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REPRESENTATIONS OF osp(1,4) IN TERMS OF THREE BOSON PAIRS AND MATRICES OF ARBITRARY EVEN ORDER.

The Basic Theorem

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INTRODUCTION AND SUMMARY OF RESULTS

1. In the first part of this paper /1/ we have constructed a class of infinite-dimensional representations of the Lie superalgebra osp(1,4) on the vector space $\mathcal{F}^N := C^\infty(M) \otimes \mathbb{C}^N$, where M is some open subset of $\mathbb{R}^3 \setminus \{0\}$. The generators X_{jk}, Y_l ($j,k,l = -2,-1,1,2$, $j \geq k$) are represented in terms of three pairs of operators p_α, q_α on $C^\infty(M)$ given by

$$(p_\alpha \psi)(\vec{x}) := \frac{\partial \psi}{\partial x_\alpha}(\vec{x}), \quad (q_\alpha \psi)(\vec{x}) := x_\alpha \psi(\vec{x}),$$

five operators $A_\pm, H, V_\pm \in \operatorname{End} \mathbb{C}^N$ and one numerical parameter $\varkappa$:

$$\hat{X}_{-2-2} = iq_2^2, \quad \hat{X}_{-1-2} = iq_1q_2, \quad \hat{X}_{1-2} = p_1q_2, \quad \hat{X}_{2-2} = q_2p_2 + \frac{1}{2},$$

$$\hat{X}_{-1-1} = i(q_1^2 + q_3^2), \quad \hat{X}_{1-1} = q_1p_1 + q_3p_3 + 1,$$

$$\hat{X}_{2-1} = q_1(p_2 - \tfrac{1}{2}q_2^{-1}) - q_2^{-1}q_3j_2 - iq_2^{-1}q_3 \otimes H,$$

$$\hat{X}_{11} = -i(p_1^2 + p_3^2) + iq_3^{-2} \otimes T,$$

$$\hat{X}_{21} = -ip_1(p_2 + \tfrac{1}{2}q_2^{-1}) + iq_2^{-1}(j_2p_3 + ip_3 \otimes H - q_1q_3^{-2} \otimes T - \tfrac{i}{2}q_3^{-1} \otimes V),$$

$$\hat{X}_{22} = -ip_2^2 - iq_2^{-2}(j_2^2 - \tfrac{15}{4} + 2ij_2 \otimes H - iq_1q_3^{-1} \otimes V + (1 - q_1^2q_3^{-2}) \otimes T - \tfrac{1}{2} \otimes W),$$

$$\hat{Y}_{-2} = \epsilon q_2 \otimes A, \quad \hat{Y}_{-1} = \epsilon(q_1 \otimes A - iq_3 \otimes B), \quad \hat{Y}_1 = -i\epsilon(p_1 \otimes A - ip_3 \otimes B + \tfrac{i}{2}q_3^{-1} \otimes Z),$$

$$\hat{Y}_2 = -i\epsilon(p_2 \otimes A + iq_2^{-1}j_2 \otimes B - \tfrac{i}{2}q_1q_2^{-1}q_3^{-1} \otimes Z - \tfrac{1}{2}q_2^{-1} \otimes \tilde{Z}),$$

with $j_2 := q_1p_3 - p_1q_3$, $\epsilon := \exp(i\pi/4)$, $W := \tilde{W} + \varkappa - 4$, $V := (V_+ + V_-)/2$,

$$\tilde{W} := \sum_{\varepsilon = \pm 1} \varepsilon A_\varepsilon (V_{-\varepsilon}A_\varepsilon + 2A_{-\varepsilon}H), \quad T := \tfrac{1}{4}(W + 8 - 2H^2 - V_+ + V_-), \quad A := A_+ + A_-,$$

$$B := A_+ - A_-, \quad Z := \tfrac{1}{2}\sum_\varepsilon [V_\varepsilon, A_{-\varepsilon}], \quad \tilde{Z} := \sum_\varepsilon \varepsilon(\{H, A_\varepsilon\} + \tfrac{1}{2}[V_\varepsilon, A_{-\varepsilon}]).$$

The mapping $\Omega : X_{jk} \mapsto \hat{X}_{jk}$, $Y_l \mapsto \hat{Y}_l$ defined in this way has the following basic features:

(a) Ω is a homomorphism of osp(1,4) into $\operatorname{End} \mathbb{C}^N$, if the operators $A_\pm, H, V_\pm$ satisfy

$$(\mathcal{H}1) \quad \begin{array}{l} \{A_\varepsilon, A_\eta\} = \delta_{\varepsilon+\eta}, \quad [H, A_\varepsilon] = \varepsilon A_\varepsilon, \quad [V_\varepsilon, A_\varepsilon] = 0, \quad [H, V_\varepsilon] = 2\varepsilon V_\varepsilon, \\ [V_+, V_-] = 4H(2H^2 - W - 8), \quad [V_\varepsilon, [V_\varepsilon, A_{-\varepsilon}]] = 8\varepsilon A_\varepsilon V_\varepsilon, \end{array}$$

and if a projection $E \in \operatorname{End} \mathbb{C}^N$ exists such that

$$(\mathcal{H}2) \quad [E, H] = [E, V_\pm] = 0, \quad EA_\pm = A_\pm E', \quad E' := I_N - E .$$

Remark: By using E we can express $\mathcal{F}^N$ as direct sum of $\mathcal{F}_0^N := C^\infty(M) \otimes E\mathbb{C}^N$ and $\mathcal{F}_1^N := C^\infty(M) \otimes E'\mathbb{C}^N$; with the help of this decomposition

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the space End $\mathcal{F}^N$ becomes graded in the standard way: even elements map each of $\mathcal{F}_0^N, \mathcal{F}_1^N$ into itself, whereas the odd ones map $\mathcal{F}_0^N$ into $\mathcal{F}_1^N$ and vice versa. The conditions $(\mathcal{H}2)$ obviously guarantee that all the $\hat{X}_{jk}$ are even and $\hat{Y}_l$ odd.

(b) The second-order Casimir operator equals $\varkappa I$; the other independent Casimir element of osp(1,4) is represented by a multiple of unity, if the set $\mathcal{H} := \{A_\pm, H, V_\pm\}$ is irreducible.

(c) Let us define involution on polynomials in $p_\alpha, q_\alpha, q_\alpha^{-1}$ by the usual extension of $q_\alpha^\star := q_\alpha$, $p_\alpha^\star := -p_\alpha$ +). By combining it with hermitian conjugation on End $\mathbb{C}^N$ we get involution on the subalgebra of End $\mathcal{F}^N$ containing all the $\hat{X}_{jk}$ and $\hat{Y}_l$. Then all the $\hat{X}_{jk}$ satisfy $\hat{X}_{jk}^\star = -\hat{X}_{jk}$, if

$$(\mathcal{H}3) \quad \varkappa \in \mathbb{R}, \; H^\star = H, \; V_\varepsilon^\star = -V_{-\varepsilon}, \; \widetilde{W}^\star = \widetilde{W}.$$

2. All the mappings that fulfil $(\mathcal{H}1 - \mathcal{H}3)$ constitute a class $\mathcal{C}$ of Schurean representations of osp(1,4) whose even part is formally skew-symmetric. The adverb "formally" reflects the fact that we have not defined any scalar product on $\mathcal{F}^N$ and, consequently, the relation of $\hat{X}_{jk}^\star$ to $\hat{X}_{jk}$ differs from what is usual in the representation theory. Instead, the following holds

$$\hat{X}_{jk}\Psi \cdot \Phi - \Psi \cdot X_{jk}^\star \Phi = \operatorname{div} \vec{\eta}\,.$$

Here $\Psi = \sum \psi_\alpha \otimes f_\alpha$, $\Phi = \sum \varphi_\beta \otimes g_\beta$, $\psi_\alpha, \varphi_\beta \in C^\infty(M)$, $f_\alpha, g_\beta \in \mathbb{C}^N$, the binary operation "$\cdot$" maps $\mathcal{F}^N \times \mathcal{F}^N$ into $C^\infty(M)$ and is defined via the scalar product in $\mathbb{C}^N$ by $\Psi \cdot \Phi := \sum \psi_\alpha \overline{\varphi}_\beta \langle f_\alpha, g_\beta \rangle$. Finally $\vec{\eta} \equiv (\eta_1, \eta_2, \eta_3)$ depends linearly on $\psi_\alpha, \varphi_\beta$ and their derivatives.

If we want a given $\Omega \in \mathcal{C}$ to be "actually" skew-symmetric, we have to introduce a scalar product on a sufficiently large subset $\mathcal{G} \subset \mathcal{F}^N$ invariant w.r.t. all the $\hat{X}_{jk}$, $\hat{Y}_l$ such that $(\hat{X}_{jk}\Psi, \Phi) = -(\Psi, \hat{X}_{jk}^\star \Phi)$ for any $\Psi, \Phi \in \mathcal{G}$. This problem is dealt with in our next paper.

3. Each representation in $\mathcal{C}$ is fully specified by $\varkappa$ and the set $\mathcal{H}$ satisfying $(\mathcal{H}1 - \mathcal{H}3)$. Thus the problem of giving the complete description of $\mathcal{C}$ can be formulated as follows:

(i) For each N=1,2,.. find $\mathcal{K}_N \subset \mathbb{R}$ such that $\varkappa \in \mathcal{K}_N$ iff there exists at least one irreducible set $\mathcal{H} \subset$ End $\mathbb{C}^N$ satisfying the conditions $(\mathcal{H}1 - \mathcal{H}3)$ for some projection $E \in$ End $\mathbb{C}^N$.
Each such set will be denoted $\mathcal{H}_\varkappa \equiv \{A_\pm, H, V_\pm\}_\varkappa$ and called "solution" (for the given $\varkappa$, E).

+) Notice that the definition is consistent with the commutation relations which are fulfilled by p_α, q_α, i.e. $[p_\alpha, q_\beta^k] = \delta_{\alpha\beta} k q_\alpha^{k-1}$, $k = \pm 1$.

(ii) For each $\varkappa \in \mathcal{K}_N$ find all the non-equivalent solutions.

The problem is completely solved by the following

<u>Theorem</u>: (a) If N is odd, then $\mathcal{K}_N = \emptyset$, i.e., no solution exists.

(b) If N=4M, M=1,2,.., then $\mathcal{K}_N = (2M(M-1)-4, +\infty)$ and for each $\varkappa \in \mathcal{K}_N$ there is just one solution.

(c) If N=4M-2, then $\mathcal{K}_2 = [\varkappa_1, +\infty)$, $\mathcal{K}_N = [\varkappa_M, \varkappa_M + \frac{1}{2})$, M=2,3,.., $\varkappa_M := 2M(M-1) - \frac{9}{2}$. There is just one solution for $\varkappa = \varkappa_M$ and just two non-equivalent solutions for all other values $\varkappa \in \mathcal{K}_N$.

(d) Let n=1,2,.., $\varkappa \in \mathcal{K}_{2n}$ and $\{A_\pm, H, V_\pm\}_\varkappa \subset$ End $\mathbb{C}^{2n}$ be a solution with the corresponding projection E. Then a regular $R_D \in$ End $\mathbb{C}^{2n}$ exists such that $\mathcal{H}_\varkappa = R_D^{-1} \mathcal{H}_\vartheta^{(D)} R_D$, $E = R_D^{-1} E^{(D)} R_D$, where

$$\vartheta \equiv \vartheta(n,\varkappa) := \begin{cases} \varkappa & \ldots n=2m \\ \pm\sqrt{2(\varkappa - \varkappa_m)} & \ldots n=2m-1 \end{cases} \; m=1,2,\ldots \qquad (1a)$$

The set $\mathcal{H}_\vartheta^{(D)} \equiv \{A_\pm^{(D)}, H^{(D)}, V_\pm^{(D)}\}_\vartheta$ and the $E^{(D)}$ have the following properties:

(i) There are four projections F^α onto subspaces $\mathcal{V}^\alpha \subset \mathbb{C}^{2n}$ orthogonal to each other, $\dim \mathcal{V}^\alpha = n_\alpha$,

$$n_1 := E(\tfrac{n}{2}+1), \quad n_2 := E(\tfrac{n-1}{2}), \quad n_3 := E(\tfrac{n+1}{2}), \quad n_4 := E(\tfrac{n}{2}) \;{}^{+)} \qquad (1b)$$

such that $\sum F^\alpha = I_{2n}$ and $E^{(D)} = F^1 + F^2$. Further the set $\mathcal{P}^{(D)} := \{H^{(D)}, V_\pm^{(D)}, W^{(D)}\}$, $W^{(D)} := \widetilde{W}^{(D)} + \varkappa - 4$, is reduced by F^α, $\alpha=1,\ldots,4$.

(ii) If $n_\alpha > 0$, then the restriction $\mathcal{P}_\alpha^{(D)} := \mathcal{P}^{(D)} \restriction \mathcal{V}^\alpha$ is irreducible and $W^{(D)} \restriction \mathcal{V}^\alpha = (\widetilde{w}_\alpha + \varkappa - 4) I_{n_\alpha}$, where

$$\widetilde{w}_1 = -\widetilde{w}_2 = \mu_0 := \begin{cases} n \\ n-\vartheta \end{cases}, \quad \widetilde{w}_3 = -\widetilde{w}_4 = \mu_1 := \begin{cases} (4\varkappa+20-n^2)^{1/2} & \ldots n=2,4,\ldots \\ n+\vartheta & \ldots n=1,3,\ldots \end{cases}$$

Further an orthonormal basis $\mathcal{E}_\alpha \subset \mathcal{V}^\alpha$ exists such that the matrices of operators $H_\alpha^{(D)} := H^{(D)} \restriction \mathcal{V}^\alpha$, $(V_\pm^{(D)})_\alpha := V_\pm^{(D)} \restriction \mathcal{V}^\alpha$ w.r.t. $\mathcal{E}_\alpha$ satisfy

$$\{[H_\alpha^{(D)}], [(V_\pm^{(D)})_\alpha]\} = \begin{cases} \mathcal{M}_1(n_\alpha, \widetilde{w}_\alpha + \varkappa - 4) & \ldots n=2,4,\ldots \\ \mathcal{M}_2(n_\alpha, \frac{1}{2}(\vartheta - (-1)^{E(\alpha/2)})) & \ldots n=1,3,\ldots \end{cases},$$

the sets $\mathcal{M}_r$ being given in /1/ Eq.(2.9).

(iii) $A_-^{(D)} = (A_+^{(D)})^\star$, $H^{(D)} = (H^{(D)})^\star$, $V_-^{(D)} = -(V_+^{(D)})^\star$, $\widetilde{W}^{(D)} = (\widetilde{W}^{(D)})^\star$.

(iv) Let $A^{\alpha\beta}$ be the operator from $\mathcal{V}^\beta$ to $\mathcal{V}^\alpha$ that is obtained by restricting $i\sqrt{\mu_0\mu_1}\, F^\alpha A_+^{(D)} F^\beta$ to $\mathcal{V}^\beta$. Then $A^{\alpha\beta} = A^{\alpha+2,\beta+2} = 0$, $\alpha,\beta = 1,2$, and for the remainig eight pairs α,β the matrices of $A^{\alpha\beta}$ w.r.t. the bases of (ii) have the following elements :

+) The definition of $E: \mathbb{R} \to \mathbb{Z} \equiv \{0, \pm 1, \ldots\}$ reads $E(x) := \sup\{n \mid n \in \mathbb{Z}, n \le x\}$.

$$a^{13}_{kl} = \delta_{l+1-k}\sqrt{2l(\mu_1-\tau_l)}\ ,\quad a^{14}_{kl} = -\delta_{l+1-k}\sqrt{2l\tau_l}\ ,$$

$$a^{23}_{kl} = \delta_{l-k}\sqrt{(\mu_0-2l)(\mu_1-\tau_l)}\ ,\quad a^{24}_{kl} = -\delta_{l-k}\sqrt{(\mu_0-2l)\tau_l}\ ,$$

$$a^{31}_{kl} = \delta_{l-k}\sqrt{(\mu_0+2-2l)\tau_l}\ ,\quad a^{32}_{kl} = -\delta_{l+1-k}\sqrt{2l\tau_{l+1}}\ ,$$

$$a^{41}_{kl} = \delta_{l-k}\sqrt{(\mu_0+2-2l)(\mu_1-\tau_l)}\ ,\quad a^{42}_{kl} = -\delta_{l+1-k}\sqrt{2l(\mu_1-\tau_{l+1})}\ ,$$

$$1 \le k \le n_\alpha,\quad 1 \le l \le n_\beta,\quad \tau_l := 2l-1+(\mu_1-\mu_0)/2.$$

The rest of the paper is devoted to proving this theorem. The notation introduced in /1/ is used mostly without explainig it anew. New formulae, lemmas, etc., are numbered by single arabic numerals while decimal numbering indicates reference to /1/.

PRELIMINARIES

4. For each pair of unit vectors $f,g \in \mathbb{C}^n$ let $U_n(f,g)$ be the operator on $\mathbb{C}^n$ (that is supposed to be equipped with the usual scalar product $\langle .,. \rangle$) whose action on any $h \in \mathbb{C}^n$ is given by

$$U_n(f,g)\, h := \langle h,g \rangle f\ . \tag{2a}$$

This operator is a partial isometry /2/ whose initial and final subspaces are the one-dimensional spaces spanned by the vectors g and f, respectively. From (2a) follow

$$U_n^*(f,g) = U_n(g,f),\ U_n(f,g)U_n(f',g') = \langle f',g \rangle\, U_n(f,g'), \tag{2b}$$

which further imply that $U_n(f,g)$ is an orthonormal projection, iff $f=g$.

Let $\{f_1^{(n)},..,f_n^{(n)}\}$ be the standard orthonormal basis in $\mathbb{C}^n$, $f_1^{(n)} := (1,0,..,0)$, $f_2^{(n)} := (0,1,0,..,0)$ etc. The following special notation will be used:+)

$$\hat{e}_{jk}^{(n)} \equiv U_n(f_j^{(n)},f_k^{(n)}),\ \varphi_\alpha \equiv f_\alpha^{(2)}\ ,\ \ \varepsilon_{\alpha\beta} \equiv \hat{e}_{\alpha\beta}^{(2)}\ ,\ \alpha,\beta=1,2$$
$$\varepsilon^+ \equiv \varepsilon_{21},\ \varepsilon^- \equiv \varepsilon_{12},\ \sigma_3 \equiv \varepsilon_{11}-\varepsilon_{22},\ \sigma_0 \equiv \varepsilon_{11}+\varepsilon_{22} = I_2\ . \tag{3}$$

5. Consider $\mathbb{C}^{2n} = \mathbb{C}^n \otimes \mathbb{C}^2$, the ordered orthonormal basis $\mathcal{B}^{(n)} := \{f_1\otimes\varphi_1,\ f_1\otimes\varphi_2,..,f_n\otimes\varphi_1,\ f_n\otimes\varphi_2\}$ in $\mathbb{C}^{2n}$ and express any $T \in \mathrm{End}\,\mathbb{C}^{2n}$ via its matrix elements w.r.t. $\mathcal{B}^{(n)}$:

$$T = \sum_{\alpha,\beta=1}^{2}\ \sum_{j,k=1}^{n} t_{jk}^{\alpha\beta}\, \hat{e}_{jk}\otimes\varepsilon_{\alpha\beta} = \sum_{\alpha\beta} \hat{t}^{\alpha\beta}\otimes\varepsilon_{\alpha\beta}\ . \tag{4}$$

+) Henceforth the upper index n will be mostly omitted.

Besides the basis $\mathcal{B}^{(n)}$ and the corresponding decomposition (4) we shall consider another basis $\tilde{\mathcal{B}}^{(n)}$ that differs from $\mathcal{B}^{(n)}$ in the ordering of vectors $f_j\otimes\varphi_\alpha$. The elements of $\tilde{\mathcal{B}}^{(n)}$ are ordered according to increasing values of $r := j+\alpha-2$, $0 \le r \le n$. If r is given, then α ranges from $\alpha_{min}^{(r)} := \max(1,r+2-n) = 1+\delta_{r-n}$ to $\alpha_{max}^{(r)} := \min(2,r+1) = 2-\delta_r$. More convenient is working with $\mu := \alpha_{max}^{(r)}+1-\alpha$, whose range is $1 \le \mu \le \mu(r) := \alpha_{max}^{(r)}+1-\alpha_{min}^{(r)} = 2-\delta_r-\delta_{r-n}$. Thus we have $\tilde{\mathcal{B}}^{(n)} := \{\psi_{r\mu} \mid r=0,1,..,n,\ \ 1 \le \mu \le \mu(r)\}$, where

$$\psi_{r\mu} := f_{r+\delta_r+\mu-1}\otimes\varphi_{3-\mu-\delta_r}\ , \tag{5a}$$

the inverse relation being

$$f_j\otimes\varphi_\alpha = \psi_{j+\alpha-2,\,3-\alpha-\delta_{j+\alpha-2}}\ . \tag{5b}$$

Let P_r be the projection onto the subspace spanned by the vectors $\psi_{r\mu}$, $1 \le \mu \le \mu(r)$, i.e., $\dim P_0 = \dim P_n = 1$, $\dim P_r = 2$, $1 \le r \le n-1$. By using (2a) and introducing

$$P_{rs;\mu\nu} := U_{2n}(\psi_{r\mu},\psi_{s\nu})\ , \tag{6a}$$

one obtains

$$P_r = \sum_{\mu=1}^{\mu(r)} P_{rr;\mu\mu},\qquad \sum_{r=0}^{n} P_r = I_{2n}\ ,\quad P_rP_s = \delta_{r-s}P_r.$$

These relations determine a "block structure" in $\mathrm{End}\,\mathbb{C}^{2n}$: for any $T \in \mathrm{End}\,\mathbb{C}^{2n}$ one has

$$T = \sum_{r,s=0}^{n} T_{rs}\ ,\quad T_{rs} := P_rTP_s. \tag{7a}$$

The blocks can be expressed via matrix elements in Eq.(4):

$$T_{rs} = \sum_{\mu=1}^{\mu(r)}\sum_{\nu=1}^{\mu(s)} t_{r+\delta_r+\mu-1,\,s+\delta_s+\nu-1}^{3-\mu-\delta_r,\,3-\nu-\delta_s}\ P_{rs;\mu\nu}\ . \tag{7b}$$

This formula ensues from (2b) and

$$P_{rs;\mu\nu} = \hat{e}_{r+\delta_r+\mu-1,\,s+\delta_s+\nu-1}\otimes\varepsilon_{3-\mu-\delta_r,\,3-\nu-\delta_s}\ , \tag{6b}$$

$$\hat{e}_{jk}\otimes\varepsilon_{\alpha\beta} = P_{j+\alpha-2,\,k+\beta-2;\,3-\alpha-\delta_{j+\alpha-2},\,3-\beta-\delta_{k+\beta-2}}\ , \tag{6c}$$

the latter relations being a consequence of (5a,b).

NECESSARY CONDITIONS

In this section we derive some necessary conditions that must satisfy the set $\mathcal{K}_N$ and any solution $\mathcal{H}_\varkappa \subset \mathrm{End}\,\mathbb{C}^N$, $\varkappa \in \mathcal{K}_N$. First we will consider the following auxiliary problems:

(i) for $n=1,2,..$ find $\mathcal{J}_n \subset \mathbb{R}$ such that $\varkappa \in \mathcal{J}_n$ iff there exists at least one irreducible set $\mathcal{N}_\varkappa \equiv \{\hat{h}, \hat{v}_\pm\}_\varkappa \subset \mathrm{End}\,\mathbb{C}^n$ satisfying

$$[\hat{h}, \hat{v}_\eta] = 2\eta v_\eta, \quad \eta = \pm 1, \tag{8a}$$

$$\{\hat{v}_+, \hat{v}_-\} = \tfrac{1}{2}\hat{h}^2 - (\varkappa + \tfrac{9}{2}), \tag{8b}$$

$$\sigma(h) \subset \mathbb{R}^{\,+)}. \tag{8c}$$

Each such $\mathcal{N}_\varkappa$ will be called "small solution".

(ii) for each $\varkappa \in \mathcal{J}_n$ find all non-equivalent small solutions.

6. <u>Lemma</u>: (a) If $n=2m$, $m=1,2,..$, then

$$\mathcal{J}_n = \mathcal{J}_n' := \mathbb{R} \setminus \{(m-2l+1)^2 + m^2 - 5 \mid l = 1,2,..,m\} \tag{9a}$$

and there is just one small solution for each $\varkappa \in \mathcal{J}_n$.

(b) If $n=2m-1$, $m=1,2,..$, then

$$\mathcal{J}_n = \mathcal{J}_n' := [\varkappa_m, +\infty) \setminus \{2m(m-1) + 2l(l-1) - 4 \mid l = 1,2,..,m-1\}^{\,++)} \tag{9b}$$

and there is just one small solution for $\varkappa = \varkappa_m$, and just two small solutions for $\varkappa \in \mathcal{J}_n \setminus \{\varkappa_m\}$.

(c) Each small solution is equivalent to $\mathcal{N}^s_\vartheta \equiv \{\hat{h}^s, \hat{v}^s_\pm\}_\vartheta$, where $\vartheta \equiv \vartheta(n,\varkappa)$ is defined by Eq.(1a) and

$$\hat{h}^s := \sum_{k=1}^{n} (2k-1-\mu_0)\, \hat{e}_{kk}\,^{++)}, \tag{10a}$$

$$\hat{v}^s_+ := \sum_{k=1}^{n-1} c_k\, \hat{e}_{k+1,k}, \quad \hat{v}^s_- := \sum_{k=2}^{n} c_{k-1}\, \hat{e}_{k-1,k}, \tag{10b}$$

the c_k, $k=1,2,..,n-1$ being non-zero complex numbers given by

$$c_k := \sqrt{d_k}, \quad 0 \le \arg c_k < \pi, \quad d_k := \begin{cases} k(k-\mu_0) & ..k \text{ even} \\ \tfrac{1}{2}(k^2 + (\mu_0-k)^2) - \varkappa - 5 & ..k \text{ odd} \end{cases}. \tag{10c}$$

<u>Proof</u>: Let $\varkappa \in \mathcal{J}_n$, $\{\hat{h}, \hat{v}_\pm\}_\varkappa$ be a small solution, β be the minimal eigenvalue of $\hat{h}$ ($\beta \in \mathbb{R}$ according to (8c)) and let $g_1 \in \mathbb{C}^n$, $g_1 \neq 0$, satisfy $hg_1 = \beta g_1$. By introducing

$$g_k := \hat{v}_+ g_{k-1} \qquad k=2,3,..,n \tag{*}$$

and using (8a), we get

$$\hat{h} g_k = (\beta + 2(k-1)) g_k, \quad \hat{h}\hat{v}_- g_1 = (\beta-2)\hat{v}_- g_1 . \tag{**}$$

Thus the vectors g_k are either eigenvectors of $\hat{h}$ or equal zero.

+) $\sigma(\hat{h})$ denotes the set of all eigenvalues of $\hat{h}$, i.e., the spectrum.

++) $\varkappa_m := 2m(m-1) - \frac{9}{2}$, $\mu_0 = n$ (if n is even) or $n - \vartheta$ (if n is odd) – cf. the Theorem.

Now β is the minimal eigenvalue of $\hat{h}$ and hence

$$\hat{v}_- g_1 = 0 . \tag{+}$$

Suppose $g_1,..,g_p$ are non-zero and $\sum_{k=1}^{p} \alpha_k g_k = 0$. Then (**) implies $\sum_{k=1}^{p} k\,\alpha_k g_k = 0$ and from these two equalities one easily concludes that $\mathcal{G}^p := \{g_1,..,g_p\}$ is a linearly independent set. Thus there is some $\bar{p}$, $1 \le \bar{p} \le n$, such that $g_1,..,g_{\bar{p}}$ are non-zero and

$$g_{\bar{p}+1} \equiv \hat{v}_+ g_{\bar{p}} = 0. \tag{***}$$

Consider the subspace $\mathcal{G}^{\bar{p}}_{\mathrm{lin}}$. Clearly, $\hat{h}$ and $\hat{v}_+$ map $\mathcal{G}^{\bar{p}}_{\mathrm{lin}}$ into itself and by using (+) and (8b) one finds for $k=1,2,..,p$

$$\hat{v}_- g_k = d_{k-1} g_{k-1}, \quad d_k := \tfrac{1}{2}(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \left[(\beta+2j)^2 - 2\varkappa - 9\right]. \tag{++}$$

This means that $\mathcal{G}^{\bar{p}}_{\mathrm{lin}}$ is an invariant subspace of $\mathcal{N}_\varkappa$ and irreducibility of $\mathcal{N}_\varkappa$ then implies $\bar{p} = n$.

Hence $g_1,..,g_n$ is a basis in $\mathbb{C}^n$ and the above equations completely determine the operators $\hat{h}, \hat{v}_\pm$ except for relating β to $\varkappa$ and n. In order to find this relation apply both sides of Eq.(8b) to g_n:

$$d_{n-1} g_n = \tfrac{1}{2}\left[(\beta + 2n - 2)^2 - 2\varkappa - 9\right] g_n, \quad \text{i.e.} \quad \sum_{j=0}^{n-1} (-1)^j \left[(\beta+2j)^2 - 2\varkappa - 9\right] = 0.$$

After performing the summation one has $\beta = 1-n$ and $\beta = 1-n+\vartheta$ for n even and odd, respectively, which, with the help of the notation introduced in (d-ii) of the Theorem, gives

$$\beta = 1 - \mu_0 \Rightarrow \mu_0 \in \mathbb{R} .$$

Similarly, the expression (++) for d_k yields Eq.(10c).

Suppose $c_k = 0$ for some k, $1 \le k \le n-1$. By (++) one has $\hat{v}_- g_{k+1} = 0$ which, together with (*) – (***), shows that $\{g_{k+1},..,g_n\}_{\mathrm{lin}}$ is an invariant subspace of $\mathcal{N}_\varkappa$. As $\mathcal{N}_\varkappa$ is irreducible, we conclude $c_k \neq 0$, i.e., also $d_k \neq 0$ for $k=1,2,..,n-1$. After substituting $\mu_0 = n$ or $n - \vartheta$ (according to the parity of n) in the formulae (10c) for d_k and realizing that $\vartheta \in \mathbb{R}$ (as $\mu_0 \in \mathbb{R}$), we get $\varkappa \in \mathcal{J}_n'$.

We have thus proven $\mathcal{J}_n \subset \mathcal{J}_n'$; for proving the opposite inclusion we only need to show that the set $\mathcal{N}^s_{\vartheta(\varkappa,n)}$ given by (10) is irreducible and fulfils the conditions (8) for any $\varkappa \in \mathcal{J}_n'$. Verification of the first of (8) is straightforward; for getting the second one first derives from (10c) the equalities

$$d_{k-1} + d_k = \tfrac{1}{2}\left[(2k-1-\mu_0)^2 - 2\varkappa - 9\right], \quad k=1,2,..,n-1, \quad d_0 := d_n := 0.$$

Further, if $\varkappa \in \mathcal{J}_n'$, then $d_k \neq 0$, $k=1,2,..,n-1$ and $\vartheta \in \mathbb{R}$. This implies on the one hand (8c) and, on the other hand, irreducibility of n^s_ϑ: suppose $\mathcal{G} \subset \mathbb{C}^n$, $\mathcal{G} \neq \{0\}$ is an invariant subspace of n^s_ϑ, let g be a non-zero element of $\mathcal{G}$, $g = \gamma_1 f_1 + \ldots + \gamma_n f_n$, and let $J := \max\{j \mid 1 \leq j \leq n, \gamma_j \neq 0\}$; then $f_1 = (\gamma_J \prod_{j=1}^{J-1} c_j)^{-1} (\hat{v}^s_-)^{J-1} g$, i.e., $f_1 \in \mathcal{G}$. Similarly, one gets $f_k = (\prod_{j=1}^{k-1} c_j)^{-1} (\hat{v}^s_+)^{k-1} f_1 \in \mathcal{G}$, $k=2,3,\ldots,n$. Thus n^s_ϑ has no non-trivial invariant subspaces and the equality $\mathcal{J}_n = \mathcal{J}'_n$ is hereby proven.

One further sees that for $n=2m-1$, $\varkappa > \varkappa_m$, the sets $n^s_{\vartheta(+)}$, $n^s_{\vartheta(-)}$, $\vartheta(\pm) := \pm\sqrt{2(\varkappa - \varkappa_m)}$ are non-equivalent: they differ, e.g., in the spectrum of $\hat{h}^s$.

For proving (c) introduce $\alpha_1 := 1$, $\alpha_{k+1} := c_k \alpha_k$ so that $\alpha_1, \ldots, \alpha_n \neq 0$. Then a regular operator $\hat{r} \in \mathrm{End}\, \mathbb{C}^n$ is defined by $\hat{r} g_k := \alpha_k f_k$ and its inverse satisfies $\hat{r}^{-1} f_k = g_k/\alpha_k$ (see sect.4 for the definition of $f_k \equiv f_k^{(n)}$). From (*)-(***),(+),(++) one then finds $n_\varkappa = \hat{r}^{-1} n^s_\vartheta \hat{r}$. ■

7. By using this lemma one gets the following partial answer to the problems formulated in sect.3.

<u>Proposition</u>: (a) If N is odd, then $\mathcal{K}_N = \emptyset$.

(b) If $N=2n, n=1,2,\ldots$, then $\mathcal{K}_N \subset \mathcal{J}'_n$ (see Eqs.(9)) and any solution $\mathcal{H}_\varkappa \equiv \{A_\pm, H, V_\pm\}_\varkappa$ is equivalent to $\mathcal{H}^s_\vartheta \equiv \{A^s_\pm, H^s, V^s_\pm\}_\vartheta$, where $\vartheta \equiv \vartheta(n,\varkappa)$ is given by Eq.(1a),

$$H^s := \tfrac{1}{2}(\hat{h}^s \otimes \sigma_0 - \hat{I}_n \otimes \sigma_3), \qquad (11a)$$

$$A^s_\eta := \hat{I}_n \otimes \varepsilon^\eta, \ \eta = \pm 1, \qquad (11b)$$

$$V^s_\eta := 2\hat{v}^s_\eta \otimes \varepsilon^\eta - \eta(\hat{v}^s_\eta)^2 \otimes \sigma_0, \qquad (11c)$$

and the set $\{\hat{h}^s, \hat{v}^s_\pm\}_\vartheta \subset \mathrm{End}\, \mathbb{C}^n$ is defined by Eqs.(10). The corresponding $\tilde{W}^s \equiv \sum_\eta \eta A^s_\eta (V^s_{-\eta} A^s_\eta + 2A^s_{-\eta} H^s)$ reads

$$\tilde{W}^s = 2\sum_\eta \eta \hat{v}^s_{-\eta} \otimes \varepsilon^\eta - \hat{h}^s \otimes \sigma_3 + I_{2n}. \qquad (11d)$$

(c) If n is even, then for each $\varkappa \in \mathcal{K}_{2n}$ there is just one solution, whereas for $n=2m-1$, $m=1,2,\ldots$, there are at most two non-equivalent solutions if $\varkappa \in \mathcal{K}_{2n} \setminus \{\varkappa_m\}$ and just one solution if $\varkappa = \varkappa_m$.

<u>Proof</u>: By using $\{A_\varepsilon, A_\eta\} = \delta_{\varepsilon+\eta}$ one easily proves (a) and existence of a regular R such that $A_\eta = RA'_\eta R^{-1}$, where

$$A'_\eta := I_n \otimes \varepsilon^\eta. \qquad (+)$$

Let $H' = \sum_{\alpha\beta} \hat{h}_{\alpha\beta} \otimes \varepsilon_{\alpha\beta}$ and $V'_\eta = \sum_{\alpha\beta} \hat{v}^\eta_{\alpha\beta} \otimes \varepsilon_{\alpha\beta}$ be the decompositions (4) for $H' := R^{-1}HR$ and $V'_\eta := R^{-1}V_\eta R$, respectively. The second and third of ($\mathcal{H}$1) and (+) then imply $\hat{h}_{12} = \hat{h}_{21} = 0$, $\hat{h}_{22} = \hat{h}_{11} + \hat{I}_n$, $\hat{v}^+_{12} = \hat{v}^-_{21} = 0$, $\hat{v}^\eta_{11} = \hat{v}^\eta_{22}$, $\eta = \pm 1$. By denoting $\hat{h} := 2\hat{h}_{11} + \hat{I}_n$, $\hat{v}_+ := \frac{1}{2}\hat{v}^+_{21}$, $\hat{v}_- := \frac{1}{2}\hat{v}^-_{12}$, $\hat{y}_\eta := \hat{v}^\eta_{11}$, we get

$$H' = \tfrac{1}{2}(\hat{h} \otimes \sigma_0 - \hat{I}_n \otimes \sigma_3), \qquad (++)$$

$$V'_\eta = 2\hat{v}_\eta \otimes \varepsilon^\eta + \hat{y}_\eta \otimes \sigma_0. \qquad (*)$$

The fourth of ($\mathcal{H}$1) now yields

$$[\hat{h}, \hat{v}_\eta] = 2\eta\hat{v}_\eta, \quad [\hat{h}, \hat{y}_\eta] = 4\eta\hat{y}_\eta. \qquad (**)$$

With the help of (+) and (*) we further obtain

$$Z'_\eta := [V'_\eta, A'_{-\eta}] = -2\eta\hat{v}_\eta \otimes \sigma_3, \quad A'_\eta V'_\eta = \hat{y}_\eta \otimes \varepsilon^\eta$$

and substituting into the last of ($\mathcal{H}$1) gives $\hat{y}_\eta = -\eta\hat{v}^2_\eta$. Hence the first of (**) implies the second one and further one has from (*)

$$V'_\eta = 2\hat{v}_\eta \otimes \varepsilon^\eta - \eta\hat{v}^2_\eta \otimes \sigma_0. \qquad (+++)$$

Let us now express $\tilde{W}' := R^{-1}\tilde{W}R = \sum_\eta \eta A'_\eta (V'_{-\eta} A'_\eta + 2A'_{-\eta} H')$ in terms of $\hat{h}$ and $\hat{v}_\eta$:

$$\tilde{W}' = 2\sum_\eta \eta \hat{v}_{-\eta} \otimes \varepsilon^\eta - \hat{h} \otimes \sigma_3 + I_{2n}. \qquad (++++)$$

Then the fifth of ($\mathcal{H}$1) implies

$$\{\hat{v}_+, \hat{v}_-\} = \tfrac{1}{2}\hat{h}^2 - (\varkappa + \tfrac{9}{2}).^{+)} \qquad (***)$$

We thus see that for any $\varkappa \in \mathcal{K}_{2n}$ the set $n := \{\hat{h}, \hat{v}_\pm\}$ fulfils the conditions (8a,b). We will show that n is irreducible and fulfils (8c) as well. Suppose n has an invariant subspace and let $\hat{p} \in \mathrm{End}\, \mathbb{C}^n$ be the corresponding projection. Then, by using (+)-(+++), one finds that $\hat{p} \otimes \sigma_0 \mathbb{C}^{2n}$ is an invariant subspace of $\mathcal{H}_\varkappa$. Hence irreducibility of $\mathcal{H}_\varkappa$ implies irreducibility of n. Finally, (++) gives

$$\{\tfrac{1}{2}(\lambda \pm 1) \mid \lambda \in \sigma(\hat{h})\} \subset \sigma(H') = \sigma(H).$$

Now, by ($\mathcal{H}$3) one has $\sigma(H) \subset \mathbb{R}$, i.e., any $\lambda \in \sigma(\hat{h})$ must be real.

Thus for any $\varkappa \in \mathcal{K}_{2n}$ the set $\{\hat{h}, \hat{v}_\pm\}$ is a small solution. By the lemma one then has $\varkappa \in \mathcal{J}'_n$ and also existence of a regular $\hat{r} \in \mathrm{End}\, \mathbb{C}^n$, such that $n = \hat{r}^{-1} n^s_{\vartheta(n,\varkappa)} \hat{r}$, is guaranteed. Now $R_s := \hat{r} \otimes I_2 \cdot R^{-1}$ is regular and, by setting $\mathcal{H}^s_\vartheta := R_s \mathcal{H}_\varkappa R_s^{-1}$, one gets from (+)-(++++) the Eqs.(11). From (10a) and (3) one further finds

$$H^s f_k \otimes \varphi_\alpha = (k + \alpha - 2 - \mu_0/2) f_k \otimes \varphi_\alpha, \quad k=1,2,\ldots,n, \ \alpha=1,2,$$

and so the spectrum of H^s reads

$$\sigma(H^s) = \{r - \mu_0/2 \mid r=0,1,\ldots,n\}.$$

+) In fact, if $A'_\pm, H', V'_\pm$ are given by (+)-(+++) and if (**) holds, then $[V'_+, V'_-] = 4H'(2H'^2 - \tilde{W}' - \varkappa - 4)$ is equivalent to (***). This can be easily verified by using the identity

$$[\hat{v}_\eta, \hat{v}^2_{-\eta}] = [\{\hat{v}_\eta, \hat{v}_{-\eta}\}, \hat{v}_{-\eta}].$$

If $n=2m-1$, $\varkappa>\varkappa_m$, ϑ assumes two values $\vartheta(\pm):=\pm\sqrt{2(\varkappa-\varkappa_m)}$ and, as $\mu_0=n-\vartheta$, one sees that $\sigma(H^s_{\vartheta(+)})\neq\sigma(H^s_{\vartheta(-)})$, i.e., the sets $\mathcal{H}^s_{\vartheta(+)}$, $\mathcal{H}^s_{\vartheta(-)}$ are non-equivalent. ∎

8. Up to now we were concerned mostly with implications of $(\mathcal{H}1)$ and irreducibility, whereas of $(\mathcal{H}3)$ only the requirement $\sigma(H)\subset\mathbb{R}$ has been used. In the rest of this section we are going to show that $(\mathcal{H}3)$ implies much stronger conditions on $\mathcal{K}_{2n}$ than those given in the Proposition; in fact, it will be shown that $\mathcal{K}_{2n}$ cannot be larger than the sets mentioned in (b) and (c) of the Theorem.

Analyzing conditions $(\mathcal{H}3)$ is complicated by the fact that the star relations implied by them for operators $H^s, \widetilde{W}^s, V^s_\pm$ involve the operators R_s, R^*_s whose interrelation is not known. On the other hand, working with $H^s, \widetilde{W}^s, V^s_\pm$ is convenient as these operators are explicitely determined by Eqs.(10,11). We must therefore start with those properties implied by $(\mathcal{H}3)$ that are invariant w.r.t. equivalence transformations.

As H and $\widetilde{W}$ are hermitian and commute (see the proof of Lemma IV.1), there exists a basis in $\mathbb{C}^{2n}$ such that the matrices of H^s and $\widetilde{W}^s$ are diagonal and real. For finding it let us decompose H^s and $\widetilde{W}^s$ according to Eq.(7a). With the help of (6b,c),(7b) one has

$$H^s_{rr'}=\delta_{r-r'}(r-\mu_0/2)P_r\,,$$
$$\widetilde{W}^s_{rr'}=\delta_{r-r'}W_r\,,\quad W_0:=\mu_0P_0,\quad W_n:=(2n-\mu_0)P_n, \tag{+}$$
$$W_r:=(2r-\mu_0)(P_{rr;11}-P_{rr;22})+2c_r(P_{rr;12}-P_{rr;21}),\quad 1\leq r\leq n-1.$$

Since $\mathrm{Tr}\,W_r=0$ for $1\leq r\leq n-1$, the eigenvalues of W_r are $\mu_r,-\mu_r$. Moreover, $\mu_r\in\mathbb{R}$ as each eigenvalue of W_r is also an eigenvalue of $\widetilde{W}^s$; this follows from (+) which further shows that W_r is diagonalizable (because $\widetilde{W}^s$ is). In other words, there exists a regular $\Omega_r\in\mathrm{End}\,\mathbb{C}^2$ such that

$$W_r^{(D)}:=\Omega_r^{-1}W_r\Omega_r=\mu_r(P_{rr;11}-P_{rr;22}). \tag{12a}$$

By taking into account that the c_r occurring in the formula for W_r is non-zero (see Lemma 6), one easily concludes that

$$\mu_r\neq 0,\quad 1\leq r\leq n-1. \tag{12b}$$

The secular equation for W_r reads

$$\mu_r^2=(2r-\mu_0)^2-4c_r^2\,; \tag{$*$}$$

by using (10c) one finds

$$\mu_{2j}=\mu_0,\ j=1,2,\dots,E(\tfrac{n}{2}),\quad \mu_{2j-1}=\mu_1,\ j=1,2,\dots,E(\tfrac{n+1}{2}) \tag{12c}$$

with (cf.(d-ii) of the Theorem)

$$\mu_0:=\begin{cases}n\\ n-\vartheta\end{cases},\quad \mu_1:=\begin{cases}\sqrt{4\varkappa+20-n^2} & \dots\ n \text{ even}\\ n+\vartheta & \dots\ n \text{ odd}\end{cases}. \tag{12d}$$

<u>Remark</u>: These formulae hold for $r=n$ as well – this can be checked, if one substitutes for μ_0 into $\mu_n=2n-\mu_0$ (see (+) and notice that P_n is one-dimensional). However, the condition (12b) applies only if $n\geq 2$; if $n=1$, then μ_0,μ_1 may equal zero.

For finding Ω_r one makes use of the decomposition (cf.(7b)) $\Omega_r=\sum_{\alpha\beta}\omega^r_{\alpha\beta}P_{rr;\alpha\beta}$ and rewrites (12a) as $W_r\Omega_r=\Omega_rW_r^{(D)}$. This equation determines the β-th column $(\omega^r_{1\beta},\omega^r_{2\beta})$, $\beta=1,2$, uniquely up to one multiplicative constant that will be fixed by requiring $|\omega^r_{1\beta}|^2+|\omega^r_{2\beta}|^2=1$. Ω_r can be written down in a compact form with the help of auxiliary quantities

$$\rho_r:=\tfrac{1}{2}(\mu_r+\mu_0)-r=\tfrac{1}{2}(\mu_r-\mu_n)+n-r\,,^{+)}\ \sigma_r:=\tfrac{1}{2}(\mu_r-\mu_0)+r=\mu_r-\rho_r,$$
$$\xi_r:=(|c_r|^2+\sigma_r^2)^{-1/2},\quad \eta_r:=(|c_r|^2+\rho_r^2)^{-1/2},\quad 1\leq r\leq n-1. \tag{13a}$$

Notice that all these quantities are real and non-zero. This is due to $c_r\neq 0$, $\mu_r\in\mathbb{R}$ and the equality

$$\rho_r\sigma_r=-c_r^2, \tag{13b}$$

which follows from $(*)$. One then has

$$\Omega_r=\xi_r(\sigma_rP_{rr;11}-c_rP_{rr;21})+\eta_r(\rho_rP_{rr;12}+c_rP_{rr;22})$$
$$\Omega_r^{-1}=\frac{c_rP_{rr;11}-\rho_rP_{rr;12}}{c_r\mu_r\xi_r}+\frac{c_rP_{rr;21}+\sigma_rP_{rr;22}}{c_r\mu_r\eta_r}. \tag{14}$$

Let us summarize the results concerning the diagonalization.

9. <u>Lemma</u>: (a) For $m=1,2,\dots$ one has $\mathcal{K}_{4m}\subset\mathcal{J}'_{2m}\cap(m^2-5,+\infty)$,

$\mathcal{K}_2\subset\mathcal{J}'_1$

$\mathcal{K}_{4m+2}\subset\mathcal{J}'_{2m+1}\setminus\{4m(m+1)-4\}=[\varkappa_{m+1},+\infty)\setminus\{2m(m+1)+2l(l-1)-4\,|\,1\leq l\leq m+1\}$.

(b) If $n=2,3,\dots$, $\varkappa\in\mathcal{K}_{2n}$, then $\widetilde{W}^s$ has four non-zero eigenvalues $\mu_0,-\mu_0,\mu_1,-\mu_1$ (see (12d). For $n=1$ the eigenvalues of $\widetilde{W}^s$ are μ_0,μ_1 and they may assume all real values. Let n_1,n_2,n_3,n_4 be the multiplicities of $\mu_0,-\mu_0,\mu_1,-\mu_1$, respectively; then Eq.(1b) holds ($n_2=n_4=0$ if $n=1$).

(c) Consider $\Omega\in\mathrm{End}\,\mathbb{C}^{2n}$ whose block structure (7a) reads

$$\Omega_{rs}:=\delta_{r-s}\Omega_r,\ r,s=1,2,\dots,n-1,\ \Omega_0=\omega_0P_0,\ \Omega_n=\omega_nP_n\,,$$

+) This equality follows from $\mu_0+\mu_n=2n$, $n=1,2,\dots$ (see (12c,d)).

where ω_0, ω_n are arbitrary non-zero complex numbers and the operators $\Omega_r \in \mathrm{End}\,\mathbb{C}^2$ are given by Eq.(14). By introducing

$$X^{(D)} := \Omega^{-1} X^s \Omega\ , \quad X^s := H^s, \widetilde{W}^s, V^s_{\pm}\ ,$$

one has

$$H^{(D)} = H^s = \sum_{r=0}^{n} (r - \mu_0/2) P_r\ , \tag{15a}$$

$$\widetilde{W}^{(D)} = \mu_0 (F^1 - F^2) + \mu_1 (F^3 - F^4), \tag{15b}$$

where F^α are orthogonal projections that are defined via one-dimensional projections $P^{r,\mu} := P_{rr;\mu\mu}$ as follows

$$F^1 := \sum_{r=1}^{n_1} P^{2r-2,1}\ , \quad F^2 := \sum_{r=1}^{n_2} P^{2r,2}\ , \quad F^3 := \sum_{r=1}^{n_3} P^{2r-1,1}, \quad F^4 := \sum_{r=1}^{n_4} P^{2r-1,2}. \tag{15c}$$

Due to the properties of $P^{r,\mu}$ one has

$$F^\alpha F^\beta = \delta_{\alpha-\beta} F^\alpha\ , \quad \sum_{\alpha=1}^{4} F^\alpha = I_{2n}\ .$$

Finally, the block structure of $V^{(D)}_{\pm}$ reads:

$$(V^{(D)}_+)_{sr} = \delta_{s-r-2} V^{(+)}_r, \quad (V^{(D)}_-)_{rs} = \delta_{s-r-2} V^{(-)}_r, \quad 0 \le r \le n-2, \tag{15d}$$

$$V^{(+)}_0 := c_1 \omega_0 \xi_2^{-1} P_{20;11}, \quad V^{(-)}_0 := 2c_1 \mu_0 \omega_0^{-1} \xi_2 P_{02;11}\ ,$$

$$V^{(+)}_{n-2} := -c_{n-1} c_{n-2} \mu_n \omega_n^{-1} \xi_{n-2} P_{nn-2;11}$$

$$n = 3, 4, \ldots$$

$$V^{(-)}_{n-2} := -2 c_{n-1} c_{n-2}^{-1} \omega_n \xi_{n-2}^{-1} P_{n-2n;11}\ ,$$

$$V^{(+)}_r := -c_r c_{r+1} (\xi_r \xi_{r+2}^{-1} P_{r+2r;11} + \eta_r \eta_{r+2}^{-1} P_{r+2r;22}), \quad 1 \le r \le n-3$$

$$V^{(-)}_r := c_r c_{r+1} (\xi_{r+2} \xi_r^{-1} \sigma_{r+2} \sigma_r^{-1} P_{rr+2;11} + \eta_{r+2} \eta_r^{-1} \rho_{r+2} \rho_r^{-1} P_{rr+2;22})\ .$$

<u>Proof</u>: (a) follows from (12b,d) and $\mu_r \in \mathbb{R}$, (b) and (15b,c) from (12a). The formulae (15d) have been obtained by using (11c),(10b), (6c),(14) and some properties of ρ, σ which follow directly from (13a,b). All the remaining assertions have just been proven above. ■

10. The operators $H^{(D)}, \widetilde{W}^{(D)}, V^{(D)}_{\pm}$ are related to the starting $H, \widetilde{W}, V_{\pm}$ via the regular transformation R_D:

$$H = R_D H^{(D)} R_D^{-1}, \ \text{etc.} \quad R_D := R_s^{-1} \Omega\ .^{+)} \tag{16a}$$

By introducing the positive regular

$$S := R_D^{*} R_D \tag{16b}$$

and realizing that $H^{(D)}, \widetilde{W}^{(D)}$ are hermitian, we find that the conditions (A3) are equivalent to

$$[H^{(D)}, S] = 0, \quad [\widetilde{W}^{(D)}, S] = 0, \quad V^{(D)*}_+ S = -S V^{(D)}_-. \tag{16c}$$

+) R_s has been introduced in the proof of Proposition 7.

By analyzing these relations we arrive at the following final form of necessary conditions.

<u>Proposition</u>: (a) for m=1,2,.. one has

$$\mathcal{K}_{4m} \subset \mathcal{K}'_{4m} := (2m(m-1)-4, +\infty),$$

$$\mathcal{K}_2 \subset \mathcal{K}'_2 := [\varkappa_1, +\infty), \quad \mathcal{K}_{4m+2} \subset \mathcal{K}'_{4m+2} := [\varkappa_{m+1}, \varkappa_{m+1} + \tfrac{1}{2}).$$

(b) To any solution $\mathcal{H}_\varkappa \subset \mathrm{End}\,\mathbb{C}^{2n}$, $n = 1, 2, \ldots$, $\varkappa \in \mathcal{K}_{2n}$, there exist four positive numbers s_α and a regular operator R_D that obeys

$$R_D^{*} R_D = \sum_{\alpha=1}^{4} s_\alpha F^\alpha \tag{16d}$$

and transforms $\mathcal{H}_\varkappa$ to $\mathcal{H}^{(D)}_\vartheta := \{A^{(D)}_{\pm}, H^{(D)}, V^{(D)}_{\pm}\}_{\vartheta}$ according to (16a). Here $H^{(D)}$ and the auxiliary operator $\widetilde{W}^{(D)} \equiv \sum_\eta \eta A^{(D)}_\eta (V^{(D)}_{-\eta} A^{(D)}_\eta + 2A^{(D)}_{-\eta} H^{(D)})$ are hermitian and the final form of $\mathcal{H}^{(D)}_\vartheta$ is given by Eqs.(15a-c) and

$$V^{(D)}_+ = \sum_{r=0}^{n-2} \sqrt{\rho_{r+1} \sigma_{r+1}} \left(\sqrt{\rho_r \sigma_{r+2}} P_{r+2r;11} + \sqrt{\rho_{r+2} \sigma_r} P_{r+2r;22} \right), \quad V^{(D)}_- = -V^{(D)*}_+, \tag{15e}$$

$$A^{(D)}_+ = -iP_{10;11}\ , \quad A^{(D)}_- = iP_{01;11} \ \ldots\ n=1,$$

$$A^{(D)}_+ = \frac{-i}{\sqrt{\mu_0 \mu_1}} \sum_{r=1}^{n} \left(\sqrt{\sigma_r \rho_{r-1}} P_{rr-1;11} - \sqrt{\sigma_r \sigma_{r-1}} P_{rr-1;12} + \sqrt{\rho_r \rho_{r-1}} P_{rr-1;21} - \sqrt{\rho_r \sigma_{r-1}} P_{rr-1;22} \right), \quad A^{(D)}_- = A^{(D)*}_+, \ n \ge 2, \tag{15f}$$

where ρ_r, σ_r are given by Eqs.(13) and satisfy $\rho_r > 0, \sigma_r > 0$, $1 \le r \le n-1$; in addition, we set

$$\sigma_0 := \rho_n := 0, \quad \rho_0 := \mu_0, \quad \sigma_n := \mu_n\ ,^{+)} \tag{13c}$$

<u>Proof</u>: By Proposition 7 and Lemma 9 there is a regular R_D such that $X^{(D)} := R_D^{-1} X R_D$, $X = H, \widetilde{W}, V_{\pm}$ are expressed via the partial isometries $P_{rs;\mu\nu}$ according to Eqs.(15a-d). Then the first two of conditions (16c) determine the block structure (7a) of $S \equiv R_D^{*} R_D$:

$$S = \sum_{r=0}^{n} S_r\ , \quad S_0 := t_0 P_0, \quad S_n := t_n P_n, \quad S_r := \sum_{\mu=1}^{2} t_{r\mu} P_{rr;\mu\mu}, \quad 1 \le r \le n-1. (+)$$

As S is regular and positive, one has

$$t_0, t_n, t_{r1}, t_{r2} > 0, .$$

+) In (15e,f) occur the operators $P_{20;22}, P_{n2;22}$ and $P_{10;22}, P_{10;12}, P_{nn-1;21}, P_{nn-1;22}$ for which the definition (6a) does not make sense. In fact, these operators can be defined arbitrarily, as they are always multiplied by zero quantities σ_0 or ρ_n. We have introduced them for bringing formulae for $V^{(D)}_{\pm}, A^{(D)}_{\pm}$ into the above compact form.

Now (15d) and the third of (16c) give

$$(V_r^{(+)})^* S_{r+2} = -S_r V_r^{(-)}, \quad r=0,1,..,n-2. \qquad (*)$$

In particular, for r=0 we get $\bar{\varepsilon}_1 \overline{\omega}_0 \xi_2^{-1} t_{21} = -2t_0 c_1 \mu_0 \omega_0^{-1} \xi_2$. Since t_{21}, t_0 and ξ_2 are positive, it must hold $c_1 \mu_0 < 0$. Now for n=2m one has $\mu_0 = n$ and (10c) then implies $2\varkappa + 9 > (1-2m)^2$, which is $\varkappa \in \mathcal{K}'_{4m}$. For n=3,5,.. (notice that (*) does not make sense for n=1) one gets

$$(\vartheta+1)(n-\vartheta) > 0 \Leftrightarrow -1 < \vartheta < n .$$

Similarly, by setting r=n-2 in (*) and using (12d), the relation $-n < \vartheta < 1$ is obtained for n=3,5,.. . Altogether one has $|\vartheta| < 1$, i.e., $0 \le \varkappa - \varkappa_{(n+1)/2} < \frac{1}{2}$, which is $\varkappa \in \mathcal{K}'_{2n}$, and thus (a) is proven.

The inclusions $\mathcal{K}_{2n} \subset \mathcal{K}'_{2n}$ imply Re $c_k = 0$, Im $c_k > 0$, $1 \le k \le n-1$ and $\mu_r > 0$, $0 \le r \le n$, $n \geq 2$. This can easily be verified with the help of Eqs.(10c),(12d). Then the relations (13) give

$$\rho_r > 0,\ \sigma_r > 0,\ 0 \le r \le n,$$
$$\xi_r = (\mu_r \sigma_r)^{-1/2},\ \eta_r = (\mu_r \rho_r)^{-1/2},\ c_r = i(\rho_r \sigma_r)^{1/2};\ 1 \le r \le n-1. \qquad (13d)$$

By substituting into (15d) and fixing the hitherto arbitrary ω_0, ω_n :

$$\omega_0 := \xi^{i},\quad \omega_n := 1,$$

the relations (15e) are obtained. Similarly, one gets from (14)

$$\Omega_r = \sqrt{\frac{\sigma_r}{\mu_r}}(P_{rr;11} + iP_{rr;22}) + \sqrt{\frac{\rho_r}{\mu_r}}(P_{rr;12} - iP_{rr;21}),\ \Omega_r^{-1} = \Omega_r^*,\ 1 \le r \le n-1. \qquad (14')$$

Further Eqs.(11b),(6c) yield

$$A_+^s = P_{10;11} + \sum_{r=2}^{n} P_{rr-1;12}, \qquad A_-^s = (A_+^s)^*$$

and then (15f) immediately follows from the definition (16a). Finally, substituting (15e) and (+) into (*) gives $t_0 = t_{21}$, $t_n = t_{n-2,1}$, $t_{r+2,1} = t_{r1}$, $t_{r+2,2} = t_{r2}$, r=1,2,..,n-3. Now Eq.(16d) follows from (15c) and (+), if one puts $s_1 := t_0$, $s_2 := t_{22}$, $s_3 := t_{11}$, $s_4 := t_{12}$. ■

SUFFICIENT CONDITIONS

11. <u>Proposition</u>: For each $\varkappa \in \mathcal{K}'_{2n}$ the set $\mathcal{H}^{(D)}_{\vartheta(n,\varkappa)} \equiv \mathcal{H}^{(D)}_{\vartheta} \equiv \{A_\pm^{(D)}, H^{(D)}, V_\pm^{(D)}\}_\vartheta$ specified by Eqs.(15a,e,f),(1a),(12d),(13a,c) is a solution, the corresponding projection being $E = F^1 + F^2$ (see (15e)); moreover, one has

$$(A_+^{(D)})^* = A^{(D)}.$$

If n=2m-1, m=1,2,.. and $\varkappa \in \mathcal{K}'_{2n} \setminus \{\varkappa_m\}$, then $\mathcal{H}^{(D)}_{\vartheta(+)}, \mathcal{H}^{(D)}_{\vartheta(-)}$, $\vartheta(\pm) := \pm\sqrt{2(\varkappa - \varkappa_m)}$, are non-equivalent solutions.

<u>Proof:</u> One verifies directly with the help of the multiplication rule (2b) that $\mathcal{H}^{(D)}_\vartheta$ satisfies ($\mathcal{H}$2) for $E = F^1 + F^2$. Let $\varkappa \in \mathcal{K}'_{2n}$, i.e., $2\varkappa > (n-1)^2 - 9$ if n is even, $|\vartheta(\pm)| < 1$ if n is odd. By (12d) one has $\mu_0, \mu_1 \in \mathbb{R}$ and, if $n \geq 2$, then (10c),(13a,b) yield

$$\mu_r > 0,\ \rho_r > 0,\ \sigma_r > 0,\ 1 \le r \le n-1.$$

Consider $\Omega \in \mathrm{End}\,\mathbb{C}^{2n}$, $\Omega_{rs} \equiv P_r \Omega P_s := \delta_{r-s} \Omega_r$, where $\Omega_0 := -iP_0$, $\Omega_n := P_n$ and Ω_r, $1 \le r \le n-1$, is defined by (14'). Using the above properties of μ, ρ, σ one sees that Ω is unitary. If the procedure that brought us from Eqs.(11) to (15a,b,e,f) is reversed, we get

$$\Omega X^{(D)} \Omega^{-1} = X^s , \quad X = H, \widetilde{W}, V_\pm, A_\pm .$$

Moreover, due to unitarity of Ω, the star properties of $X^{(D)}$ imply

$$(H^s)^* = H^s,\ (\widetilde{W}^s)^* = \widetilde{W}^s,\ (V_+^s)^* = -V_-^s . \qquad (*)$$

Now $H^s, \widetilde{W}^s, V_\pm^s$ are expressed via $\hat{h}^s, \hat{v}_\pm^s$ which are defined by (10a,b,c); as $\mathcal{K}'_{2n} \subset \mathcal{J}'_n$ (see (9a,b)), the set $\{\hat{h}^s, \hat{v}_\pm^s\}_\vartheta$ is irreducible and fulfils the relations (8a,b). By using them, one verifies directly that $\mathcal{H}^s_\vartheta := \{A_\pm^s, H^s, V_\pm^s\}_\vartheta$ satisfies ($\mathcal{H}$1). For proving irreducibility of $\mathcal{H}^s_\vartheta$ we make use of the star properties (*) which imply that the linear envelope of $\mathcal{H}^s_\vartheta$ is a symmetric set: $(\mathcal{H}^s_\vartheta)^*_{lin} \subset (\mathcal{H}^s_\vartheta)_{lin}$. Then $\mathcal{H}^s_\vartheta$ is irreducible if its commutant $(\mathcal{H}^s_\vartheta)'$ contains only multiples of unity. Let $C \in (\mathcal{H}^s_\vartheta)'$; by using the decomposition (4) $C = \sum_{\alpha\beta} \hat{c}_{\alpha\beta} \otimes \varepsilon_{\alpha\beta}$, $\hat{c}_{\alpha\beta} \in \mathrm{End}\,\mathbb{C}^n$, one finds that $[C, A_\pm^s] = 0$ implies $\hat{c}_{12} = \hat{c}_{21} = 0$, $\hat{c}_{11} = \hat{c}_{22} \equiv \hat{c}$, i.e. $C = \hat{c} \otimes \sigma_0$. From $[C, H^s] = 0$ and $[C, V_\pm^s] = 0$ one then gets $[\hat{c}, \hat{h}^s] = 0$, $[\hat{c}, \hat{v}_\pm^s] = 0$. Since $\{\hat{h}^s, \hat{v}_\pm^s\}$ is irreducible, we have $\hat{c} = \gamma \hat{I}_n$, i.e. $C = \gamma I_{2n}$, $\gamma \in \mathbb{C}$. Thus $\mathcal{H}^s_\vartheta$ is a solution and, as Ω is unitary, $\mathcal{H}^{(D)}_\vartheta = \Omega^{-1} \mathcal{H}^s_\vartheta \Omega$ is a solution as well.

If n=2m-1, m=1,2,.., then Tr $H^{(D)} = n\vartheta$; now $\vartheta(+) \neq \vartheta(-)$ for $\varkappa \neq \varkappa_m$ and hence the solutions $\mathcal{H}^{(D)}_{\vartheta(+)}, \mathcal{H}^{(D)}_{\vartheta(-)}$ cannot be equivalent. ■

REDUCTION OF $\mathcal{P}^{(D)}(\vartheta) := \{H^{(D)}, V_\pm^{(D)}, W^{(D)}\}_\vartheta$

12. Up to now we have proven the first three statements ((a)-(c)) of the Theorem. The last statement (d) concerns essentially reduction of the set $\mathcal{P}^{(D)}(\vartheta)$. The star properties of $\mathcal{H}^{(D)}_\vartheta$ imply that $(\mathcal{P}^{(D)}(\vartheta))_{lin}$ is a symmetric set and hence $\mathcal{P}^{(D)}(\vartheta)$ is fully reducible. It further satisfies the conditions (2.7) of sect.II.4 (see ($\mathcal{H}$1) and Lemma IV.1). According to Proposition II.4 $\mathcal{P}^D(\vartheta)$ equals direct sum of the irreducible sets $\mathcal{S}_r(n(\alpha), \tau(\alpha))$ given by Eqs.(2.8,2.9), the corresponding subspaces $\mathcal{V}^\alpha$, $\dim \mathcal{V}^\alpha = n(\alpha)$, $\mathcal{S}_r(n(\alpha), \tau(\alpha)) \subset \mathrm{End}\,\mathbb{C}^{n(\alpha)}$, being orthogonal

to each other. In particular, there exist $\widetilde{w}_\alpha \in \mathbb{R}$ such that

$$\widetilde{W}^{(D)} \restriction \mathcal{V}^\alpha = \widetilde{w}_\alpha \, I_{n(\alpha)} .$$

It is convenient to introduce a basis in $\mathbb{C}^{2n}$ in such a way that the matrix of $\widetilde{W}^{(D)}$ w.r.t. it equals direct sum of unit matrices $[I_{n(\alpha)}]$ multiplied by $\widetilde{w}_\alpha$. By using (15b,c) one sees that the structure of such a basis should be

$$\{\chi_k^\alpha \mid \alpha = 1,..,4, \quad k = 1,..,n(\alpha)\} ,$$

the dimensions being $n(\alpha) = n_\alpha$ (see (1b)). The vectors χ_k^α can be obtained, e.g., by reordering the basis (5a) as follows

$$\chi_k^1 := \psi_{2k-2,1}, \quad \chi_k^2 := \psi_{2k,2}, \quad \chi_k^3 := \psi_{2k-1,1}, \quad \chi_k^4 := \psi_{2k-1,2}. \qquad (17)$$

The corresponding partial isometries (2a) will be denoted $F_{kl}^{\alpha\beta}$, i.e.,

$$F_{kl}^{\alpha\beta} := U_{2n}(\chi_k^\alpha, \chi_l^\beta) .$$

Then the projections (15b) become

$$F^\alpha = \sum_{k=1}^{n_\alpha} F_{kk}^{\alpha\alpha}$$

and the multiplication rule (2b) yields

$$F^\alpha F_{kl}^{\alpha'\beta'} F^\beta = \delta_{\alpha-\alpha'} \delta_{\beta-\beta'} F_{kl}^{\alpha\beta} . \qquad (18)$$

For the partial isometries $P_{rs;\mu\nu}$ that occur in Eqs.(15e,f) one has

$$P_{2r,2s;\mu\nu} = F_{r+\delta_{\mu-1},s+\delta_{\nu-1}}^{\mu\nu}, \quad P_{2r-1,2s-1;\mu\nu} = F_{rs}^{\mu+2,\nu+2}, \text{ etc.}$$

13. The searched decomposition of $\mathcal{P}^{(D)}(\vartheta)$ can now be simply obtained if one rewrites Eqs.(15a,e) in terms of $F_{kl}^{\alpha\beta}$:

$$H^{(D)} = \sum_{\alpha=1}^{4} \sum_{k=1}^{n_\alpha} (2k - d_\alpha - \mu_0/2) F_{kk}^{\alpha\alpha}, \quad d_1 := 2, \ d_2 := 0, \ d_3 := d_4 := 1 , \qquad (19a)$$

$$V_+^{(D)} = \sum_{\alpha=1}^{4} \sum_{k=1}^{n_\alpha - 1} \omega_{\alpha k} F_{k+1\,k}^{\alpha\alpha}, \quad \omega_{1k} := |c_{2k-1}| \sqrt{\rho_{2k-2}\sigma_{2k}},$$

$$\omega_{2k} := |c_{2k+1}| \sqrt{\rho_{2k+2}\sigma_{2k}}, \quad \omega_{3k} := |c_{2k}| \sqrt{\rho_{2k-1}\sigma_{2k+1}}, \qquad (19b)$$

$$\omega_{4k} := |c_{2k}| \sqrt{\rho_{2k+1}\sigma_{2k-1}} .$$

With the help of (18) one then finds:

$$F^\alpha H^{(D)} F^\beta = \delta_{\alpha-\beta} H_\alpha, \quad H_\alpha := \sum_{k=1}^{n_\alpha} (2k - d_\alpha - \mu_0/2) F_{kk}^{\alpha\alpha} ,$$

$$F^\alpha V_+^{(D)} F^\beta = \delta_{\alpha-\beta} V_\alpha^{(+)}, \quad V_\alpha^{(+)} := \sum_{k=1}^{n_\alpha - 1} \omega_{\alpha k} F_{k+1\,k}^{\alpha\alpha} ,$$

$$F^\alpha W^{(D)} F^\beta = \delta_{\alpha-\beta} \widetilde{w}_\alpha F^\alpha, \quad \widetilde{w}_1 := -\widetilde{w}_2 := \mu_0, \quad \widetilde{w}_3 := -\widetilde{w}_4 := \mu_1 .$$

Thus the set $\mathcal{P}^{(D)}(\vartheta)$ is reduced by all the subspaces $\mathcal{V}^\alpha \equiv F^\alpha \mathbb{C}^{2n}$. In addition, the matrices $[H_\alpha]$, $[V_\alpha^{(+)}]$ of operators $H_\alpha, V_\alpha^{(+)}$ w.r.t. the basis $\{\chi_k^\alpha \mid k = 1,..,n_\alpha\}$ in $\mathcal{V}^\alpha$ are obtained: the element in the k-th row and l-th column equals the coefficient at $F_{kl}^{\alpha\alpha}$.

Let us show that all the sets $\mathcal{P}_\alpha^{(D)} := \mathcal{P}^{(D)}(\vartheta) \restriction \mathcal{V}^\alpha$ are irreducible. Notice that $(\mathcal{P}_\alpha^{(D)})_{lin}$ is symmetric and so the usual argument concerning the commutant $(\mathcal{P}_\alpha^{(D)})'$ applies. Since $[H_\alpha]$ is a diagonal matrix and its diagonal elements are different from each other, one has for any $C \in \operatorname{End} \mathcal{V}^\alpha, C \in (\mathcal{P}_\alpha^{(D)})'$

$$C = \sum_{k=1}^{n_\alpha} \gamma_k F_{kk}^{\alpha\alpha} ;$$

further the relation $[V_\alpha^{(+)}, C] = 0$ yields $\gamma_1 = \gamma_2 = .. = \gamma_{n_\alpha}$ (all the $\omega_{\alpha k}$ are positive). Hence $C = F^\alpha \equiv I_{n_\alpha}$.

Each of the sets $\mathcal{P}_\alpha^{(D)}$ is thus equivalent to some $\mathcal{S}_{r_\alpha}(n_\alpha, \gamma_\alpha) \equiv \{\underline{h}^{r_\alpha}, \underline{v}_\pm^{r_\alpha}, w_{r_\alpha} \underline{I}_{n_\alpha}\}$ and we need to determine r_α and γ_α for given n_α, $w_r = \widetilde{w}_\alpha + \varkappa - 4$ and $\operatorname{Tr} \underline{h}^{r_\alpha} = \operatorname{Tr} H_\alpha$. To this purpose the following properties of $\mathcal{S}_r(n, \gamma)$ derived in sect.II.4 will be used: $\operatorname{Tr} \underline{h}^r = 0 \Leftrightarrow r = 1$, for r=1 one has $\gamma = w_r$ and

$$\gamma > 2(n_\alpha - 1)^2 - 8 \quad {}^{+)}; \qquad (*)$$

for r=2 one has $\gamma = n_\alpha^{-1} \operatorname{Tr} \underline{h}^r$ and

$$0 < |\gamma| < 1 \ {}^{+)}, \quad 2\gamma^2 + 2n_\alpha^2 - 10 = w_r . \qquad (**)$$

We shall consider separately the cases of even and odd n.

(i) n=2m , m=1,2,... : By substituting $\mu_0 = n$, $\mu_1 = \sqrt{4\varkappa + 20 - n^2}$, we find

$$\operatorname{Tr} H_\alpha = n_\alpha(n_\alpha + 1) - n_\alpha(m + d_\alpha) = 0, \quad \alpha = 1,2,3,4.$$

Thus $r_\alpha = 1$, $\gamma_1 = 2m + \varkappa - 4$, $\gamma_2 = -2m + \varkappa - 4$, $\gamma_3 = \mu_1 + \varkappa - 4 = (\sqrt{\varkappa + 5 - m^2} + 1)^2 + m^2 - 10$, $\gamma_4 = -\mu_1 + \varkappa - 4 = (\sqrt{\varkappa + 5 - m^2} - 1)^2 + m^2 - 10$ and one easily verifies that $\varkappa \in \mathcal{K}_{4m}$ i.e. $\varkappa > 2m(m-1) - 4$, implies (*) for all α and m. Hence

$$\mathcal{P}_\alpha^{(D)} = T \mathcal{S}_1(n_\alpha, \widetilde{w}_\alpha + \varkappa - 4) T^{-1}, \quad 1 \le \alpha \le 4, \quad T \in \operatorname{End} \mathcal{V}^\alpha, \text{ regular.}$$

In view of the star properties of $\mathcal{P}^{(D)}$ and $\mathcal{S}$ one finds that T^*T belongs to the commutant of $\mathcal{S}$. Now $\mathcal{S}$ is irreducible and thus one can suppose T unitary. Consider the matrix $[H_\alpha]$; only diagonal elements h_{kk} are non-zero and as they strictly increase with k, one has $[H_\alpha] = [\underline{h}^1]$. This further implies that the unitary matrix T must be diagonal. Then the elements $[V_\alpha^{(+)}]_{jk}$, $[\underline{v}_+^1]_{jk}$ may differ at most in a phase factor, $1 \le j,k \le n_\alpha$. Now Eqs.(19b) and (2.8b) show that all the-

+) This need not hold if $n_\alpha = 1$.

se elements are non-negative and so we conclude

$$\{[H_\alpha],[V_\alpha^{(\pm)}]\}=\mathcal{M}_1(n_\alpha,\widetilde{w}_\alpha+\varkappa-4),\ 1\le\alpha\le 4.$$

(ii) $n=2m-1$, $m=1,2,..$: In this case $\mu_0=n-\vartheta$, $\mu_1=n+\vartheta$, which yields

$$\mathrm{Tr}\,H_\alpha=n_\alpha\gamma_\alpha,\quad \gamma_1:=\gamma_4:=\frac{\vartheta-1}{2},\quad \gamma_2:=\gamma_3:=\frac{\vartheta+1}{2}. \qquad (+)$$

Further one easily verifies that the second of (**) holds for $1\le\alpha\le 4$, $m=1,2,..$. If $m\ge 2$, then $\varkappa\in\mathcal{K}_{4m-2}\Leftrightarrow|\vartheta|<1$ implies $\gamma_1,\gamma_4\in(-1,0)$, $\gamma_2,\gamma_3\in(0,1)$ and hence $r_\alpha=2$, $1\le\alpha\le 4$. For $m=1$ only $\alpha=1,3$ have to be considered ($n_2=n_4=0$). Now γ_1,γ_3 may equal zero since ϑ assumes all real values. However, in case that $\gamma_\alpha=0$, the corresponding $w_{r_\alpha}=-8$ and in view of the equality $\mathcal{S}_1(1,0)=\mathcal{S}_2(1,-8)$ (cf.(2.8),(2.9)), we conclude

$$\mathcal{P}_\alpha^{(D)}\sim\mathcal{S}_2(n_\alpha,\gamma_\alpha),\ 1\le\alpha\le 4,\quad m=1,2,\ldots,$$

γ_α being given by(+). By the same argument as in the case (i) we then find

$$\{[H_\alpha],[V_\alpha^{(\pm)}]\}=\mathcal{M}_2(n_\alpha,\tfrac{1}{2}(\vartheta-(-1)^{E(\alpha/2)})),\ 1\le\alpha\le 4.$$

14. Finally, the matrix representation of $A_+^{(D)}$ given in (d-iv) of the Theorem is obtained, if one rewrites Eq.(15f) in terms of $F_{kl}^{\alpha\beta}$:

$$A_+^{(D)}=-iF_{11}^{31},\quad n=1$$

$$A_+^{(D)}=\frac{i}{\sqrt{\mu_0\mu_1}}\sum_{\alpha=1}^{2}\sum_{\beta=3}^{4}\Big((-1)^\beta\sum_{k=1}^{n_\alpha+\alpha-2}\sqrt{a_k^\alpha a_k^\beta}\,F_{k+2-\alpha,k}^{\alpha\beta}+(-1)^\alpha\sum_{k=1}^{n_\beta+1-\alpha}\sqrt{b_k^\alpha b_{k+\alpha-1}^\beta}\,F_{k+\alpha-1,k}^{\beta\alpha}\Big),\quad n=2,3,..,$$

with $a_k^1:=b_k^2:=\sigma_{2k}=2k$, $a_k^2:=b_{k+1}^1:=\rho_{2k}=\mu_0-2k$,
$a_k^4:=b_k^3:=\sigma_{2k-1}=\frac{1}{2}(\mu_1-\mu_0)+2k-1=:\tau_k$, $a_k^3:=b_k^4:=\rho_{2k-1}=\mu_1-\tau_k$.

By applying (18) one then finds for $A^{\alpha\beta}:=i\sqrt{\mu_0\mu_1}\,F^\alpha A_+^{(D)}F^\beta$

$$A^{\alpha\beta}=A^{\alpha+2,\beta+2}=0,\quad \alpha,\beta=1,2,$$

and gets the explicit expressions for the remaining eight pairs $\alpha\beta$. Hereby the Theorem is completely proven.

APPENDIX

The matrices of operators $H^{(D)},\widetilde{W}^{(D)},V_+^{(D)},A_+^{(D)}\in\mathrm{End}\,\mathbb{C}^{2n}$ w.r.t. the basis (17) will be given for n=1,2,3,4 explicitely as functions of $\mu_1=\sqrt{4\varkappa+20-n^2}$ (if n=2,4) or $\vartheta=\pm\sqrt{2\varkappa+10-n^2}$ (n=1,3). The remaining operators in $\mathcal{H}^{(D)}$ are given by $V_-^{(D)}=(-V_+^{(D)})^*$, $A_-^{(D)}=(A_+^{(D)})^*$. Notice

that in the decomposition of $\mathcal{P}^{(D)}$ there are four non-trivial terms $\mathcal{P}_\alpha^{(D)}$, i.e., terms for which $n_\alpha\ge 1$, iff $n\ge 3$; for n=1,2 there are only two and three non-trivial terms, respectively.

(i) For n=1 one has $n_1=n_3=1$, $n_2=n_4=0$, $\vartheta=\pm\sqrt{2\varkappa+9}$. There is just one solution iff $\varkappa=-9/2$ and just two non-equivalent solutions iff $\varkappa>-9/2$.

$$(+)\quad [H^{(D)}]=\begin{pmatrix}(\vartheta-1)/2 & 0\\ 0 & (\vartheta+1)/2\end{pmatrix},\ [\widetilde{W}^{(D)}]=\begin{pmatrix}1-\vartheta & 0\\ 0 & 1+\vartheta\end{pmatrix},\ [A_+^{(D)}]=\begin{pmatrix}0 & 0\\ -i & 0\end{pmatrix},\ V_+^{(D)}=0.$$

<u>Remark</u>: We have just obtained this result in our preliminary study /3/; for identifying it with (+) one has to replace ϑ by 2c and multiply the matrices (+) on the left and right by $(\sigma_1-i\sigma_2)/\sqrt{2}$.

(ii) For n=2 one has $n_1=2$, $n_2=0$, $n_3=n_4=1$, $\mu_1=2\sqrt{\varkappa+4}$. There is just one solution iff $\varkappa>-4$.

$$[H^{(D)}]=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad [\widetilde{W}^{(D)}]=\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&\mu_1&0\\0&0&0&-\mu_1\end{pmatrix},\quad [\widetilde{V}_+^{(D)}]=\begin{pmatrix}0&0&0&0\\\mu_1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$[A_+^{(D)}]=\frac{i}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&-1&1\\-1&0&0&0\\-1&0&0&0\end{pmatrix}.$$

In the remaining cases n=3,4 rectangular n_α by n_β blocks $X^{\alpha\beta}$, $\alpha,\beta=1,2,3,4$ are given instead of $[X]$. Let us recall that

$$X^{\alpha\beta}=\delta_{\alpha-\beta}X_\alpha,\quad X=H^{(D)},\widetilde{W}^{(D)},V_+^{(D)},$$
$$A_+^{\mu\nu}=A_+^{\mu+2,\nu+2}=0,\quad \mu,\nu=1,2.$$

(iii) For n=3 one has $n_1=n_3=2$, $n_2=n_4=1$, $\vartheta=\pm\sqrt{2\varkappa+1}$. There is just one solution iff $\varkappa=-1/2$ and just two non-equivalent solutions iff $\varkappa\in(-\frac{1}{2},0)$.

$$H_1=\begin{pmatrix}(\vartheta-3)/2&0\\0&(\vartheta+1)/2\end{pmatrix},\ H_2=\frac{\vartheta+1}{2},\ H_3=\begin{pmatrix}(\vartheta-1)/2&0\\0&(\vartheta+3)/2\end{pmatrix},\ H_4=\frac{\vartheta-1}{2},$$

$$\widetilde{W}_1=(3-\vartheta)\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \widetilde{W}_2=\vartheta-3,\ \widetilde{W}_3=(3+\vartheta)\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \widetilde{W}_4=-(3+\vartheta),$$

$$V_1^+=2\sqrt{(3-\vartheta)(1+\vartheta)}\begin{pmatrix}0&0\\1&0\end{pmatrix},\ V_2^+=V_4^+=0,\ V_3^+=2\sqrt{(3+\vartheta)(1-\vartheta)}\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

$$A_+^{13}=\frac{-2i}{\sqrt{9-\vartheta^2}}\begin{pmatrix}0&0\\1&0\end{pmatrix},\ A_+^{14}=i\sqrt{\frac{2(1+\vartheta)}{9-\vartheta^2}}\begin{pmatrix}0\\1\end{pmatrix},\ A_+^{23}=-i\sqrt{\frac{2(1-\vartheta)}{9-\vartheta^2}}\,(1\ \ 0),$$

$$A_+^{24}=i\sqrt{\frac{1-\vartheta^2}{9-\vartheta^2}},\ A_+^{31}=\frac{-i}{\sqrt{9-\vartheta^2}}\begin{pmatrix}\sqrt{(3-\vartheta)(1+\vartheta)}&0\\0&\sqrt{(3+\vartheta)(1-\vartheta)}\end{pmatrix},\ A_+^{32}=i\sqrt{\frac{2}{3-\vartheta}}\begin{pmatrix}0\\1\end{pmatrix},$$

$$A_+^{41}=-i\sqrt{\frac{2}{3+\vartheta}}\,(1\ \ 0),\quad A_+^{42}=0.$$

(iv) For $n=4$ one has $n_1=3$, $n_2=1$, $n_3=n_4=2$, $\mu_1=2\sqrt{\varkappa+1}$. There is just one solution iff $\varkappa>0$.

$$H_1=\begin{pmatrix}-2&0&0\\0&0&0\\0&0&2\end{pmatrix},\quad H_2=0,\quad H_3=H_4=\begin{pmatrix}-1&0\\0&1\end{pmatrix},$$

$$\widetilde{W}_1=4\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \widetilde{W}_2=-4,\quad \widetilde{W}_3=-\widetilde{W}_4=\mu_1\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$V_1^+=\sqrt{2(\mu_1^2-4)}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix},\quad V_2^+=0,\quad V_3^+=(\mu_1+2)\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad V_4^+=(\mu_1-2)\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

$$A_+^{13}=-k\begin{pmatrix}0&0\\\sqrt{\mu_1+2}&0\\0&\sqrt{2\mu_1-4}\end{pmatrix},\quad A_+^{14}=k\begin{pmatrix}0&0\\\sqrt{\mu_1-2}&0\\0&\sqrt{2\mu_1+4}\end{pmatrix},\quad A_+^{23}=-k(\sqrt{\mu_1+2}\;\;0),$$

$$A_+^{24}=k(\sqrt{\mu_1-2}\;\;0),\quad A_+^{31}=-k\begin{pmatrix}\sqrt{2\mu_1-4}&0&0\\0&\sqrt{\mu_1+2}&0\end{pmatrix},\quad A_+^{32}=k\begin{pmatrix}0\\\sqrt{\mu_1+2}\end{pmatrix},$$

$$A_+^{41}=-k\begin{pmatrix}\sqrt{2\mu_1+4}&0&0\\0&\sqrt{\mu_1-2}&0\end{pmatrix},\quad A_+^{42}=k\begin{pmatrix}0\\\sqrt{\mu_1-2}\end{pmatrix},\quad k:=\frac{1}{2\sqrt{\mu_1}}\;.$$

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Беднарж М., Бланк И., Экснер П., Гавличек М. E2-83-150

Представления osp(1,4) при помощи трех бозонных пар
и матриц произвольного четного порядка. Основная теорема.

Продолжается рассмотрение семейства бесконечномерных шуровских инволютивных представлений супералгебры Ли osp(1,4), введенных в первой части настоящей работы/1/. Приводится подробный анализ матричных соотношений, определяющих структуру данного семейства; все их неэквивалентные решения приведены в явном виде.

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Bednář M., Blank J., Exner P., Havlíček M. E2-83-150

Representations of osp(1,4) in Terms of Three Boson Pairs
and Matrices of Arbitrary Even Order. The Basic Theorem.

The study of the class of infinite-dimensional Schur-irreducible star representations of the Lie superalgebra osp(1,4), introduced in the first part of the paper/1/, is continued. The matrix relations determining the structure of the class are analyzed in detail and all non-equivalent solutions of these relations are given explicitly.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1983