement; with any wider H the array of essential, unremovable parameters of the coset space would include other fields besides u(x).

Let us explain why the algebra $\mathcal{G}$ should be infinite dimensional for our construction to be valid. At the first sight, we might restrict ourselves to the maximal finite dimensional subalgebra so(2,2) of $\mathcal{G}$ with the generators $\{L^{-1}_{\pm}, L^0_{\pm}, L^1_{\pm}\}$. The relevant Cartan forms are given by the first three of expressions (13) in which one has to put $z_2^{\pm}=0$. The equations by which the subspace **SL(2,R)/SO(1,1) is singled out coincide with (17) and (18a). One again obtains the Liouville equation for u(x) but it is followed now by the additional constraints**

$$\partial_{\pm}^2 u(x)+[\partial_{\pm}u(x)]^2=0 \quad (25)$$

(these originate from the relations (19b) after setting there $z_2^{\pm}=0$). The conditions (25) are compatible with eq. (2) but they strictly fix the coordinate dependence of u(x), selecting a class of particular solutions of eq. (2).*). Thus, in order to obtain the Liouville equation without extra restrictions the original structure of 1-forms $\omega_1^{\pm}$ should be the same as in eqs. (13), i.e. one should include from the begining the parameters $z_2^{\pm}(x)$ in the coset space, and, hence, the generators $L^2_{\pm}$ in the algebra $\mathcal{G}$. But adding of $L^2_{\pm}$ to the generators $L^{-1}_{\pm}, L^0_{\pm}, L^1_{\pm}$ inevitably produces the whole algebra $\mathbb{K}^+(1)\oplus\mathbb{K}^-(1)$ because commuting of $L^2_{\pm}$ with $L^1_{\pm}$ gives $L^3_{\pm}$ and so on.

Once the explicit structure of Ω_0^{Red} is established, one may readily write the linear set for eq. (2)./5/, i.e. the system of equations for which (2) serves as the integrability condition. This set looks as

$$d\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\frac{1}{i}\,\Omega_0^{Red}\begin{pmatrix}v_1\\ v_2\end{pmatrix} \quad (26)$$

*) These solutions are of the form $u_0(x)=\ln[c_1x^+x^-+c_2x^++c_3x^-+c_4]$, $c_2c_3-c_1c_4=m^2$.

with Ω_0^{Red} being the matrix 2x2 in the fundamental representation of SL(2,R) with the generators:

$$R_+=\begin{pmatrix} 0 & im \\ 0 & 0 \end{pmatrix}, \quad R_-=\begin{pmatrix} 0 & 0 \\ -im & 0 \end{pmatrix}, \quad U=-\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{27}$$

Explicitly:

$$\partial_+\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=\begin{pmatrix} \frac{1}{2}u_+ & m\eta e^{-u} \\ 0 & -\frac{1}{2}u_+ \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \partial_-\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=\begin{pmatrix} -\frac{1}{2}u_- & 0 \\ -\frac{me^{-u}}{\eta} & \frac{1}{2}u \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{28}$$

Here we have introduced the spectral parameter η by a constant right SO(1,1)-rotation of the coset element (6).

To close this Section, we briefly discuss the Euclidean case. $\mathcal{G}$ is now the complex algebra $\mathbb{K}_c(1)$, the corresponding group G is isomorphic to the conformal group of Euclidean plane, the generators L_+^n, L_-^m are mutually conjugated, the same is true for the coordinates x^+, x^- and the parameter-fields $\mathcal{Z}_n^{\pm}$ ($n \geqslant 1$). When H is chosen to be the group of two dimensional rotations SO(2), the single essential parameter of the coset space G/H is again the relevant dilaton. The Euclidean analog of subalgebra so(2,2) is so(1,3) while subalgebra sl(2,R)$\approx$so(1,2) with generators (4) has two analogs, so(1,2) and so(3), turning into each other with $m^2 \to -m^2$ (in the pseudoeuclidean case, the change $m^2 \to -m^2$ in formulas (4) yields again sl(2,R)). Hence, there exist covariant reductions of the coset G/H to three connected subspaces with the same isotropy group SO(2): sphere SO(3)/SO(2), the Lobachevski surface SO(1,2)/SO(2) and the Euclidean plane $\mathcal{P}_2$/SO(2). These reductions yield, respectively, two Euclidean Liouville equations which differ in sign before m^2, and the free equation.

3. In this Section, we show how to construct the general solution of eq. (2) within the present scheme. The method we apply has been used in refs./12/ and proceeds as follows. Once the 1-form $i\Omega^{Red} = g^{-1}dg$ belongs to sl(2,R) and meets the zero-curvature condition, it is representable as

$$i\,\Omega^{Red} = g_0^{-1}\,dg_0\,, \tag{29}$$

with g_0 being some element of SL(2,R). Taking g_0 in the parametrization

$$g_0 = e^{i\alpha R_+}e^{i\beta R_-}e^{i\gamma U}, \tag{30}$$

where α, β, γ are arbitrary functions of x^+, x^- for the moment, we express u(x) through these functions by the condition (29). Since the structure of Ω_0^{Red} is fixed by the formula (20), not all of the above functions turn out to be independent, there arise some relations between them. Besides, their coordinate dependence is specified in a definite way. Indeed, the r.h. side of eq. (29) is as follows:

$$\begin{cases} \omega^{R_+} = e^{-\gamma}d\alpha \\ \omega^{R_-} = e^{\gamma}(d\beta + m^2\beta^2 d\alpha) \\ \omega^{U} = d\gamma - 2m^2\beta\, d\alpha\,. \end{cases} \tag{31}$$

Comparing eqs. (31) and (20) gives rise to the following restrictions on α, β, γ :

$$\alpha = \Psi(x^+),\ \beta = (m^2\Psi(x^+) + \varphi(x^-))^{-1},\ \gamma = u + \ln \Psi_+(x^+)\,, \tag{32}$$

$$\exp(-2u) = -\frac{\varphi_-(x^-)\,\Psi_+(x^+)}{(m^2\Psi(x^+) + \varphi(x^-))^2}\,, \tag{33}$$

where $\Psi(x^+)$ and $\varphi(x^-)$ are arbitrary functions of x^+ and x^-, respectively. The expression (33) gives the desirable general solution of the Liouville equation.

It is worth noting that the functions $\Psi(x^+)$ and $\varphi(x^-)$ entering into the general solution (33) simultaneously solve the equations of motion of the nonlinear σ-model on the coset SL(2,R)/SO(1,1) associated with the Liouville equation through the relation (29)*). In terms of 1-forms (31), these equations read as

$$\begin{cases} \partial_+\omega_-^{R_+} + \omega_+^{U}\omega_-^{R_+} = 0 \\ \partial_+\omega_-^{R_-} - \omega_+^{U}\omega_-^{R_-} = 0, \end{cases} \tag{34}$$

where $\omega_-^{R_\pm}, \omega_+^{U}$ are the coefficients of dx^- and dx^+ in $\omega^{R_\pm}, \omega^{U}$. To see that eqs. (34) are satisfied when (31) is equated to Ω_0^{Red} (20), it suffices to write (34) in terms of u(x). Relation between the Liouville equation and nonlinear σ- model on the coset SL(2,R)/SO(1,1) has also been discussed in refs./13/ in context of the relativistic string theory.

---

*) It seems that an analogous correspondence exists between supersymmetric extensions of the Liouville equation /5,6/ and nonlinear σ-models on certain internal supergroups.

4. Now we turn to considering the Bäcklund transformations. The knowledge of them is crucial for determining infinite series of conserved currents[14], exposing the relevant hidden symmetries [15], and so on.

The standard Bäcklund transformation can be defined as a one-parameter family of mappings which project the solutions of a given integrable equation onto the solutions of the same or some other equation. These transformations do not affect the space coordinates and are realized on the field u(x) and its derivatives. The Bäcklund transformations for the Liouville equation are well known (see, e.g.[5]). Here we derive them within our method.

Let us begin with transformations relating different solutions of eq. (2) to each other. In the present scheme, they are implemented as the right gauge shifts of the coset element (6):

$$\tilde{g} = g\, e^{i m b^{\pm}(x) L^{1}_{\pm}}\, e^{i a(x)(L^{0}_{+} + L^{0}_{-})} \tag{35}$$

restricted by the requirement of preserving the reduction constraint (16):

$$\tilde{\Omega}_1 = 0, \tag{36}$$

where

$$\tilde{\Omega}_1 = -i(\tilde{g}^{-1} d\tilde{g} - i\tilde{\Omega}_0) = \tilde{\Omega} - \tilde{\Omega}_0 \tag{37}$$

and $\tilde{\Omega}_0 \in sl(2,R)$. The condition (36) results in the following equations for gauge parameters $b^{\pm}(x)$ and $a(x)$:

$$\begin{cases} m e^{-a} \omega_{\eta}^{R+} = e^{a}(db^{-} + m\omega_{\eta}^{R+} + m(b^{-})^2 \omega_{\eta}^{R-} + \omega_{\eta}^{U} b^{-}) \\ m e^{-a} \omega_{\eta}^{R-} = e^{a}(db^{+} + m\omega_{\eta}^{R-} + m(b^{+})^2 \omega_{\eta}^{R+} - \omega_{\eta}^{U} b^{+}) \\ da = m(b^{+}\omega_{\eta}^{R+} + b^{-}\omega_{\eta}^{R-}) \end{cases} \tag{38}$$

with $\omega_{\eta}^{R+}, \omega_{\eta}^{R-}, \omega_{\eta}^{U}$ defined by eq. (20) into which the spectral parameter η is introduced. Bearing in mind that the transformed field $\tilde{u}(x)$ is connected with u(x) as

$$\tilde{u} = u + a$$

one may check that eqs. (38) are reduced to the following system:

$$\begin{cases} \tilde{u}_{+} + u_{+} = 2\eta m\, sh(\tilde{u} - u) \\ \tilde{u}_{-} - u_{-} = \frac{m}{\eta} \exp[-(\tilde{u} + u)] \end{cases} \tag{39}$$

It is not difficult to be convinced that the integrability condition of the system (39) is just the Liouville equation for $\tilde{u}(x)$

$$\tilde{u}_{+-} = m^2 \exp(-2\tilde{u}) \tag{40}$$

so the relations (39) define the Bäcklund transformation (the fact that $\tilde{u}(x)$ obeys the Liouville equation follows directly from the form of the constraint (36) to which all considerations of Sect. 2 are applicable).

If, instead of the conditions (36), one imposes on $\tilde{\Omega}$ constraints of the type (23) which single out the pseudoplane $\mathcal{P}(1,1)/SO(1,1)$:

$$\tilde{\omega}_0^{+} + \tilde{\omega}_0^{-} = 0, \quad \tilde{\omega}_n^{\pm} = 0 \quad (n \geq 1), \tag{4I}$$

the relation between $\tilde{u}(x)$ and u(x) takes a slightly different form:

$$\begin{cases} \tilde{u}_{+} + u_{+} = m\eta\, e^{\tilde{u} - u} \\ \tilde{u}_{-} - u_{-} = \frac{m}{\eta} e^{-(\tilde{u} + u)}. \end{cases} \tag{42}$$

The field $\tilde{u}(x)$ is subject now to the free equation

$$\tilde{u}_{+-} = 0$$

which is the consistency condition for the system (42). Thus, the relations (42) yield the Bäcklund transformations from solutions of the Liouville equation to those of the free one.

The present approach essentially clarifies the geometric and group-theoretical meaning of Bäcklund transformations. They convert into each other different geodesic hypersurfaces of the coset G/SO(1,1)(a pseudosphere into a pseudosphere or pseudoplane) and have a uniform representation by the right constrained gauge transformations acting in the covering space. The distinction between Bäcklund transformations of the first and second kind has its origin in the difference between the constraints (36) and (4I) fixing geometry on the hypersurface to which one passes*).

Note that the transformation (35) is the most general right gauge shift which does not affect the coordinates $x^{\pm}$, does not spoil the parametrization (6) of the coset G/SO(1,1) and is compatible with the constraints (36) or (4I). The first two properties agree also with the gauge shifts generated by $L^{n}_{\pm}$ (n $\geq$ 2) but the constraints (36) or (4I) force the corresponding gauge parameters to vanish.

---

*) In the contraction limit $m^2 \to 0$, the transformations (39), (42) go over to the "Bäcklund transformation" of the free equation:

$$\tilde{u}_{\pm} \pm u_{\pm} = 0, \quad \tilde{u}_{+-} = 0, \quad u_{+-} = 0.$$

The properties that $x^\pm$ do not shift under the transformation (35) and the element $\tilde{g}$ has the same appearance as g imply that the transformed 1-form $\tilde{\Omega}_0^{Red}$ looks just as Ω_0^{Red} (20) but with $\tilde{u}(x)$ instead of $u(x)$ (in the case of the conditions (41), one has also to replace the generators $R_\pm$ by $L^{-1}_\pm$). So, as far as the 1-form Ω_0^{Red} is considered, the Bäcklund transformation of the first kind is effectively reduced to a certain restricted gauge SL(2,R)-transformation. This fact has been mentioned in refs./16/. In the case of Bäcklund transformation of the second type, the relation between $\tilde{\Omega}_0^{Red}$ and Ω_0^{Red} proves to be more complicated; it involves right shifts with the generators which lie outside of SL(2,R) (in particular, with $L^0_+ + L^0_-$).

Note that the reduction constraints (16) and (23) possess from the beginning an evident freedom with respect to right gauge SL(2,R) and $\mathcal{P}(1,1)$ transformations, respectively. But these transformations are of purely kinematic character, as they maintain (16), (23) from the very beginning, with placing no restrictions on the gauge parameters (the freedom we talk about reflects an arbitrariness in the choice of origin of coordinate sets in the coset spaces G/SL(2,R) and $G/\mathcal{P}(1,1)$). They act on the form Ω_0^{Red}(20) as ordinary Yang-Mills transformations, inserting into its components three arbitrary functions (the latter can be chosen, e.g. so as to alter the u(x)-dependence of Ω_0^{Red}). The zero-curvature condition (21) is invariant in the obvious way with respect to such redefinitions and is always equivalent to the Liouville equation (or to the free equation in the case of the reduced $\mathcal{P}(1,1)$-form). The Bäcklund transformations radically differ from these right SL(2,R)-and $\mathcal{P}(1,1)$-shifts in that they contain, when realized on the coset (6), the other generators of G beyond those of the subgroups SL(2,R) or $\mathcal{P}(1,1)$.

To conclude this Section, we mention that in the Euclidean case there exist several types of Bäcklund transformations relating to each other the relevant geodesic subspaces SO(3)/SO(2), SO(1,2)/SO(2) and $\mathcal{P}$(2)/SO(2).

5. In the present paper we have shown that the simplest integrable system, the Liouville equation, has an adequate description in the universal language of nonlinear realizations and Cartan forms. Actually, one may treat the theory of this equation as a kind of nonlinear σ-model associated with the conformal group in two dimensions (the coordinates $x^\pm$ and parameters-fields $z_n^\pm(x)$ are direct analogs of Goldstone fields of ordinary σ-models). The zero-curvature representation and Bäcklund transformations naturally arise in this picture and admit a transparent group-theoretic interpretation.

It provides grounds for belief that the other properties of the Liouville equation connected with its complete integrability, such as the existence of infinite series of conservation laws (different from those caused by G-invariance), the existence of transform reducing eq. (2) to a system of linear equations, etc., will also get a simple explanation in the present approach. We note, in particular, that the linearization of eqs. (17), (18) is achieved by passing to some new special parametrization of the coset elements G/SO(1,1). These problems will be discussed elsewhere.

In conclusion, let us indicate some further lines of thinking. An interesting task is to generalize the present construction to other integrable systems and their supersymmetric extensions*). As have been already mentioned, the main question one faces in carrying out this program is as to what are the corresponding (super) algebras $\mathcal{G}$, analogs of the algebra (3). We know which superalgebras are connected with various supersymmetric extensions of the Liouville equation; these are the contact superalgebras $K(1|N)$ (an explicit construction has been given yet only for the cases of N=1 and N=2 /6/). We have also verified that the simplest bosonic extension of eq. (2), the complex Liouville equation/18/, arises when choosing $\mathcal{G}$ to be the complexification $K_c^+(1) \oplus K_c^-(1)$ of the algebra (3). It still remains to learn which $\mathcal{G}$ correspond to more interesting bosonic systems such as the sine-Gordon and KdV equations, chiral models, the Leznov-Saveliev systems/12/, etc. One may hope to obtain, in perspective, a kind of group-theoretic classification of completely integrable systems according to their (super) algebras $\mathcal{G}$.

We believe that the presented method, being algorithmic enough, will allow one to understand on a common ground the connections between different two-dimensional integrable models, their relation to realistic four-dimensional theories and will help in searching for integrable systems in dimensions higher than 2. In particular it would be interesting to interpret in this spirit the self-dual sector of the Yang-Mills theory. Let us emphasize that the Yang-Mills theory, analogously to the Liouville equation, can be treated as a nonlinear realization of a certain infinite parameter symmetry/19/. Bearing in mind a possible integrability of gauge theories, this analogy seems to deserve attention.

We are thankful to P.P.Kulish, D.A.Leites and V.I.Ogievetsky for valuable discussions.

---

*) The statement that any integrable system is connected with some nonlinearly realized infinite parameter symmetry has been formulated as the existence theorem in ref./17/.

Appendix

We prove here the self-consistency of the reduction conditions (17) and (18). At first glance, the infinite sequence of equations (17) and (18) might constrain the field u(x) more strictly than the Liouville equation alone. However, it does not come about. To demonstrate this, we write down the Maurer-Cartan equation (15) in terms of the 1-forms $\omega^n_\pm$:

$$d\omega^{-1}_\pm = \omega^{-1}_\pm \wedge \omega^0_\pm \tag{A1a}$$

$$d\omega^0_\pm = -2\omega^{-1}_\pm \wedge \omega^1_\pm \tag{A1b}$$

$$d\omega^1_\pm = -\omega^1_\pm \wedge \omega^0_\pm + 3\omega^{-1}_\pm \wedge \omega^2_\pm \tag{A1c}$$

$$d\omega^2_\pm = -2\omega^2_\pm \wedge \omega^0_\pm + 4\omega^{-1}_\pm \wedge \omega^3_\pm \tag{A1d}$$

. . . .

$$d\omega^n_\pm = \sum_{k,m=-1,\ k+m=n,\ m>k}^{+\infty} (m-k)\,\omega^k_\pm \wedge \omega^m_\pm . \tag{A1n}$$

Let us represent $\omega^n_\pm$ as

$$\omega^n_\pm = \nabla_+ \omega^n_\pm\, \omega^{-1}_+ + \nabla_- \omega^n_\pm\, \omega^{-1}_- , \tag{A2}$$

where $\nabla_\pm \omega^n$ are covariant derivatives. Then, that part of eqs. (18b) whose role is to eliminate higher parameters-fields $\chi^\pm_n (n \geqslant 3)$ can be covariantly written as

$$\nabla_\pm \omega^n_\pm = 0 \qquad (n \geqslant 2) \tag{A3}$$

whence

$$\omega^n_\pm = (\nabla_\mp \omega^n_\pm)\, \omega^{-1}_\pm . \tag{A4}$$

Inserting of the conditions (17$\pm$), (18a) into (A1c) yields

$$d\omega^{-1}_\mp = \omega^{-1}_\mp \wedge \omega^0_\mp + 3\omega^{-1}_\pm \wedge \omega^2_\pm . \tag{A5}$$

Comparing (A4) with (A1a), we find

$$\omega^{-1}_\pm \wedge \omega^2_\pm = 0 . \tag{A6}$$

Since $\omega^2_\pm = (\nabla_\mp \omega^2_\pm)\,\omega^{-1}_\mp$ in virtue of eq. (A3), it follows from (A6) that

$$\nabla_- \omega^2_+ = 0 , \tag{A7}$$

i.e.

$$\omega^2_\pm = 0 . \tag{A8}$$

Analogously, taking into account (A8) one obtains from (A1d):

$$\omega^3_\pm = 0 . \tag{A9}$$

Proceeding further by induction, it is possible to prove

$$\omega^n_\pm = 0 \qquad (n \geqslant 2) . \tag{A10}$$

Thus, the whole system of the reduction conditions (17$\pm$), and (18) is satisfied provided the equations (17$\pm$), (18a) and (A3) hold. These equations have no other consequences apart from elimination of parameter-fields $\chi^\pm_n(x)\ (n \geqslant 1)$ and the Liouville equation for u(x).

References

1. A.C.Scott, F.Y.F.Chu, D.W.McLaughlin, Proc. IEEE, 1973, v. 61, p. 1443;
   V.E.Zakharov et al., Theory of Solitons: Inverse Scattering Method, M., "Nauka", 1980.
2. M.J.Ablowitz, D.J.Kaup, A.C.Newell, H.Segur, Phys.Rev.Lett., 19 (1967) 1095.
3. V.E.Zakharov, A.V.Mikhailov, Sov. Phys. JETP, 47 (1978) 1953.
   R.D'Auria, T.Regge, S.Sciuto, Nucl.Phys., B171 (1980) 167.
4. R.D'Auria, S.Sciuto, Nucl.Phys., B171 (1980) 189.
5. M.Chaichian, P.P.Kulish, Phys.Lett., 78B (1978) 413.
6. E.A.Ivanov, S.O.Krivonos, JINR, E2-83-104, Dubna, 1983.
7. V.G.Kac, Comm.Math.Phys., 53 (1977) 31; Adv. Math., 26 (1977) 8.
8. M.Ademollo et al., Nucl.Phys., B111 (1976) 77.
9. S.Coleman, J.Wess, B.Zumino, Phys.Rev., 177 (1969) 2239;
   C.L.Callan et al., ibid., 2247.
   D.V.Volkov, Part. and Nucl., 4 (1973) 3.
10. E.A.Ivanov, V.I.Ogievetsky, Teor.Mat. Fiz., 25 (1975) 164.
11. L.P.Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Dover, N-Y, 1960.
12. A.N.Leznov, M.V.Saveliev, Part. and Nucl., 11 (1980) 40; 12 (1981) 161.

13. A.A.Zheltukhin, Yad.Fiz., 33 (1981) 1723,
V.V.Nesterenko, JINR, E2-82-745, Dubna, 1982.
14. M.Wadati et.al., Prog.Theor.Phys., 53 (1975) 419.
15. S.Kumei, J.Math.Phys., 16 (1975) 2461.
16. R.Sasaki, Nucl.Phys., B154 (1979) 343.
C.Bhattacharya, H.Bohr, ICTP/81/82-113.
17. B.G.Konopelchenko, Lett.Math.Phys., 3 (1979) 67.
18. B.M.Barbashov, V.V.Nesterenko, A.M.Chervyakov, Comm.Math.Phys., 84 (1982) 471.
19. E.A.Ivanov, V.I.Ogievetsky, Pis'ma JETP, 23 (1976) 661.

Received by Publishing Department
on April 29, 1983.

Иванов Е.А., Кривонос С.О. E2-83-286

Нелинейная реализация конформной группы двумерия
и уравнение Лиувилля

Показано, что уравнение Лиувилля $u_{+-} = m^2 e^{-2u}$ имеет адекватное описание на языке нелинейной реализации бесконечнопараметрической конформной группы двумерия G. Координаты двумерного пространства Минковского x^+, x^- и поле $u(x)$ отождествляются с определенными параметрами фактор-пространства G/H, где $H = SO(1,1)$ - группа Лоренца двумерия. Уравнение Лиувилля возникает как одно из ковариантных условий редукции фактор-пространства G/H к его связному геодезическому подпространству $SL(2,R)/H$. Альтернативная редукция к подпространству $\mathcal{P}(1,1)/H$, где $\mathcal{P}(1,1)$ - двумерная группа Пуанкаре, приводит к свободному уравнению на $u(x)$. Соответствующие представления нулевой кривизны и преобразования Бэклунда приобретают в данном подходе простой теоретико-групповой смысл. Обсуждается возможность обобщения предложенной конструкции на другие интегрируемые системы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1983

Ivanov E.A., Krivonos S.O. E2-83-286

Nonlinear Realization of the Conformal Group in Two Dimensions
and the Liouville Equation

The Liouville equation $u_{+-} = m^2 e^{-2u}$ is shown to have an adequate description in terms of the nonlinear realization of infinite parameter conformal group G in $(1+1)$ dimensions. The $(1+1)$-Minkowski space coordinates x^+, x^- and the field $u(x)$ are identified with certain parameters of the coset G/H, $H = SO(1,1)$ being the $(1+1)$-Lorentz group. The Liouville equation appears as one of the covariant constraints reducing this coset space to its connected geodesic subspace $SL(2,R)/H$. . An alternative reduction to the subspace $\mathcal{P}(1,1)/H$ ($\mathcal{P}(1,1)$ is $(1+1)$-Poincaré group), yields the free equation for $u(x)$. . We demonstrate that the relevant zero-curvature representations and Bäcklund transformations get a simple group-theoretic interpretation within this approach and discuss a possibility of its extension to other integrable systems.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1983