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E2-83-259

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ON FIERZ IDENTITIES IN NON-INTEGER DIMENSIONS

Submitted to "ТМФ"

1983

Fierz identities [1] are of great use for different purposes. They are most essential for supersymmetric theories [2]. But Fierz reorderings of spinor indices allow many simplifications in performing the γ-matrix algebra for non-supersymmetric theories as well. Due to this very reason, Siegel's regularization by dimensional reduction (RDR), proposed [3] as a supersymmetric modification of dimensional regularization, has been applied to non-supersymmetric computations[4]. But as shown in ref.[5], it is the four-dimensional Fierz identities that make this regularization scheme mathematically inconsistent [6]. However, maybe, a more general form of Fierz reorderings exists that can be formulated in a consistent fashion for γ matrices in dimensional regularization. Kennedy [7] has proposed a variant of such reorderings for scalar direct products of γ matrices. In the present paper this possibility is systematically studied. The conditions are formulated, under which the Fierz identities do not contradict commutation relations for γ matrices, and hence, can be used in actual calculations. Consequences essential for supersymmetric dimensional regularization are emphasized.

First, it is worth mentioning some important properties of the formal space used in dimensional regularization. Its "metric tensor" $g_{\mu\nu}$ is defined by the relations:

$$g_{\mu\nu} = g_{\nu\mu}\,, \quad g_{\mu\nu} g_{\nu\lambda} = g_{\mu\lambda}\,, \quad g_{\mu\mu} = d\,. \tag{1}$$

The space has a non-integer "dimensionality" $d = 4 - 2\varepsilon$. But as argued in ref.[8], such a space should be infinite-dimensional because the number of distinct values of its Lorentz indices cannot be finite. This follows from the fact that the antisymmetrization operator over any number m of indices is not identical zero,

$$g^{m}_{\mu\nu} \equiv \frac{1}{m!} \sum_{\pi \in S_m} (-)^{P(\pi)} g_{\mu_1 \nu_{\pi(1)}} \cdots g_{\mu_m \nu_{\pi(m)}} \neq 0, \tag{2}$$

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since for non-integer d and any integer m eqs. (1) imply that

$$g^m_{\mu\mu} = \binom{d}{m} \equiv \frac{1}{m!}\prod_{n=1}^{m}(d-m+n) \neq 0.$$

Operator (2) is normalized so that it is a projection operator:

$$g^m_{\mu\nu}\, g^m_{\nu\lambda} = g^m_{\mu\lambda}\,. \qquad (3)$$

Let us make a comment on the quasi-D-dimensional space (QDS) /5,8/ used in the consistent version of RDR. By definition, this space has a non-integer-dimensional subspace. Therefore, nevertheless its own "dimensionality" d=D in eq. (1) is a finite integer number, QDS must be infinite-dimensional in the sense of relation (2). Our formulae will include QDS as a special case when d is set equal to the integer number D.

The definition of γ matrices in d dimensions is the following:

$$\{\gamma_\mu,\gamma_\nu\}_+ = 2g_{\mu\nu}\mathbb{1},\ \mathbb{1}\gamma_\mu=\gamma_\mu\mathbb{1}=\gamma_\mu,\ \mathbb{1}^2=\mathbb{1},\ g_{\mu\nu}\gamma_\nu=\gamma_\mu\,. \qquad (4)$$

It is convenient to expand any γ-matrix product over the basis of antisymmetrized products

$$M^m \equiv \frac{1}{\sqrt{m!}}\, g^m_{\mu\mu'}\,\gamma_{\mu'_1}\dots\gamma_{\mu'_m}\,, \qquad (m=0,1,2,\dots). \qquad (5)$$

This expansion is always possible because of the commutation relation (4). And the trace formula, following /9/ from eqs. (4) and the cyclicity of the traces,

$$\mathrm{Tr}\,(M^m N^n) = \delta^{mn} s_m\, g^m_{\mu\nu}\,\mathrm{Tr}\,\mathbb{1}\,, \qquad s_m \equiv (-)^{m(m-1)/2}, \qquad (6)$$

makes the expansion unambiguous. The basis (5) is infinite if relation (2) holds for all m.

Due to Lorentz covariance, the most general form of Fierz reorderings for scalar (with no free Lorentz indices) direct products of basic elements (5) is

$$M^m_{AB}M^m_{CE} = \frac{1}{\mathrm{Tr}\,\mathbb{1}}\sum_{n=0}^{\infty} f^{mn}_d\, N^n_{AE}N^n_{CB}\,, \qquad (7)$$

where the repeated indices $\mu_1\dots\mu_m$ of the basic elements M^m in the l.h.s., as well as $\nu_1\dots\nu_n$ in the r.h.s., are contracted.

The following necessary condition can be formulated for this rearrangement to be true: Eq. (7) should become an identity after any contraction of spinor indices A,B,C,E to traces through γ-matrix products when both sides of the equation can be computed using the trace formula (6).

Multiply eq. (7) by the product $K^k_{BC}\Lambda^l_{EA}$ of basic elements of the form (5). Using eqs. (6) and (3), we get

$$\mathrm{Tr}\,(M^m K^k M^m \Lambda^l) = s_k f^{mk}_d\,\mathrm{Tr}\,(K^k\Lambda^l). \qquad (8)$$

Because l is arbitrary and the basic elements Λ^l (5) are linearly independent, eq. (8) implies that

$$M^m K^k M^m = s_k f^{mk}_d K^k. \qquad (9)$$

Formula (9) fixes the values of the Fierz coefficients f^{mk}_d because the calculation of the contraction over $\mu_1\dots\mu_m$ in the l.h.s. with the use of eqs. (1) and (4) is a purely combinatorial problem. It has been solved in ref./10/ for any integer even $d \geqslant m,k$:

$$f^{mk}_d = s_{m+k}\sum_{l=0}^{\min(m,k)}(-)^l\binom{k}{l}\binom{d-k}{m-l}. \qquad (10)$$

This formula also holds for non-integer d: It is obvious from eq. (9) that f^{mk}_d is always a polynomial in d of degree m; hence, to fix its coefficients, integer even points are enough. Thus, the form (7) and (10) of the Fierz reorderings /7/ is determined unambiguously.

Now contract eq. (7) with $K^k_{BA}\Lambda^l_{EC}$, to obtain a product of two traces in the l.h.s. With eqs. (9), (6) and (3) the result is reduced to

$$\delta^{mk}\delta^{kl}g^k_{\varkappa\lambda}(\mathrm{Tr}\,\mathbb{1})^2 = \delta^{kl}g^k_{\varkappa\lambda}\sum_{n=0}^{\infty} f^{mn}_d f^{nk}_d\,. \qquad (11)$$

Because of relation (2), we can cancel a non-zero Lorentz structure $g^k_{\varkappa\lambda}$ in both sides of eq. (11). This leads to a consistency condition for the Fierz identity (7):

$$\delta^{mk}(\mathrm{Tr}\,\mathbb{1})^2 = \sum_{n=0}^{\infty} f^{mn}_d f^{nk}_d\,. \qquad (12)$$

Since the basis (5) is complete, any contraction of spinor indices in eq. (7) through γ-matrix products, which is calculable with the use of eqs. (4) and (1), is reduced to the two cases considered, (8) and (11). Thus, if and only if eq. (12) is satisfied for the coefficients (10), the rearrangement (7) is compatible with eqs. (1) and (4).

Setting $m=k=0$ in eq. (12) and using the results of formula (10), $f^{0n}_d = s_n$ and $f^{n0}_d = s_n\binom{d}{n}$, immediately yields:

$$(\mathrm{Tr}\,\mathbb{1})^2 = \sum_{n=0}^{\infty}\binom{d}{n} = 2^d. \qquad (13)$$

Consequently, the Fierz rearrangement (7) is impossible for the usual choice $\mathrm{Tr}\,\mathbb{1}=4$ when $d=4-2\varepsilon$.

So, we are left with $\mathrm{Tr}\,\mathbb{1}=2^{d/2}$. For infinite-dimensional spaces (2), any non-negative integer numbers m and k are allowed in the consistency condition (12). And there are cases when the series in the r.h.s. of eq. (12) is divergent. Consider, for example, the case of Q4S ($\mathrm{Tr}\,\mathbb{1}=d=4$) with $m=0$ and $k=5$. Due to eq. (10), $f^{n5}_4=\zeta_n\sum_{\ell=0}^{n}\binom{5}{\ell}$. Hence, the r.h.s. of eq. (12) takes the form

$$r^{05}_4\equiv\sum_{n=0}^{\infty}f^{0n}_4 f^{n5}_4=80+\sum_{n=5}^{\infty}2^5, \tag{14}$$

whereas the l.h.s. must be zero. The same divergent series (14) is obtained for $d=4-2\varepsilon$ to zeroth order in ε (eq. (12) must be satisfied identically in ε). Of course, in finite-dimensional spaces, where d is integer and $g^m_{\mu\nu}=0$ for $m>d$, there is no such difficulty: In eq. (11) one cannot cancel $g^k_{\pi\lambda}$ for $k>d$, so the consistency condition (12) includes only $m,k\leqslant d$, and the sums in eqs. (7), (11) and (12) terminate at $n=d$. But in non-integer or quasi-integer dimensions the rearrangement (7) necessarily requires a special procedure of summing up divergent series.

Analytic continuation in d is the most natural way to achieve it. Using the expression (10) for the Fierz coefficients, one can reduce the r.h.s. of eq. (12) to

$$r^{mk}_d=\zeta_m\zeta_k\sum_{j=0}^{k}(-)^j\binom{k}{j}\sum_{n=j}^{\infty}(-)^{n(m+k)}\binom{d-k}{n-j}\sum_{\ell=0}^{m}(-)^{\ell}\binom{n}{\ell}\binom{d-n}{m-\ell}. \tag{15}$$

The sum over ℓ in formula (15) is a polynomial both in d and in n of the whole degree m. Hence, the use of the relation $n\binom{d}{n}=d\cdot\binom{d-1}{n-1}$ allows one to reduce the series over n in formula (15) to a limited number of the following sums:

$$\sigma_\ell=p_m(d)\sum_{n=0}^{\infty}(-)^{n(m+k)}\binom{d-k-\ell}{n}, \tag{16}$$

where $\ell=0,\dots,m$, and $p_m(d)$ are polynomials in d of degree at most equal to m. For any complex d with $\mathrm{Re}\,d>k+\ell$, the series in eq. (16) absolutely converges/11/, and for $m+k$ even, $\sigma_\ell=p_m(d)2^{d-k-\ell}$, else $\sigma_\ell=0$. Consequently, for $\mathrm{Re}\,d>m+k$, formula (15) represents an analytic function of d of the form

$$r^{mk}_d=p^{(k)}_m(d)\,2^d, \tag{17}$$

where $1+m$ coefficients of the polynomial $p^{(k)}_m(d)$ somehow depend on m and k.

On the other hand, in ref./10/ it has been shown that the condition (12) is satisfied for any integer even $d\geqslant m,k$. For fixed m and k the polynomial in eq. (17) is hence determined in an infinite number of points. It must be $p^{(k)}_m(d)=\delta^{mk}$, and the consistency condition takes place for non-integer $d>m+k$ as well.

Thus, provided $\mathrm{Tr}\,\mathbb{1}$ is normalized in accordance with eq. (13), the Fierz rearrangement (7) with the coefficients (10) will never contradict the definitions (1) and (4) if series like (12) will be summed by analytic continuation from $\mathrm{Re}\,d>m+k$. Treated so, the Fierz identities (7) can be added in the definitions of γ matrices and used in practical calculations.

However, for any choice of $\mathrm{Tr}\,\mathbb{1}$, irrespective of whether relation (7) is possible or not, the contraction formula (9) remains always true, since it is a direct consequence of eqs. (1) and (4). It is worth mentioning that formula (9) is sufficient to perform actual calculations carried out in ref./7/ with the use of Fierz identities.

Consider an example:

$$\Delta(M,N)=\mathrm{Tr}(\gamma_\lambda M)\,\mathrm{Tr}(\gamma_\lambda N)+\mathrm{Tr}(\gamma_\lambda[M-(-)^m\tilde{M}]\gamma_\lambda N), \tag{18}$$

where M and N are products of m and n γ matrices, respectively, and $\tilde{M}$ is M in the reverse order. The quantities of the form (18) represent multiplicative factors for contributions to supersymmetry Ward identities from the variation

$$\delta S=\int dx\,\frac{g}{2}\,t^{abc}\,(\bar{\xi}\gamma_\lambda\psi^a)(\bar{\psi}^b\gamma_\lambda\psi^c) \tag{19}$$

of the vector-supermultiplet action regularized by the consistent RDR version in terms of component fields /5,8/. In eq. (19) ψ are Majorana spinors in the adjoint representation of the gauge group, t^{abc} are the antisymmetric structure constants of the group, γ_λ are the Dirac matrices in Q4S. The products M and N in eq. (18) are determined by particular diagrams. It is interesting to know whether the contributions (18) are equal to zero, and if they are not, to find minimum numbers m and n, for which $\Delta(M,N)\neq 0$. This will allow one to point out lowest-order diagrams which can break supersymmetry Ward identities in the RDR scheme.

It is sufficient to substitute the basic elements (5) into eq. (18). Since $\tilde{M}^m=\zeta_m M^m$,

$$\Delta(M^m, N^n) = Tr(\gamma_\lambda M^m)Tr(\gamma_\lambda N^n) + (1-\zeta_{m+1})Tr(\gamma_\lambda M^m \gamma_\lambda N^n) = \\ = [\delta^{m1}\, Tr\,\mathbb{1} + (1-\zeta_{m+1}) f_d^{1m}]\, \delta^{mn} g^m_{\mu\nu}\, Tr\,\mathbb{1}, \tag{20}$$

where we used the trace and contraction formulae (6) and (9). Due to eq. (10), $f_d^{1m} = \zeta_{m+1}(d-2m)$. Hence, eq. (20) yields

$$\Delta(M^m, N^n) = [\delta^{m1}\, Tr\,\mathbb{1} + (1-\zeta_{m+1})(2m-d)]\, \delta^{mn} g^m_{\mu\nu}\, Tr\,\mathbb{1}. \tag{21}$$

According to formula (21), for $Tr\,\mathbb{1} = d = 4$ we have $\Delta = 0$ if m or $n \leq 4$. In the usual four-dimensional space these are all the cases possible. But in Q4S also $m, n > 4$ are allowed, and we get the simplest non-zero example of the form

$$\Delta(M^5, N^5) = 48\, g^5_{\mu\nu}\,, \tag{22}$$

in agreement with refs. /5,8/. The same result is obtained if in eq. (20) we Fierz rearrange the direct product of two γ_λ's by formula (7). To reveal this, it is sufficient to use eqs. (9), (12) and (6), remembering that $Tr\,\mathbb{1}$ satisfies eq. (13).

It is quantities like (22), unequal to zero in Q4S, that cause non-invariance of RDR discovered explicitly in ref. /12/. The Q4S Fierz identities do not help to prove invariance because they involve additional higher terms as compared with the usual four-dimensional identities. Thus, the consistent RDR version proves to be superinvariant only in several first orders of perturbation theory /8,12,13/. To get a really supersymmetric dimensional regularization, one probably has to abandon the use of the infinite-dimensional algebra of covariants (1), (4).

For the Wess-Zumino model this program has been accomplished (using the α representation for Feynman diagrams) in ref. /14/, in which one deals with the Lorentz and spinor algebra in four dimensions only. When rewritten in the momentum representation, this recipe implies that the (infinite-dimensional) space (1) with $d = 4 - 2\varepsilon$, necessary for regularization of momentum integrals, has a subspace of four dimensions, and all factors in numerators of supergraphs belong to this finite-dimensional subspace. Here the four-dimensional Fierz identities ensure a manifest superinvariance, while spinor covariants in d dimensions do never come into play. But transformations of subintegral expressions, such as cancellation of the (four-dimensional) squared combinations of momenta in numerators with the d-dimensional ones in denominators, are impossible in this

scheme. As a result, theories with local gauge symmetries cannot be regularized in an invariant fashion with the use of such a recipe: One is unable to prove the gauge-invariance Ward identities in the usual form. Therefore, the scheme fails for the most interesting case of supersymmetric gauge theories.

Another modification of dimensional regularization using no infinite-dimensional algebra has been proposed in ref. /15/. The idea consists in generalizing supersymmetry to every integer even dimensionality d and then performing analytic continuation just as in the conventional dimensional regularization. Here, the lagrangian explicitly depends on d and for every finite-dimensional case is supersymmetric due to appropriate Fierz identities. But in actual calculations the scheme proves to be extremely cumbersome because of the necessity to perform complicated algebraic manipulations with supercovariant derivatives in d dimensions for arbitrary d. Furthermore, there are doubts /3,6/ of whether such a scheme preserves unitarity.

Thus, the problem of supersymmetric dimensional regularization has so far no satisfactory solution, and difficulties arising are fundamental.

I am grateful to A.A.Vladimirov for the discussion of the work.

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Received by Publishing Department
on April 20,1983.

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О тождествах Фирца в пространствах нецелой размерности

Изучаются преобразования Фирца, меняющие порядок спинорных индексов у прямых произведений γ-матриц. Показано, что особенности тождеств Фирца в бесконечномерных пространствах /к числу которых следует относить пространство нецелой "размерности" $d=4-2\epsilon$/ сильно ограничивают возможности построения суперсимметричной размерной регуляризации.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1983

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On Fierz Identities in Non-Integer Dimensions

Fierz reorderings of spinor indices for direct products of γ matrices are studied. It is shown that special features of Fierz identities in infinite-dimensional spaces (such as one of $d=4-2\epsilon$ dimensions) strongly restrict the possibilities of constructing supersymmetric dimensional regularization.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1983