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SUMMATION OF ASYMPTOTIC EXPANSIONS: THE gx^{2N}-ANHARMONIC OSCILLATOR

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1. INTRODUCTION

In a previous paper [1] we proposed a new technique for summing up divergent but Borel-symmable series. We argued that the Borel transform, when suitably defined to be free from singularities in the finite complex plane, can mimic the asymptotic behaviour of the underlying function and that a finite number of terms are sufficient to probe this behaviour. This possibility offers arguments in favour of considering modified Borel transformations which lead to a certain entire function as the Borel transform of the divergent series under study. The unavoidable approximation of the Borel transform is a much easier task for such functions because it is no longer necessary to perform an analytic continuation. In [1], as well as in the present paper, we use a power-law approximation of the generalized Borel transform although, in principle, alternative options are available.

In what will follow we shall give further evidence of the applicability of our method by setting it to work on the perturbation theory expansions for the ground state energy levels of the quantum mechanical anharmonic oscillators

$$H(x)=\frac{1}{2}p^2+\frac{1}{2}x^2+gx^{2N},\quad N=2,3,4,\dots \tag{1.1}$$

While the $N=2$-case has always been the number-one favourite among the inventors of new summation procedures, the series generated by the perturbation theory for N greater than 2 are not so popular because their coefficients grow too rapidly for the Borel-method to be applicable in its standard form. Our method, however, which, as we said, is based on a generalization of the Borel summation formula, is much more democratic in this respect. Otherwise, even for a higher-order anharmonicity, the perturbation theory expansion for the ground state energy level is still the perfect test-ground for any new summation technique both because of the plentiful nonperturbative results available and of the very large number of calculated expansion coefficients. These will be reviewed briefly in the third paragraph of this paper, after first giving a description of our summation method in our second paragraph.

2. A REVIEW OF THE SUMMATION METHOD

Consider a series $\sum_{k=0}^{\infty} f_k(x)^k$. Following Borel (cf.e.g., [7]) we ascribe a "sum" $f(x)$ to it by means of the integral

$$f(x)=\int_0^{\infty} e^{-t}\,\tilde{B}(xt)\,dt \tag{2.1}$$

provided it is convergent. The function $\tilde{B}(z)$ is the analytic continuation of the so-called Borel transform of the series

$$B(z)=\sum_{k=0}^{\infty}\frac{f_k}{k!}z^k\ . \tag{2.2}$$

This is an example of a regular summation method which means that, when applied to a convergent series, it yields the correct result. Formulae (2.1) and (2.2) can be generalized in several ways, one of which is this:

$$\begin{aligned} f(x)&=\int_0^{\infty} e^{-t}\,t^{\mu}\tilde{B}_{\mu,\nu}(xt^{\nu})\,dt\ ,\\ B_{\mu,\nu}(z)&=\sum_{k=0}^{\infty}\frac{f_k z^k}{\Gamma(\nu k+\mu+1)}\ . \end{aligned} \tag{2.3}$$

The Borel summation method and its modifications provide us with a tool for tackling the so-called reconstruction problem which consists in trying to recover a function from its asymptotic expansion. The existence of functions whose coefficients in their asymptotic expansion are all equal to zero points to the necessity of additional information about the objective function if a unique solution of the reconstruction problem is what we need. The Watson-Nevanlinna theorem justifies the Borel summation as the unique solution of the reconstruction problem when $f(x)$ is analytic in a sufficiently large domain and the difference between $f(x)$ and the truncated series is suitably controlled.

From now on we shall use the term "summation" as a synonym of "reconstruction" and in the context of the perturbation theory we shall be concerned with the summation of divergent asymptotic expansions of which but a few coefficients are exactly calculated and for the rest only the large-order asymptotic behaviour is known [4,5].

It is by looking at the leading term of the asymptotic formula for the expansion coefficients that we know what values of ν and μ in (2.3) will lead to an entire-function Borel transform $B_{\mu,\nu}(z)$. We believe, and in this we are backed by the numerical analysis [1], that when $f(x)$ obeys a power law

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$$f(x) \sim x^{\rho}, \quad x \to \infty \tag{2.4}$$

then this is felt by the generalized Borel transform $B_{\mu,\nu}(z)$ (2.3). The opposite is obviously true, i.e., when $B_{\mu\nu}(z) \sim z^{\rho}$, $z \to \infty$, then the Borel sum $f(x)$ (2.3) behaves powerwise too, as in (2.4). Unfortunately, in real life, we are not in a position to study the large-z behaviour of the true Borel transform $B_{\mu,\nu}(z)$ since we know but a limited number of exact coefficients f_k. Instead, we can construct an approximate Borel transform

$$\bar{B}_{\mu,\nu}(z) = \sum_{k\leq M} \frac{f_k z^k}{\Gamma(\nu k+\mu+1)} + \sum_{k=M+1}^{M+L} \frac{\bar{f}_k z^k}{\Gamma(\nu k+\mu+1)}, \tag{2.5}$$

where the additional L coefficients $\bar{f}_k$ are found from an extrapolation of the exact coefficients, based on the whole available information about the asymptotic behaviour of f_k. The expression (2.5) is a reasonable approximation of the actual Borel transform only for values of z which are below some z_{max}. The latter depends on the number of terms involved, on μ and ν as well as on the accuracy of approximation required and for alternating series is determined from the size of the first of the neglected terms. Next, for $z \leq z_{max}$ we try a power-law fit for $\bar{B}_{\mu,\nu}$:

$$\bar{B}_{\mu,\nu}(z) \sim C z^{\rho} \tag{2.6}$$

and find a power ρ which turns out to be a function of μ and ν. We emphasize this fact because it helps to wave off the arbitrariness we brought about by introducing the two new parameters μ and ν with the eqs. (2.3). If we want to link the ρ in (2.6) with the power-behaviour (2.4) of the objective function $f(x)$ then any (μ,ν)-dependence is undesirable. We explain the appearance of such dependence as being the effect of the truncation procedure doomed to vanish when we add more and more new terms. But, as was already mentioned, additional terms are available only through the extrapolation procedure and it is apt to introduce error when overdone. Hence we give up the hope of seeing the dependence on μ and ν die a natural death and turn to the principle of minimal sensitivity, proposed by P.M.Stevenson [2]. It instructs us to study the function $\rho(\mu,\nu)$ and find its stationary points or, more generally, its points of minimum variation, the idea being that at such points we are nearer to the ultimate (μ,ν)-invariance. Should it turn out that there are more than one stationary points and, moreover, the values of ρ vary significantly as we go from one such point to another, then, obviously, we could not depend on the principle of minimal

sensitivity alone and would need some additional criterion. Fortunately, in the examples we have studied the numerical analysis seems to indicate the existence of but one stationary point. Let us denote it by (μ_0,ν_0) and the corresponding power by ρ_0. We postulate that the approximate Borel transform is given by the finite sum (2.5) for values of $z \leq z_{max}$ and by the extrapolated power-law fit (2.6) for $z > z_{max}$. Finally we perform the integration in (2.3) and get the result of the summation.

3. THE ANHARMONIC OSCILLATOR

The quantum-mechanical anharmonic oscillator

$$H(x) = \frac{1}{2}p^2 + \frac{1}{2}x^2 + gx^{2N} \tag{3.1}$$

has a ground state energy $E_0(g,N)$ for which a simple scaling argument, credited to Symanzik, gives the following power-law asymptotic behaviour:

$$E_0(g,N) = \gamma g^{a}[1 + O(g^{-2a})], \tag{3.2}$$

where

$$a = 1/(N+1). \tag{3.3}$$

Very accurate numerical results are available [6] for the ground state in a wide range of values of g and for various N. The analytic structure of $E_0(g,N)$ as a function of g has been exhaustively studied [7] and it has been found that the singularities are not positioned in a way that would interfere with the requirements of the Watson-Nevanlinna theorem. Perturbation theory has also been carried out up to a very high order [8] and the large order asymptotic of the coefficients in the expansion in the powers of g has been found [5]:

$$E_0(g,N) = \frac{1}{2} + \sum_{k=1}^{\infty} A_k g^k,$$
$$A_k = [(N-1)k]!\,a^k k^b c[1 + O(\tfrac{1}{k})], \tag{3.4}$$

where

$$a = -\frac{1}{2}\left[\frac{\Gamma(2N/(N-1))}{\Gamma^2(N/(N-1))}\right]^{N-1}, \tag{3.5}$$

$$b = -\frac{1}{2}\,,$$

$$c = -\frac{1}{\pi^{3/2}}\sqrt{\frac{(N-1)\Gamma(2N/(N-1))}{\Gamma^2(N/(N-1))}}\,. \tag{3.5}$$

In Table 1 we give the absolute values of the first several coefficients A_k for $N = 2,\dots,5$. The numbers which correspond to the quartic $(N = 2)$ anharmonic oscillator are the same as in [8]. For the time being we shall leave aside the question of whether the summation procedure is unique and we shall proceed with its actual implementation. As it was explained in the preceding paragraph, in order to implement our summation scheme we have to study the behaviour of the power ρ (see eq. (2.6)) as a function of μ and ν with reference to any stationary points. Our numerical analysis, performed on 25 terms in the approximate Borel-transform, shows that there is a saddle-point of $\rho(\mu,\nu)$ for all of the studied values of N. Graphically this is shown on fig.1a,b, for $N = 3$.

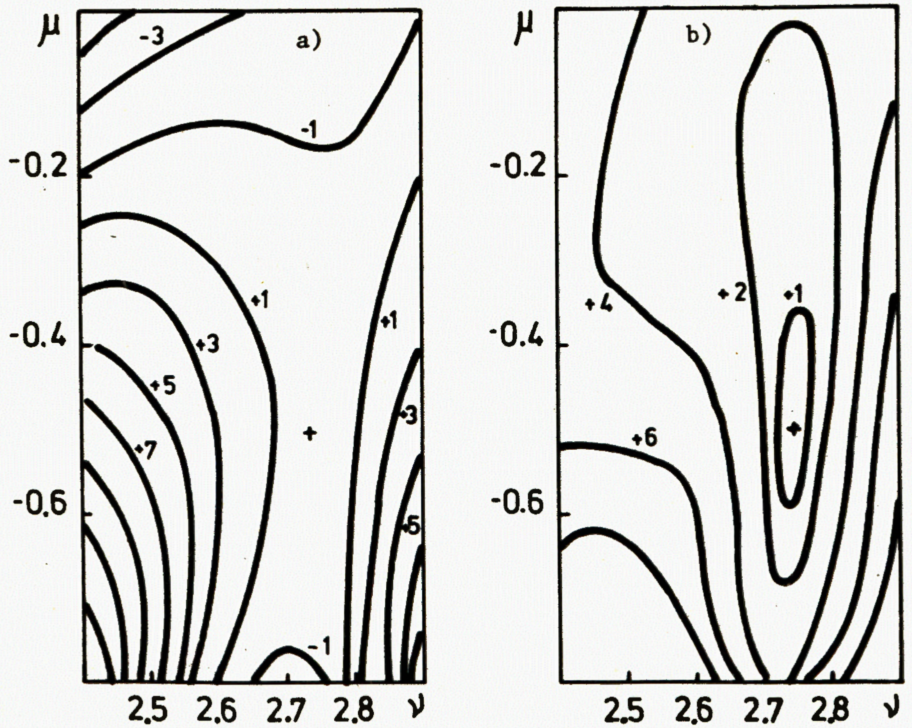

Fig.1. The function $\rho(\mu,\nu)$ in the case $N = 3$ (a), and its variation $\sqrt{(\partial\rho/\partial\mu)^2 + (\partial\rho/\partial\nu)^2}$ (b). The step in both (a) and (b) is 0.017.

Table 1

The absolute values of the first several coefficients A_k in the perturbation theory series (3.4). In our notations, e.g., $(2).546 = 0.546\text{x}10^2$

k	N = 2	N = 3
1	.75	(1).1875
2	(1).2625	(2).54609375
3	(2).208125	(4).484248046875
4	(3).2412890625	(6).815996912842
5	(4).358098046875	(9).221275751656
6	(5).639828134766	(11).882689884590
7	(7).132973372705	(14).487941808745
8	(8).314482146928	(17).357700987914
9	(9).833541603263	(20).336138986250
10	(11).244789407028	(23).394149172293
11	(12).789333316003	(26).564244023521
12	(14).277387769635	(29).968563504957
13	(16).105564665831	(33).196385531864
14	(17).432681068354	(36).464374894191
15	(19).190081719760	(40).126662552918
16	(20).891210175364	(43).394748846698
17	(22).444255088999	(47).139403316931
18	(24).234646430681	(50).553751811712
19	(26).130915026105	(54).245816339035
20	(27).769399985382	(58).121233211849
21	(29).475124077343	(61).660789166175
22	(31).307579295113	(65).396156847401
23	(33).208301009434	(69).260108567638
24	(35).147290492138	(73).186298347574
25	(37).108552296005	(77).145029638553
26	(38).832483627531	(81).122306719529

Table 1 (continued)

k	N = 2	N = 3
27	(40).663329371112	(85).111391832870
28	(42).548392431329	(89).109251775090
29	((44).469782420783	(93).115087134189
30	(46).416502699806	(97).129890801920

k	N = 4	N = 5
1	(1).65625	(2).2953125
2	(4).210984375	(6).136774907226
3	(7).313710058594	(10).477358783720
4	(11).124141097987	(15).630594517692
5	(15).103187117942	(21).226960465827
6	(19).156008963407	(27).182192256092
7	(23).388856541451	(33).285286662138
8	(28).148824783055	(39).791825237512
9	(32).828991567444	(46).362592298265
10	(37).644612572672	(53).259081857592
11	(42).676742466618	(60).276265835393
12	(47).933387374939	(67).423895641311
13	(53).165321756794	(74).907915664710
14	(58).368863047903	(82).264564828799
15	(64).101976557053	(90).102602938985
16	(69).344369698233	(97).519572011856
17	(75).140283032202	(105).337865309279
18	(80).681784211044	(113).278009958644
19	(86).391461706620	(121).285702435182
20	(92).263220647532	(129).362427286572
21	(98).205640264007	(137).561573251045
22	(104).185331826340	(146).105276566733

Table 1 (continued)

K	N = 4	N = 5
23	(110).191436145026	(154).236719729378
24	(116).225296191399	(162).633402307120
25	(122).300454278797	(171).200225164574
26	(128).451778135228	(179).742770937729
27	(134).762410905654	(188).321376910922
28	(141).143784957932	(197).161258418118
29	(147).301838279834	(205).933424918883
30	(153).702694671501	(214).620219703731

The behaviour of $\rho(\mu,\nu)$ along the two perpendicular lines $\mu=\mu_0$ and $\nu=\nu_0$ is shown on Fig.2a,b. The picture is qualitatively the same for all of the studied values of N. In Table 3 we give the values of the power ρ we have found as compared to the exact values (3.3).

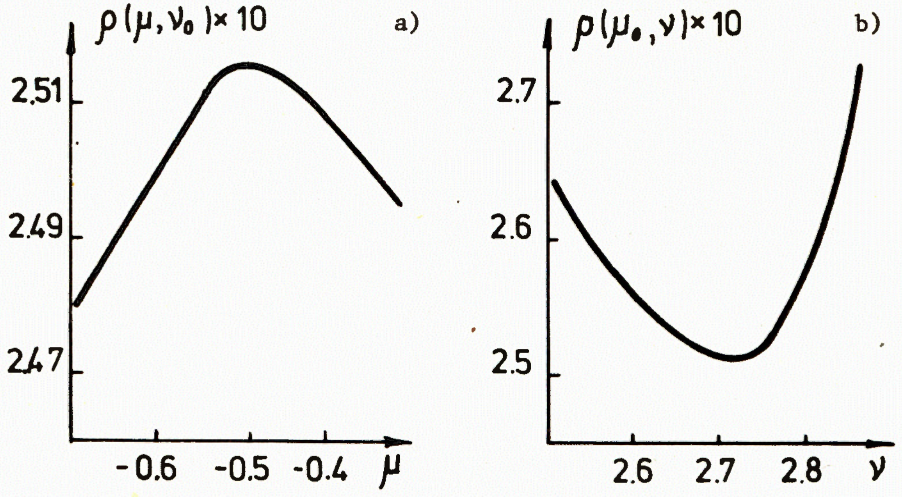

Fig.2. The function $\rho(\mu,\nu)$ in the case N = 3 in the vicinity of the stationary point (μ_0,ν_0); μ_0 = -0.51, ν_0 = = 2.733.

The loss of accuracy for bigger values of N is probably due to the insufficient number of input coefficients. Finally, in table 3 we present the results of the summation.

Table 2

The values of ρ at the stationary point

N	μ_0	ν_0	$\rho(\mu_0, \nu_0)$	Exact values
2	-.4	2.318	.334$\pm$.002	1/3
3	-.51	2.733	.251$\pm$.002	1/4
4	.345	3.83	.17$\pm$.05	1/5
5	1.54	5.256	.12$\pm$.05	1/6

Table 3

The results of the summation of the perturbation theory series for the ground state energy $E_0(g, N)$ (a), and the non-perturbative results (b). The estimated errors are given in the brackets

g	N = 2		N = 3		N = 4	
	a	b	a	b	a	b
1	.8037(1)	.8038	.8018(8)	.8050	.806(30)	.8207
10	1.507(2)	1.505	1.272(10)	1.282	1.12(13)	1.1909
100	3.147(20)	3.131	2.159(27)	2.193	1.62(36)	1.816
1000	6.78(10)	6.694	3.77(9)	3.851	2.35(86)	2.833

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Суммирование асимптотических рядов:
ангармонический осциллятор gx^{2N}

Предложенная нами ранее [1] техника суммирования применена к рядам теории возмущений для основного уровня энергии ангармонических осцилляторов $V(x) = gx^{2N}$ в квантовой механике. Показано, что чрезвычайно быстрый рост коэффициентов разложения в этих примерах не препятствует применению нашей процедуры суммирования. Особое внимание было уделено использованию принципа наименьшей чувствительности, предложенного П.М.Стивенсоном в работе [2].

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Summation of Asymptotic Expansions:
the gx^{2N}-Anharmonic Oscillator

The summation technique proposed by us in [1] has been applied to the perturbation theory expansions for the ground state energy of the quantum mechanical anharmonic oscillators $V(x) = gx^{2N}$. We show that the extremely rapid growth of the expansion coefficients in the above example is not an obstacle for our summation procedure. We have elaborated the utilization of the principle of minimal sensitivity proposed by P.M.Stevenson in [2].

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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