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P.Exner

ON THE REPRESENTATIONS OF POINCARÉ GROUP ASSOCIATED WITH UNSTABLE PARTICLES

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The Poincaré group $\mathcal{P}$ of special-relativistic space-time transformations plays undoubtedly a central role in the high-energy physics. In particular, its unitary irreducible representations may be used for classification of the (stable) elementary particles according to their mass and spin *). A relativistically-covariant description is needed for unstable particles too. For practical purposes, it is frequently sufficient to describe them as classical point particles which decay exponentially in their proper time. Maybe this is the reason, why some quantum aspects of the problem are not yet fully understood.

One is naturally tempted to generalize the idea of stable-particle classification and associate suitable non-unitary irreducible representations of $\mathcal{P}$ with the unstable particles. Such representations were actually constructed and used by many authors /3-8/. Typically the homogeneous Lorentz transformations are represented by unitary operators, while the space-time translations are non-unitary and characterized by some complex four-momentum vector **). The generalization from stable to unstable particles should not be taken too literally, otherwise one is faced to interpretative difficulties as growing norms for negative times. It seems reasonable to associate the direct physical meaning with the operators representing the sub-

*) Classification of the unitary irreducible representations of $\mathcal{P}$ started from the paper by E.P.Wigner /1/. For their description and application to classification of elementary particles see, e.g., Ref.2, § 17.2 .

**) Beside the mentioned irreducible representations, some non-unitary indecomposable representations (i.e., reducible but not completely reducible) were proposed for description of unstable particles : see Refs.9,10 , and Ref.2, § 17.4 . However, these attempts aimed mainly to yield decay laws related to higher-order poles which have been never observed.

set $\mathcal{P}_+ \subset \mathcal{P}$ which consists of the homogeneous Lorentz group and translations to the forward light cone ; it is called sometimes the Poincaré semigroup[9]. Other authors tried to overpass the difficulty by modifying basic postulates of the quantum theory[7,10].

In fact, there is no a priori reason why non-unitary representations of $\mathcal{P}$ should be associated with unstable particles Explained in a standard way, the principle of relativistic invariance means that the state Hilbert space of any isolated quantum system is carrier space of some unitary (strongly continuous) representation of $\mathcal{P}$, under which dynamical variables of the system transform in a specific way. In particular, some important observables are identified directly with generators of the corresponding representation of $L_{\mathcal{P}}$, the Lie algebra of $\mathcal{P}$: the total Hamiltonian $H \equiv P_0$ with the generator of time translations, components P_j of the momentum with the generators of space translations, etc.

Hence one should start with a larger isolated system which contains the unstable particle under consideration as well as its decay products, and to choose on its state Hilbert space $\mathcal{H}$ a suitable unitary representation $U : \mathcal{P} \to \mathcal{B}(\mathcal{H})$. This representation is presumably reducible but it should be characterized by a sharp value of spin ; examples of such representations are known[11,12]. Having determined U , one may return to the subspace $\mathcal{H}_u \subset \mathcal{H}$ which belongs to the unstable particle alone, and study the operator-valued function $V : \mathcal{P} \to \mathcal{B}(\mathcal{H}_u)$ defined by

$$V(\Lambda,a) = pr_u U(\Lambda,a) \tag{1}$$

for all elements $(\Lambda,a) \in \mathcal{P}$. The following questions arise naturally :

(i) do the operators $V(\Lambda,a)$ fulfil the composition law of $\mathcal{P}$, at least for some subgroup or subset of elements ?

(ii) if so, what can be said about the relations between such a representation and the corresponding restriction of the above-mentioned non-unitary representations ?

The only serious attempt to find an answer, and to reconcile thereby the two approaches, was undertaken by Williams[13] ; but he failed on the well-known difficulty with below-unbounded energy spectrum. Our aim in the present paper is to clarify the matter.

The boosts should not be represented unitarily

To begin with, let us recall few basic facts about the Hilbert-space kinematics of decay processes[13-18]. Assume that the Hilbert spaces $\mathcal{H}_u$, $\mathcal{H}$ referring to the unstable particle and a larger isolated system, respectively, and a strongly continuous unitary representation U of $\mathcal{P}$ on $\mathcal{H}$ are given. Let U_t denote the operators which represent the one-parameter subgroup of time translations, $U_t = \exp(-iHt)$. A natural requirement implied by the non-stability is

$$U_t \mathcal{H}_u \not\subset \mathcal{H}_u \quad , \quad t > 0 \quad ; \tag{2}$$

or more explicitly, there is no $t > 0$ for which $\mathcal{H}_u$ is invariant under U_t .

The reduced propagator is defined by $V_t = pr_u U_t \equiv E_u U_t \upharpoonright \mathcal{H}_u$, where E_u is the projection referring to $\mathcal{H}_u$. It is easy to see (cf. Refs.16-19) that the function $t \mapsto V_t$ is positive definite and continuous (weakly or strongly, it amounts to the same here), and fulfils $V_0 = I_u$. On the other hand, it appears that these properties of V_t are sufficient to ensure existence of solution to the inverse decay problem, i.e., to reconstruct a tripple $\{\mathcal{H}, U_t, E_u\}$ such that $V_t = pr_u U_t$ for all t , and moreover, that this solution is essentially unique under a natural minimality condition[14]. Technically, these results are achieved by means of the unitary-dilations theory[19].

Experience suggests that the operators V_t might fulfil the semigroup condition, $V_t V_s = V_{t+s}$ for all $t,s \geq 0$. Unfortunately, in such a case the Hamiltonian H referring to the solution of the inverse decay problem contains the whole real axis in its spectrum *). Nonetheless, the semigroup reduced propagators represent a very useful approximation. The unphysical character of the energy spectrum makes no harm, since it has no observable consequences[17] ; it may be removed when preparation of the unstable particle is completed by an energy-filtering procedure[21,22]. In fact, the inevitable deviations from the semigroup behaviour are likely to be unobservable even if they are amplified by repeated non-decay measurements performed on the particle and an artificial energy filtering[23].

*) The semigroup condition is equivalent to absence of the regeneration. The conclusion about the spectrum remains valid even if the regeneration ceases after a finite time - cf. Ref.16 . A necessary and sufficient condition for energy semiboundedness was given in Ref.18 . The problem has been discussed many times ; for a more complete bibliography see Ref.20 .

Let us finally mention definition of the decay law. For an unstable particle which is described initially by a density matrix ρ, $\mathrm{Ran}\,\rho \subset \mathcal{H}_u$, the non-decay probability equals $P_\rho(t) = \mathrm{Tr}(\rho V_t^* V_t)$. In particular, if the initial state is pure and described by a unit vector $\psi \in \mathcal{H}_u$, its decay law is

$$P_\psi(t) = \|V_t\psi\|^2 = \|E_u U_t \psi\|^2 \quad . \tag{3a}$$

Situation is especially simple in the case of a one-dimensional $\mathcal{H}_u$ (spanned by ψ) when

$$P_\psi(t) = |v(t)|^2 \quad , \quad v(t) = (\psi, U_t\psi) \quad ; \tag{3b}$$

the semigroup condition imposed on $\{V_t\}$ now requires the decay law (3b) to be exponential.

Now we shall return to the Poincaré group. Its space-time transformations are given by

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu \quad , \tag{4}$$

where Λ belongs to SO(3,1) and a is a four-vector. For simplicity, we shall consider the connected component of $\mathcal{P}$ only avoiding discussion of the space and time inversions. The composition law of the transformations (4) implies

$$U(\Lambda,a)U(\Lambda',a') = U(\Lambda\Lambda', a+\Lambda a') \tag{5}$$

for all $(\Lambda,a), (\Lambda',a') \in \mathcal{P}$. Unitarity of U together with the definition (1) yield the relation

$$V(\Lambda,a)^* = V(\Lambda^{-1}, -\Lambda^{-1}a) \tag{6}$$

for all $(\Lambda,a) \in \mathcal{P}$. Suppose that V fulfils the group law analogous to (5), then $V(\Lambda,a)^*V(\Lambda,a) = V(\Lambda,a)V(\Lambda,a)^* = I_u$ so $V(\Lambda,a)$ is unitary. However, this is equivalent to the fact that $U(\Lambda,a)$ commutes with E_u; particularly for the time translations, it would mean that the condition (2) was violated. Thus the operators $V(\Lambda,a)$ cannot fulfil the group composition law for all $(\Lambda,a) \in \mathcal{P}$, i.e., V cannot be a (non-unitary) representation of $\mathcal{P}$.

This conclusion is not yet disastrous. Motivated by the above sketched description of the time evolution, we are ready to accept the following possibility : there is a non-unitary representation $\tilde{V}$

of $\mathcal{P}$, presumably some of the ones mentioned in the introduction, such that $\tilde{V}(\Lambda,a) = V(\Lambda,a)$ within some reasonable subset of $\mathcal{P}$, say $\mathcal{P}_+$. Unfortunately, even this point of view cannot be retained. The reason is that it does not respect the Euclidean invariance (the first and maybe the most important among the laws on which physics is built - E.Wigner dixit). It is quite natural to assume that two observers, whose reference frames are obtained one from the other by space translations and rotations, will determine exactly the same decay law and other characteristics for a given unstable particle. Hence, in particular, the operators $V(I,a)$ with $a = (0,\vec{a})$ should be unitary, and this is not true for the representations we have in mind.

Furthermore, the translational invariance implies that the operators $V(\Lambda,0)$ referring to the pure Lorentz transformations (boosts) must not be unitary. In order to see it, notice that the relation (5) yields the identity

$$U(I,\Lambda a)U(\Lambda^{-1},0)U(I,-a)U(\Lambda,0) = U(I,\Lambda a - \Lambda^{-1}a) \quad . \tag{7}$$

Let $\Lambda = \Lambda_{\vec{\beta}}$ be a boost with a velocity $\vec{\beta}$ and $a = (0,\vec{a})$, where $\vec{a}$ is parallel to $\vec{\beta}$. In such a case, one has

$$\Lambda a - \Lambda^{-1}a = (-2\xi|\vec{a}|\sinh|\vec{\beta}|, \vec{0}) \quad , \quad \xi = \mathrm{sgn}\,\vec{\beta}.\vec{a} \quad . \tag{8}$$

We have pointed out that $V(\Lambda,a)$ is unitary for some (Λ,a) iff the corresponding $U(\Lambda,a)$ commutes with E_u. Thus if the boosts were represented unitarily, the same would be the rhs of (7). Since ξ, $|\vec{\beta}|, |\vec{a}|$ may be chosen arbitrarily, the relation (8) shows that E_u must commute with the operators representing time translations. Of course, this contradicts to (2), so the conclusion is proved [*)].

Notice finally that up to now no requirement specific for unstable particles was used. The above considerations apply therefore by the same right to free unstable nuclei and other decaying objects for which a relativistically-covariant description is appropriate.

The representations U related to unstable particles

Since the unstable particles may be characterized by spin quantum numbers, the most natural choice for U is a direct integral over mass of the unitary irreducible representations $U^{(m,s,+)}$

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*) A formal Lie-algebraic version of this argument was given in Ref.17 .

(cf. Ref.2, § 17.2). Notice that the same U was used in Refs.11,12, but it was not accompanied there by an Euclidean-invariant choice of $\mathcal{H}_u$. The carrier space of such a representation is given by

$$\mathcal{H} = L^2([m_0,\infty)\times\mathbb{R}^3, dm\otimes\frac{d^3p}{2(m^2+\vec{p}^2)^{1/2}})\otimes\mathbb{C}^{2s+1} \quad , \tag{9}$$

where m_0 is a threshold mass. It is useful sometimes to separate fully the kinematical variables from mass. To this end, one has to employ the four-velocity $k = p/m$, i.e., to introduce the Hilbert space

$$\tilde{\mathcal{H}} = L^2([m_0,\infty))\otimes L^2(\mathbb{R}^3, d^3k/2k_0)\otimes\mathbb{C}^{2s+1} \quad , \tag{10}$$

where $k_0 = (1+\vec{k}^2)^{1/2}$; the two spaces are isomorphic by means of the relation

$$\tilde{\psi}_j(m,\vec{k}) = m\psi_j(m,m\vec{k}) \tag{11}$$

valid for all $j = -s,-s+1,\dots,s$, $m\in[m_0,\infty)$ and $\vec{k}\in\mathbb{R}^3$.

The representation U acts on the space (9) according to the following prescription

$$(U(\Lambda,a)\psi)(m,\vec{p}) = e^{-ip.a}\, S(m,s;\Lambda)\,\psi(m,\vec{p}_\Lambda) \quad , \tag{12}$$

where $a.p = a_\mu p^\mu$, further $\vec{p}_\Lambda$ is the three-vector part of $\Lambda^{-1}p$, and the matrix S expresses by means of representations of the little group SU(2) . For the space-time translation on $x = (t,\vec{x})$, we have $S = I$ so

$$(\varphi,U(I,x)\psi) = \sum_{j=-s}^{s}\int_{m_0}^{\infty}dm\int_{\mathbb{R}^3}\frac{d^3p}{2(m^2+\vec{p}^2)^{1/2}}\exp\{-i(t(m^2+\vec{p}^2)^{1/2} - x.p)\}\bar{\varphi}_j(m,\vec{p})\psi_j(m,\vec{p}) \quad . \tag{13}$$

In particular, for the pure time translations and $\varphi = \psi$ we have

$$(\psi,U_t\psi) = \sum_{j=-s}^{s}\int_{m_0}^{\infty}dm\int_{\mathbb{R}^3}\frac{d^3p}{2(m^2+\vec{p}^2)^{1/2}}\exp\{-it(m^2+\vec{p}^2)^{1/2}\}|\psi_j(m,\vec{p})|^2 \quad . \tag{14a}$$

Changing the variables $(m,\vec{p})$ to $(\lambda,\vec{p})$ with $\lambda = p_0 = (m^2+\vec{p}^2)^{1/2}$, we may rewrite the last expression in the form

$$(\psi,U_t\psi) = \int_{m_0}^{\infty}d\lambda\, e^{-i\lambda t}\Big\{\sum_{j=-s}^{s}\int_{V_\lambda}\frac{d^3p}{2(\lambda^2-\vec{p}^2)^{1/2}}|\psi_j((\lambda^2-\vec{p}^2)^{1/2},\vec{p})|^2\Big\} \quad , \tag{14b}$$

where $V_\lambda = \{\vec{p} : |\vec{p}| \le (\lambda^2-m_0^2)^{1/2}\}$.

Effective one-dimensionality of $\mathcal{H}_u$

Now the crucial point lies in the choice of the subspace $\mathcal{H}_u$ which would be ascribed to the unstable particle alone. If this space was one-dimensional (spanned by some $\psi\in\mathcal{H}$), then (14) would yield according to (3b) the non-decay amplitude. However, we have argued above that $\mathcal{H}_u$ should be invariant particularly with respect to the space translations. This is impossible for a one-dimensional $\mathcal{H}_u$, because the momentum operators P_j have purely continuous spectra so ψ cannot by their eigenvector. Nevertheless, we are going to formulate an argument which shows that in most cases the relations (14) may be accepted as expressions of the non-decay amplitude in a reasonable approximation.

We shall consider first the scalar particles, $s = 0$. Our most important hypothesis is that there is a state of the unstable particle described by a wave function which factorizes

$$\psi(m,\vec{p}) = f(m)g(\vec{p}) \quad . \tag{15}$$

Next we addopt various simplifying assumptions. First of all, we set

$$\text{supp } f = (M-\eta, M+\eta)\subset[m_0,\infty) \quad , \tag{16a}$$

$$\int_{m_0}^{\infty}|f(m)|^2 dm = \int_{M-\eta}^{M+\eta}|f(m)|^2 dm = 1 \quad , \tag{16b}$$

where η is supposed to be a positive number much less than M . Further we assume

$$\text{supp } g = B_\varepsilon = \{\vec{p} : |\vec{p}| < \varepsilon\} \tag{16c}$$

so the support of g is centered at $\vec{p} = 0$. For small enough ε , this is practically equivalent to the assumption that the particle dwells in its rest system. According to (12), the space translations give $\psi_{\vec{a}} : \psi_{\vec{a}}(m,\vec{p}) = e^{i\vec{p}.\vec{a}}\psi(m,\vec{p})$ when acting on $\psi = \psi_{\vec{0}}$. Since $\psi_{\vec{a}}$ should belong to $\mathcal{H}_u$ for all $\vec{a}\in\mathbb{R}^3$, and the exponentials form a complete set in $L^2(B_\varepsilon)$, we may set

$$\mathcal{H}_u = \{ \psi : \psi(m,\vec{p}) = f(m)g(\vec{p}) \;,\; g \in L^2(B_\varepsilon) \} \;. \tag{17}$$

As a set, this $\mathcal{H}_u$ coincides with $\mathbb{C}(f) \otimes L^2(B_\varepsilon)$, where $\mathbb{C}(f)$ is the complex linear span of f . The scalar product is, however, different : the norm of ψ is according to (9),(16) given by

$$\|\psi\|^2 = \int_{M-\eta}^{M+\eta} dm \; |f(m)|^2 \int_{B_\varepsilon} \frac{d^3p}{2(m^2+\vec{p}^2)^{1/2}} \, |g(\vec{p})|^2 \;. \tag{18a}$$

Let $\|\cdot\|_2$ denote the norm in $L^2(B_\varepsilon)$:

$$\|g\|_2^2 = \int_{B_\varepsilon} |g(\vec{p})|^2 \, d^3p \;. \tag{18b}$$

We may use it to estimate the norm (18a) from both the sides, or vice versa, to derive the inequalities

$$2(M-\eta)\|\psi\|^2 \leq \|g\|_2^2 \leq 2\left[(M+\eta)^2+\varepsilon^2\right]^{1/2}\|\psi\|^2 \;. \tag{19}$$

It shows particularly that a sequence $\{\psi_n\} \subset \mathcal{H}_u$ is Cauchy iff the same is true for the corresponding sequence $\{g_n\} \subset L^2(B_\varepsilon)$; hence $\mathcal{H}_u$ defined by (17) is a (closed) subspace in $\mathcal{H}$. The inequality (33a) below shows that $\varepsilon \ll M$ and the same restriction was imposed on η , so the function g corresponding to a unit vector $\psi \in \mathcal{H}_u$ fulfils $\|g\|_2^2 \approx (2M)^{1/2}$.

Let us inspect now action of the time-translation operators on a unit ψ from the chosen subspace (17). According to (12), they multiply $\psi(m,\vec{p})$ by $\exp\{-it(m^2+\vec{p}^2)^{1/2}\}$. This expression does not factorize, but for ε small enough one may try to approximate it by e^{-imt} . Since $\varepsilon \ll M$ [*], we may restrict ourselves to the first two terms of the expansion

$$\exp\{-it(m^2+\vec{p}^2)^{1/2}\} = e^{-imt}\left\{1 - i\,\frac{\vec{p}^2 t}{2m} + O(\vec{p}^4)\right\} \;. \tag{20}$$

The evolution operator is correspondingly written as $U_t = U_t^{(0)} + U_t^{(1)}$ with neglection of the remainder. In order to estimate influence of the second term, we take an arbitrary unit vector $\varphi \in \mathcal{H}_u$, $\varphi(m,\vec{p}) = f(m)h(\vec{p})$, and express

$$(\varphi, U_t^{(1)}\psi) = \int_{M-\eta}^{M+\eta} dm \; e^{-imt} |f(m)|^2 \int_{B_\varepsilon} \frac{d^3p}{2(m^2+\vec{p}^2)^{1/2}} \left(-i\,\frac{\vec{p}^2 t}{2m}\right) \overline{h}(\vec{p}) g(\vec{p}) \tag{21}$$

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*) According to (33a), we have $\varepsilon^2 \ll M\Gamma$, and therefore in most cases $(\varepsilon/M)^2 \ll 10^{-11}$.

The relations (16),(19) and (21) yield the following inequalities

$$|(\varphi, U_t^{(1)}\psi)| \leq \frac{1}{2(M-\eta)} \, \frac{\varepsilon^2 t}{2(M-\eta)} \|h\|_2 \|g\|_2 \leq \frac{\left[(M+\eta)^2+\varepsilon^2\right]^{1/2}}{M-\eta} \, \frac{\varepsilon^2 t}{2(M-\eta)} \;. \tag{22}$$

Hence we may estimate the norm

$$\|E_u U_t^{(1)}\psi\| = \sup\left\{ |(\varphi, U_t^{(1)}\psi)| \;:\; \varphi \in \mathcal{H}_u \;,\; \|\varphi\| = 1 \right\} \;. \tag{23}$$

Since both ε, η are supposed to be much less than M , we find (23) to be $\lesssim \varepsilon^2 t/2M$; the approximation mentioned above is therefore possible under the condition

$$\frac{\varepsilon^2 t}{2M} \ll 1 \;. \tag{24}$$

In that case, norm of the difference between $E_u U_t \psi = V_t \psi$ and $E_u U_t^{(0)}\psi$ is very small, and we are allowed to write $V_t\psi \approx E_u U_t^{(0)}\psi$.

In the next step, we shall verify that the last expression is close to $(\psi, U_t^{(0)}\psi)\psi$. To this end, we take an arbitrary unit vector $\varphi \in \mathcal{H}_u$, $\varphi(m,\vec{p}) = f(m)h(\vec{p})$, which is perpendicular to ψ . This orthogonality together with (16b) makes it possible to estimate $(h,g)_2$ from the identity

$$\frac{1}{2M}(h,g)_2 = \int_{M-\eta}^{M+\eta} dm\,|f(m)|^2 \int_{B_\varepsilon} \left(\frac{1}{2M} - \frac{1}{2(m^2+\vec{p}^2)^{1/2}}\right) \overline{h}(\vec{p}) g(\vec{p}) \, d^3p \;. \tag{25}$$

Since ε, η are much less than M , we have the following estimate

$$\left| \frac{1}{2M} - \frac{1}{2(m^2+p^2)^{1/2}} \right| \lesssim \frac{1}{2M}\left(\frac{\eta}{M} + \frac{\varepsilon^2}{2M^2}\right) \tag{26}$$

(up to higher order terms). Combining it with the Hölder inequality, we obtain

$$|(h,g)_2| \lesssim \left(\frac{\eta}{M} + \frac{\varepsilon^2}{2M^2}\right) \|h\|_2 \|g\|_2 \;. \tag{27}$$

Now we are able to estimate the scalar product $(\varphi, U_t^{(0)}\psi)$:

$$\begin{aligned} |(\varphi, U_t^{(0)}\psi)| \leq{} & \left| \frac{1}{2M} \int_{M-\eta}^{M+\eta} dm \; |f(m)|^2 \; e^{-imt} \; (h,g)_2 \right| + \\ & + \int_{M-\eta}^{M+\eta} dm \; |f(m)|^2 \int_{B_\varepsilon} \left| \frac{1}{2(m^2+\vec{p}^2)^{1/2}} - \frac{1}{2M} \right| |h(\vec{p})| \, |g(\vec{p})| \, d^3p \;. \end{aligned} \tag{28}$$

Applying (26) and (27) to the second and the first term on the rhs of (28), respectively, and using the Hölder inequality again, we get

$$|(\varphi, U_t^{(0)}\psi)| \lesssim \frac{2}{2M}\left(\frac{\eta}{M} + \frac{\varepsilon^2}{2M^2}\right) \|h\|_2 \|g\|_2 \int_{M-\eta}^{M+\eta} |f(m)|^2\, dm \quad .$$

However, φ and ψ are assumed to be unit vectors so $\|h\|_2 \approx \|g\|_2 \approx$ $\approx (2M)^{1/2}$. Finally, the normalization condition (16b) yields

$$|(\varphi, U_t^{(0)}\psi)| \lesssim 2\frac{\eta}{M} + \frac{\varepsilon^2}{M^2} \quad . \tag{29}$$

Since φ is an arbitrary unit vector from $\mathcal{H}_u$ orthogonal to ψ , we see that $U_t^{(0)}\psi$ stays nearly parallel to ψ . Hence we may write

$$(V_t\psi)(m,\vec{p}) \approx \psi(m,\vec{p}) \int_{m_0}^{\infty} d\mu\; e^{-i\mu t} |f(\mu)|^2 \int_{B_\varepsilon} \frac{d^3\varkappa}{2(m^2+\vec{\varkappa}^2)^{1/2}} |g(\vec{\varkappa})|^2 \quad . \tag{30a}$$

Moreover, the inequality (26) allows to replace the denominator in the last integral by $2M$; the corresponding error is again at most comparable with the rhs of (27) . Thus we have also

$$(V_t\psi)(m,\vec{p}) \approx \psi(m,\vec{p}) \int_{m_0}^{\infty} e^{-i\mu t} |f(\mu)|^2\, d\mu \quad . \tag{30b}$$

Concluding the above discussion, we may say that <u>if the three-momentum spread of ψ is sufficiently narrow, the decay goes effectively as if $\mathcal{H}_u$ would be one-dimensional</u>. In that case, the non-decay amplitude is given by (14), and it may be approximated by the integrals appearing in (30). Of course, the approximation needs also $\eta \ll M$ but it can be achieved as we shall see in a while.

The presented argument generalizes easily for particles with a non-zero spin. One has only to use the rotational invariance of $\mathcal{H}_u$ too, then the following choice is natural

$$\mathcal{H}_u = \{\psi : \psi(m,\vec{p}) = f(m)g(\vec{p}) \quad , \; g \in L^2(B_\varepsilon) \otimes \mathbb{C}^{2s+1}\} \quad . \tag{31}$$

Mimicking the above reasoning, we arrive again at the approximation (30b) .

Hence we must ask under which circumstances the conditions (24) and $\eta \ll M$ are valid. In any realistic description of unstable particles, the function $|f(.)|^2$ should have a sharp peak of more or less Breit-Wigner shape. Its position may be identified with the mass M of the particle. On the other hand, the mean life is defined by

$$T = \int_0^{\infty} P_\psi(t)\, dt \quad ; \tag{32}$$

its inverse Γ characterizes width of the peak. For all real unstable particles, M is much larger than Γ : the ratio M/Γ varies from 1.06×10^5 for Σ^0 to 1.31×10^{27} for neutrons (with exception of π^0, η and Σ^0 , its lower bound is 10^{11}). Hence we can choose η so the inequalities $\Gamma \ll \eta \ll M$ hold. The first of them ensures that truncation of the mass distribution $|f(.)|^2$ to the interval $(M-\eta, M+\eta)$ will cause a negligible change in the decay law*,**.

Of course, the condition (24) cannot hold for all values of t , but it seems reasonable to demand its validity in the region where the decay law is actually measured, i.e., up to few T . Thus the three-momentum spread $\Delta p \equiv \varepsilon$ must obey $(\Delta p)^2 \ll M\Gamma$ or

$$\Delta p \ll c^{-1}(M\Gamma)^{1/2} \tag{33a}$$

when we return to the conventional system of units. In order to appreciate this restriction, let us rewrite it by the uncertainty relation to the form

$$\Delta q \gg \hbar c\,(M\Gamma)^{-1/2} \quad . \tag{33b}$$

Thus we come to the following result : the conclusion about the effectively one-dimensional $\mathcal{H}_u$ is applicable <u>provided the unstable particle is not spatially localized too sharply to violate (33b)</u>. This condition is, however, fulfilled almost always in actual experimental arrangements as the below listed values show[24] :

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*) Cf., e.g., Refs.20,23 . Simple estimates similar to those performed there show that for all practical purposes it is enough to choose $\eta \approx 10^2\Gamma$. Thus we may assume $\eta/M \lesssim 10^{-9}$ in most cases (see the footnote on p.8). In fact, truncation of the mass distribution might change T substantially, because the modified decay law has a power-like decrease, eventually as t^{-1} , for large values of t . However, from the practitioneer's point of view the infinity may be replaced in the integral (32) by the range, where the decay law is actually measured, say $10\,\Gamma^{-1}$, so the tail effect is suppressed. Notice that validity of the condition (24) may be discussed under a similar restriction on t.

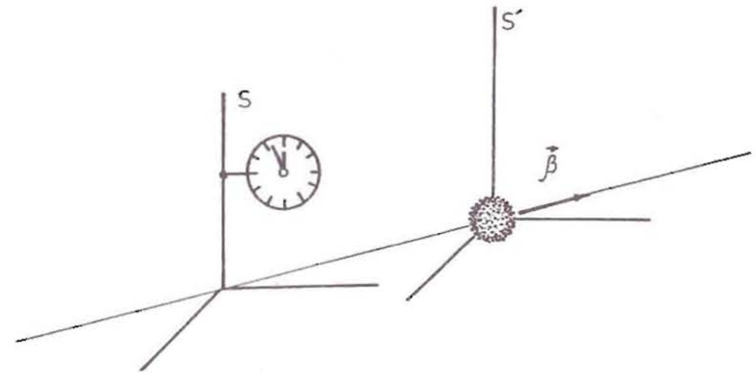
**) We restrict our attention to real unstable (metastable) particles. Our assumptions may not be fulfilled for scattering resonances. However, time evolution of a resonance as a separate object cannot be studied experimentally (as noticed by many authors, particularly in Ref.11) and even associating some $\mathcal{H}_u$ with it is a speculative matter.

particle	$\hbar c(M\Gamma)^{-1/2}$ [cm]	particle	$\hbar c(M\Gamma)^{-1/2}$ [cm]	particle	$\hbar c(M\Gamma)^{-1/2}$ [cm]
μ	1.11×10^{-4}	$\pi^{\pm}$	1.05×10^{-5}	n	0.763
τ	1.2×10^{-8}	π^{0}	6.03×10^{-10}	Λ	3.74×10^{-7}
		η	1.3×10^{-11}	Σ^{+}	3.53×10^{-7}
		$K^{\pm}$	3.85×10^{-6}	Σ^{0}	5.4×10^{-12}
		K_S	3.26×10^{-7}	Σ^{-}	2.71×10^{-7}
		K_L	7.85×10^{-6}	Ξ^{0}	3.6×10^{-7}
		$D^{\pm}$	1.7×10^{-8}	Ξ^{-}	2.71×10^{-7}
		D^{0}	1.2×10^{-8}	Ω^{-}	1.70×10^{-7}
		F	8.0×10^{-8}	Λ_c^{+}	5.3×10^{-9}

Notice finally that the above considerations apply to the "coordinate" part of the wave function only. If the part of the decay problem related to internal degrees of freedom cannot be decomposed fully, we have $\dim \mathcal{H}_u > 1$ even in the sense of the discussed approximation. So for neutral kaons, e.g., the space $\mathcal{H}_u$ is effectively two-dimensional provided the conditions (33) are valid.

Decay of a moving particle

We are obliged to show that the proposed description by means of the representation (12) and its restriction to a subspace of the type (31) will yield a correct result for an unstable particle which is not at rest. Let a reference frame S belong to the observer, and suppose the rest system of the particle to move with a velocity $\vec{\beta}$ respectively to S , as it is sketched on the figure. Of course, we may not only sandwich the propagator between $U(\Lambda_{\pm\vec{\beta}},0)$; similarly as a simple-minded look on the factor which multiplies the time variable in Lorentz transformation does not yield the time dilatation. From the viewpoint of the reference frame S , we are interested in the space time shift on $x=(t,\vec{\beta}t)$. If the condition (33a) is valid, i.e., if we are allowed to characterize the particle by a single vector $\psi \in \mathcal{H}$ which refers to its rest system, then the observer will ascribe to it the vector $U(\Lambda_{\vec{\beta}},0)^{-1}\psi$. The corresponding non-decay amplitude equals



$$v(t;\vec{\beta}) = (U(\Lambda_{\vec{\beta}},0)^{-1}\psi, U(I,x)U(\Lambda_{\vec{\beta}},0)^{-1}\psi) \quad . \qquad (34)$$

Using the relations (5) and (13), we may rewrite (34) as follows

$$v(t;\vec{\beta}) = (\psi, U(I,\Lambda_{\vec{\beta}}x)\psi) =$$

$$= \sum_{j=-s}^{s} \int_{m_0}^{\infty} dm \int_{\mathbb{R}^3} \frac{d^3p}{2(m^2+\vec{p}^2)^{1/2}} \exp\{-ip.\Lambda_{\vec{\beta}}x\} \, |\psi_j(m,\vec{p})|^2 \quad .$$

However, the Lorentz transformation gives $\Lambda_{\vec{\beta}}x = (t(1-\vec{\beta}^2)^{1/2},\vec{0})$ so

$$v(t;\vec{\beta}) = v(t(1-\vec{\beta}^2)^{1/2};\vec{0}) \quad . \qquad (35)$$

This provides us the relation

$$P_\psi(t;\vec{\beta}) = P_\psi(t(1-\vec{\beta}^2)^{1/2};\vec{0}) \quad , \qquad (36)$$

which is valid as far as the approximation identifying the decay law with the square of (34) may be used. The relation (36) is, of course, the desired result. It is tested by numerous experiments ; and it was even used for a direct proof of the relativistic time dilatation from cosmic-ray muons thirty years ago (cf.Ref.25, section IV.4.3).

Conclusions

Let us compare the above discussed description of unstable particles with the one based on non-unitary representations of $\mathcal{P}$. We have already mentioned the Williams' construction[13] of minimal unitary dilation for the non-unitary representation proposed by Zwanziger[3]. He obtained the Hilbert space (10) with $m_0 = -\infty$ and a unitary representation $\tilde{U}$ of $\mathcal{P}$ on $\tilde{\mathcal{H}}$ which coincides with (12) when transformed by means of (11). The principal difference concerns the choice of $\mathcal{H}_u$: the Zwanziger's representation is recovered by projection of $\tilde{U}$ to the subspace

$$\tilde{\mathcal{H}}_u^Z = \mathbb{C}(f)\otimes L^2(\mathbb{R}^3, d^3k/2k_0)\otimes \mathbb{C}^{2s+1} , \qquad (37)$$

where $f(m) = (2\pi/\Gamma)^{-1/2}(m - M + \frac{i}{2}\Gamma)^{-1}$.

Williams himself regarded the below unbounded mass spectrum as the main defect, but it can be rectified by a mass-filtering procedure without any observable consequences[21-23]; essentially the same argument we have used through the condition (16a). Except of that, in a theory pretending for completeness the function f should be obtained as a solution to the dynamical problem, with the Breit-Wigner shape of $|f(.)|^2$ resulting from the pole approximation to this solution. However, it seems that we will not have such a theory soon. In spite of a substantial progress achieved in the perturbation theory of embedded eigenvalues during the last decade *), one can hardly proceed beyond the Fermi golden rule since even finding of the "unperturbed" eigenvalues represents a difficult problem for the theory of strong interactions.

A difference between the two approaches is now obvious. In both of them, it is only the mass distribution which is essential for expression of the decay law, while effect of the momentum (velocity) dependence of the wave function is suppressed. In the approach treated here, this conclusion is obtained by realizing that the momentum distribution is actually very narrow **). On the contrary, with the choice (37) the mentioned independence is achieved because it makes all velocity distributions possible. Both the approaches

*) Cf., e.g., Refs.26,27 , and a more complete bibliography in Ref.28, notes to section XII.6 .

**) This fact was already noted, particularly in Ref.11 . However, the main problem is to use this observation to prove that the decay law defined naturally by (3a) may be approximated by the much more simple expressions which follow from (3b) and (30b).

yield the same decay law, simply because they have been constructed so. However, the first one has the advantage of producing the translationally-invariant description.

One might say, that in a subspace $\mathcal{H}_u$ of the type (37) a lot of space is left unemployed. The present qualitative considerations show that one really needs only *)

$$\tilde{\mathcal{H}}_u' = \mathbb{C}(f)\otimes L^2(B_\varkappa, d^3k/2k_0)\otimes \mathbb{C}^{2s+1} , \qquad (38)$$

where $\varkappa \approx \varepsilon/M$. The subspace $\mathcal{H}_u' \subset \mathcal{H}$ isomorphic to (38) through (11) is "intermediate" in a sense between (31) and $\mathcal{H}_u^Z$ referring to (37). For $\mathcal{H}_u'$, one can derive a conclusion about the effective one-dimensionality with more ease. On the other hand, (38) is not longer translationally-invariant, though the violation is manifested on large distances only **).

Finally, let us mention that frequently the possibility of neglecting $\vec{p}$-spread of the wave function is even better than the condition (33b) together with the table show. We have in mind the situations when the unstable particle suffers repeated non-decay measurements[20,23], e.g., by monitoring its track. Since the decay starts anew after each measurement (which has given the positive result), we need not require (24) to hold for times comparable with Γ^{-1} but merely with the mean time between the neighbouring measurements which is usually few orders of magnitude shorter. As an example, consider the decay of charged kaons treated in Ref.23 : there the mean time between measurements is $\sim 10^{-4}\Gamma^{-1}$. Instead of (33b), we obtain then the condition $\Delta q \gg 10^{-8}$ cm , but actually the kaons are localized within the range of bubble diameter, i.e., about 10^{-2} cm . Similar conclusions may be obtained for the other unstable particles and track-monitoring devices too. On the other hand, the conclusions about the effective one-dimensionality of $\mathcal{H}_u$ can be used to justify the basic reduction postulate of the repeated-measurements theories.

*) Of course, the condition (33b) does not really require the momentum (velocity) distribution to be supported by some ball. In order to take possible tails of these distributions into account (preserving at the same time translational invariance of $\mathcal{H}_u$), a mathematically more sophisticated treatment is needed ; we hope to discuss it elsewhere.

**) Assume again the condition (16a) to be fulfilled. Then it is easy to see that the non-invariance becomes essential for $\varkappa^{-1}(\eta/M)a\Delta p \gtrsim 1$, i.e., $a \gtrsim (M/\eta)\Delta q$. Thus if $M/\eta \gtrsim 10^9$ and $\Delta q = 10^{-2}$ cm , we get $a \gtrsim 100$ km .

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Экснер П. E2-83-1
О представлениях группы Пуанкаре, связанных с нестабильными частицами

Обсуждается проблема релятивистски-ковариантного описания нестабильных частиц. Мы придерживаемся подхода, который связывает унитарное приводимое представление группы Пуанкаре с подходящей изолированной системой, и сравниваем его с подходом, в котором одной нестабильной частице приписывается неунитарное неприводимое представление. Показано, что проблема основывается на выборе подпространства $\mathcal{H}_u$ в гильбертовом пространстве состояний, которое можно сопоставить нестабильной частице. Показано, что трансляционная инвариантность $\mathcal{H}_u$ несовместима с унитарностью лоренцевских бустов. Предложен конкретный вид $\mathcal{H}_u$ и приводятся рассуждения, показывающие, что для большинства реальных экспериментальных ситуаций это подпространство является эффективно одномерным. Получено правильное замедление для распада движущейся частицы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Exner P. E2-83-1
On the Representations of Poincaré Group Associated with Unstable Particles

The problem of relativistically-covariant description of unstable particles is reexamined. We follow the approach which associates a unitary reducible representation of Poincaré group with a larger isolated system, and compare it with the one ascribing a non-unitary irreducible representation to the unstable particle alone. It is shown that the problem roots in choice of the subspace $\mathcal{H}_u$ of the state Hilbert space which could be related to the unstable particle. Translational invariance of $\mathcal{H}_u$ is proved to be incompatible with unitarity of the boosts. Further we propose a concrete choice of $\mathcal{H}_u$ and argue that in most cases of the actual experimental arrangements, this subspace is effectively one-dimensional. A correct slow-down for decay of a moving particle is obtained.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1983